The Geometry and Limits of Young Partition Flow Polytopes

Advay Goel[1], Mentor: Zoe Wellner[2]

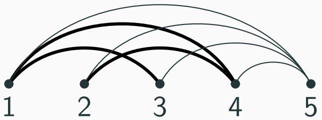
MIT PRIMES Conference

October 2022

[1]Acton-Boxborough Regional High School
[2]Carnegie Mellon University

Digraphs

Consider G to be a loopless directed graph with vertices $\{1, \ldots, n, n+1\}$. We direct edges by vertex order, so for $i < j$, the edge x_{ij} is directed from i to j. Here is an example for $n = 4$.



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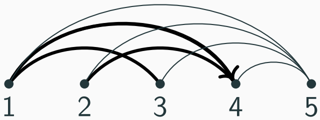
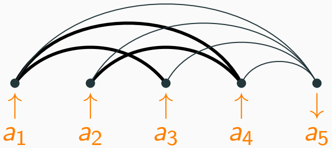
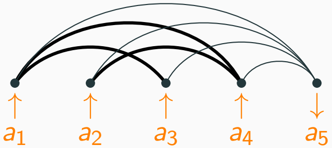


Netflow Vectors

Along with a digraph, we also need $\mathbf{a} \in \mathbb{R}^{n+1}$ called the **netflow vector**. Each element of the netflow vector corresponds to a vertex in the digraph as we can see applied to the previous example.



Forming the Flow Polytope



At each vertex, we must satisfy **conservation of flow**. For our example this gives the following equations:

$$x_{13} + x_{14} + x_{15} = a_1$$

$$x_{24} + x_{25} = a_2$$

$$x_{35} = a_3 + x_{13}$$

$$x_{45} = a_4 + x_{14} + x_{24}$$

The CRY Polytope

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Theorem (Zeilberger 1999)

The volume of the CRY polytope is given by

$$\mathrm{vol} CRY_{n+1} = \prod_{i=1}^{n-1} C_i$$

where $C_i = \frac{1}{i+1}\binom{2i}{i}$ *is the* i*th Catalan number.*

The First Proof of the CRY Conjecture

Definition (The Morris Constant Term)

For $n, a, b \in \mathbb{Z}^+$ and $c \in \mathbb{Z}_{\geq 0}$, define the constant term

$$M_n(a,b,c) := \mathrm{CT}_x \prod_{i=1}^{n} (1-x_i)^{-b} x_i^{-a+1} \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-c},$$

where $\mathrm{CT}_x := \mathrm{CT}_{x_n} \ldots \mathrm{CT}_{x_1}$.

Theorem (Zeilberger 1999)

The Morris Constant Term can be expressed as

$$M_n(a,b,c) = \prod_{j=0}^{n-1} \frac{\Gamma(a-1+b+(n-1+j)\frac{c}{2})\Gamma(\frac{c}{2}+1)}{\Gamma(a+j\frac{c}{2})\Gamma(b+j\frac{c}{2})\Gamma(\frac{c}{2}(j+1)+1)}.$$

Furthermore, by setting $a = b = c = 1$, the volume of CRY_{n+1} is given by $M_n(1,1,1) = \prod_{i=1}^{n-1} C_i$ where C_i is the ith Catalan number.

Motivation For Studying Flow Polytopes

- Why do the Catalan numbers appear?
- What combinatorial relations do flow polytopes have with other objects?

Partitions

Example (Partition)

$\lambda = (4, 2, 1, 1)$ is a partition of 8 since $4 + 2 + 1 + 1 = 8$. The **length** of $\ell(\lambda)$ is the number of parts, so in this case 4.

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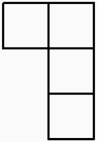
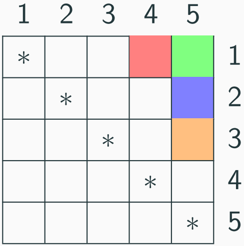
A right-justified Young Diagram contains $\ell(\lambda)$ rows. The length of row *i* is given by the *i*th index in the partition.

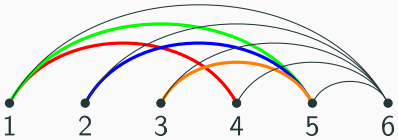
Below, we see the right-justified Young Diagram for the partition $\lambda = (2, 1, 1)$:



Flow Polytopes of Young Partitions

From left to right: the left-justified Young diagram of $\lambda = (2, 1, 1)$, the diagram in a 5×5 matrix, and the corresponding graph on six vertices.





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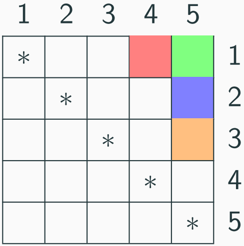
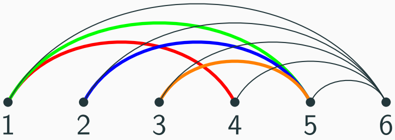


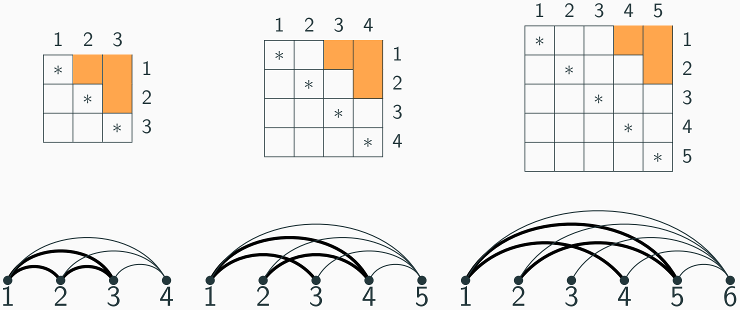


To form the flow polytope, we go as follows:

$$\text{Partition} \to \text{Digraph} \to \text{Flow Polytope}$$

Definition (Family of Flow Polytopes)

For constant λ and netflow $\mathbf{a}$, we define $\mathcal{F}_{(\lambda,\mathbf{a})}$ as a family of flow polytopes, containing all polytopes as n varies.



The Limiting Polytope

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For $n \geq \ell(\lambda) + \lambda_1$, $\mathcal{F}_{G(\lambda,n)}$ is integrally equivalent to $\mathcal{F}_{G(\lambda,\ell(\lambda)+\lambda_1)}$.

Definition (The Limiting Polytope)

For a partition λ and netflow $\mathbf{a} \in \mathbb{Z}^n_{>0}$, the **limiting polytope of the family** $\mathcal{F}_{(\lambda,\mathbf{a})}$, denoted as $\mathcal{F}^{\lim}_{(\lambda,\mathbf{a})}$, is the polytope $\mathcal{F}_{G(\lambda,\ell(\lambda)+\lambda_1)}$.

The Limiting Polytope

Theorem (Mészáros-Simpson-Wellner 2017)

Let λ be a partition and $\mathbf{a}$ a netflow vector. Then, the limiting polytope of $\mathcal{F}_{(\lambda,\mathbf{a})}$ has normalized volume:

$$\mathrm{vol}\mathcal{F}^{\lim}_{(\lambda,\mathbf{a})} = \left(\sum_{i\in[\ell(\lambda)]} \lambda_i\right)! \prod_{i\in[\ell\lambda]} \frac{a_i^{\lambda_i}}{\lambda_i!}.$$

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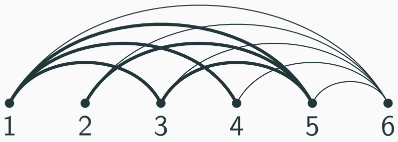
- What about other polytopes in the family?
- How do the volumes in the family change as the polytopes gets closer and closer to limiting?

The Inequalities of a Flow Polytope

Recall that a flow polytope can be expressed as a set of equations.



$$
\begin{aligned}
x_{13} + x_{14} + x_{15} + x_{16} &= a_1 \\
x_{25} + x_{26} &= a_2 \\
x_{35} + x_{36} &= x_{13} + a_3 \\
x_{46} &= x_{14} + a_4 \\
x_{56} &= x_{15} + x_{25} + a_5
\end{aligned}
$$

The Inequalities of a Flow Polytope

Recall that a flow polytope can be expressed as a set of equations.

Now, we turn them into inequalities.

$$
\begin{aligned}
x_{13}+x_{14}+x_{15}+x_{16}=a_1 \rightarrow x_{13}+x_{14}+x_{15} &\leq a_1 \\
x_{25}+x_{26}=a_2 \rightarrow x_{25} &\leq a_2 \\
x_{35}+x_{36}=x_{13}+a_3 \rightarrow x_{35} &\leq x_{13}+a_3 \\
x_{46} &= x_{14}+a_4 \\
x_{56} &= x_{15}+x_{25}+a_5
\end{aligned}
$$

Finding Volumes From Inequalities

We have 3 inequalities:

$$x_{13} + x_{14} + x_{15} \leq a_1$$

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In general, there are two main types of inequalities:

1. **Type A:** $\sum x \leq a_i$
2. **Type B:** $\sum x \leq a_i + \sum x$

Type A Inequalities

Assume we have a Type A Inequality: $\sum_{i \leq n} x_i \leq a_k$. What geometric shape does this represent?

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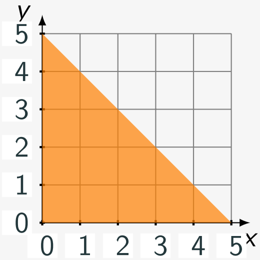
Example (n=2)

For $x + y \leq 5$ where $x, y > 0$ we have a solution set as follows:



Type B Inequalities

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There are actually several subcases to consider.

Type B Inequalities

Example (Type B)

Take the inequality $x_{35} \leq a_3 + x_{13}$. We can split this into 2 disjoint cases:

1. $x_{35} \leq a_3$
2. $a_3 \leq x_{35} \leq a_3 + x_{13}$

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Type B Inequalities

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Case 1 is a Type A Inequality, so we can deal with that. What about Case 2?

Let $x_{35} = a_3 + t_1$. Then, if $x_{13} = t_1 + t_2$ for some $t_1, t_2 > 0$, the inequality holds.

Type B Inequalities

Recall, this flow polytope is described by the following

$$x_{13} + x_{14} + x_{15} \leq a_1$$

$$x_{25} \leq a_2$$

$$x_{35} \leq x_{13} + a_3$$

Substituting $x_{13} = t_1 + t_2$ from our Type B bijection, we get

$$t_1 + t_2 + x_{14} + x_{15} \leq a_1.$$

Now the Type A inequality now has volume $\frac{1}{24}a_1^4$.

Volume Formulas for Hooks

We developed a computer algorithm to deal with Type B calculations.

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1. Turn a flow polytope into a set of inequalities.
2. Take inequality n and create a list of all m cases that it has.
3. For i going from 1 to m, consider case j of inequality n.
    1. If case j is a Type A inequality: add the simplex it represents to the value.
    2. Else:
        1. Analyze how case j affects the other inequalities of the flow polytope
    3. Create a new flow polytope f' that is f with inequality n removed and other inequalities changed based on the bijections from case j.
    4. Print output.

Volume Formulas for Hooks

Using this algorithm, we found and proved a general formula for volumes of the families of polytopes from **hook partitions**, partitions of the form $\lambda = (n, 1, 1, \dots, 1)$.

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Theorem (Goel-Wellner)

Let $\lambda = (a, 1, 1, \cdots 1)$ where there are b 1's and $a > b$. Then,

$$\mathrm{vol}\mathcal{F}_{(G,a+1)}(\mathbf{a}) = \sum_{j=0}^{b} \frac{1}{(a+j)!} a_1^{a+j} e_{b-j}(a_2, a_3, \ldots, a_{b+1})$$

and

$$\mathrm{vol}\mathcal{F}_{(G,a+1+x)}(\mathbf{a}) = \sum_{j=0}^{b-x} \frac{1}{(a+j)!} a_1^{a+j} \prod_{i=2}^{x+1} a_i e_{b-x-j}(a_{x+2}, a_{x+3}, \ldots a_{b+1})$$

for $x > 0$ where e_i is the ith elementary symmetric sum.

Future Directions

The limiting process still seems like a promising avenue to gain more combinatorial information about flow polytopes. We noticed that often in each limit step, we will have terms in our volume formulas vanish. In the future, we hope to study more families to better understand how these terms vanish at each step.

Acknowledgements

I'd like to thank

- My mentor Zoe Wellner for all her help through this journey
- The MIT PRIMES program for providing this opportunity and for all of its guidance
- Professor Karola Mészáros for her suggestions of the project onto its current path
- Professor Florian Frick for his initial discussions and introducing us to the methods for geometric decompostion.
- Connor Simpson for his aid in using SageMath
- My parents for all of their support

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Thank you! Any questions?