Representations and Quantum Systems

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Physical Background

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- (Quantized) Only specific angular momentum are allowed.
- (Fixed) Although direction could be altered, its magnitude never change

Spin Quantum Numbers

Spin Quantum Numbers

For each type of particle, we associate $s = \frac{n}{2}$ where n is an integer. It only depends on the type of particles and cannot be altered in any known way.

Fermions: types of particles with half-integers spins $(eg.\frac{1}{2}, \frac{3}{2}\cdots)$
Bosons: types of particles with integers spins $(eg.1, 2\cdots)$

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Bosons: types of particles with integers spins $(eg.1, 2\cdots)$
electron: $S_e = \frac{1}{2}$

Quantum System for Electrons

We developed a system to make sense of the direction of the spins and probabilities

State of Electron

The spin can be described as a unit vector in $\mathbb{C}^2$ with inner product $\langle \psi_1, \psi_2 \rangle = \psi_1^* \cdot \psi_2$.

$\mathbb{C}^2$ is called the state space of electrons. It describes all possible "direction" of the spin.

Linear Operators

Definition (Linear Operator)

An operator is a transformation that transform a state into another. Suppose the operator is A, and states ψ_1, ψ_2 then

$$A\psi_1 \rightarrow \psi_2$$

In addition, a linear operator has to satisfy

$$A(\alpha\psi_1) = \alpha A\psi_1, A(\psi_1 + \psi_2) = A\psi_1 + A\psi_2.$$

This is how we alter the direction of the spin.

Unitary Operators

Definition (Unitary Operator)

An operator U is said to be unitary if it preserves the inner product. That is,

$$\langle \psi_1, \psi_2 \rangle = \langle U\psi_1, U\psi_2 \rangle$$

It preserves the "length" and the "angles" of the vectors.

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Time-evolution Operator

Time evolution operators are all unitary.

Rationale

- Time evolution operators ought to be unitary since the magnitude of the spin cannot change and it shouldn't matter when we measure the system.
- We naturally care about unitary representation since unitary operators are important in quantum mechanics.
- Since unitary operators preserves length and angle, it essentially is a type of rotation.

Representation

Definition (Representation)

A Representation (π, V) *of a group* G *and vector space* V *is a homomorphism*

$$\pi : G \to GL(V)$$

where $GL(V)$ *is the group of invertible linear maps* $V \to V$*.*

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It could be viewed as matching a group element with a transformation(linear map) of the vector space.

Representation

Definition (Unitary Representation)

An representation (π, V) on a complex vector space V with Hermitian inner product $\langle \cdot, \cdot \rangle$ is a unitary representation if preserves the inner product with the map

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That is,

$$\langle v_1, v_2 \rangle = \langle \pi(g) v_1, \pi(g) v_2 \rangle$$

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The linear map preserves the "distance" of the two vectors in the space.

Rotation group SO(3)

As we could imagine the transformations as rotations, we want to realize the rotation group in some way on quantum mechanics. It would be nice if one can represent a unitary operator by a familiar rotation in 3D spacetime, which is finding a unitary representation of $SO(3)$.

Definition (Rotation Group SO(3))

The group of 3×3 orthogonal matrix with determinant 1. That is, for matrix A,

$$AA^T = A^T A = I, det(A) = 1$$

This group is all the rotations about the origin in $\mathbb{R}^3$. we want to find a representation of $SO(3)$, $(\pi, \mathbb{C}^2)$.

SU(2), SO(3)

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Unitary Transformation of the Two-State System

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Definition (Special Unitary group SU(2))

The group of 2×2 unitary matrix with determinant 1. That is, for matrix U,

$$UU^* = U^*U = I, det(U) = 1$$

where U^* is its conjugate transpose.

General form

In general it has the form $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$ with $|\alpha|^2 + |\beta|^2 = 1$.

Spinor Representation of $SU(2)$

a representation of $SU(2)$, $(\pi_{spinor}, \mathbb{C}^2)$ is given naturally by

$$\pi_{spinor}(g) = g \in GL(\mathbb{C}^2)$$

Representation of SU(2) and SO(3)

Theorem (Double Covering Map)

There is a 2-to-1 surjective homomorphism

$$\phi : SU(2) \to SO(3)$$

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There is no representations of $SO(3)$ that can represent the unitary operators on $\mathbb{C}^2$.

Rotations of $\mathbb{C}^2$

- As the spin of electron evolves through time, it is not the same as rotations in $\mathbb{R}^3$.
- Because of the 2-to-1 aspect of the homomorphism. There are two rotations in $SU(2)$ that correspond to the same rotation in $SO(3)$
- Rotating 360 doesn't get back to the identity. You have to rotate 720 to be back at the same place.
- This apply to all particles with spin number $\frac{1}{2}$.

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Thank You!