The Hilbert Series of the Irreducible Quotient for the Polynomial Representation of the Rational Cherednik Algebra of Type A

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October 15-16, 2022 MIT PRIMES Conference

Vector Space

Definition (Vector Space)

A vector space V is a set of elements equipped with addition and multiplication by scalars from a field.

Example

•
$$V = \mathbb{k}^{n}$$

• Addition: $\begin{bmatrix} a_{1} \\ \vdots \\ a_{n} \end{bmatrix} + \begin{bmatrix} b_{1} \\ \vdots \\ b_{n} \end{bmatrix} = \begin{bmatrix} a_{1} + b_{1} \\ \vdots \\ a_{n} + b_{n} \end{bmatrix}$
• Scalar Multiplication: $\lambda \begin{bmatrix} a_{1} \\ \vdots \\ a_{n} \end{bmatrix} = \begin{bmatrix} \lambda a_{1} \\ \vdots \\ \lambda a_{n} \end{bmatrix}$ where $\lambda \in \mathbb{k}$
• $V = \mathbb{C}^{n}$

Vector Space Examples

Example

- $V = \Bbbk[x_1, x_2]$
 - Addition:

$$\sum_{i,j\geq 0}a_{ij}x_1^ix_2^j+\sum_{i,j\geq 0}b_{ij}x_1^ix_2^j=\sum_{i,j\geq 0}(a_{ij}+b_{ij})x_1^ix_2^j$$

• Scalar Multiplication:

$$\lambda \sum_{i,j \ge 0} a_{ij} x_1^i x_2^j = \sum_{i,j \ge 0} \lambda a_{ij} x_1^i x_2^j$$

for $\lambda \in \mathbb{k}$ • $\mathbb{k}[x_1, \dots, x_n]$

Dimension

Definition (Dimension)

The dimension of a finite-dimensional vector space V is the least number of vectors needed to span V.

Example

•
$$V_1 = \mathbb{R}^n = \operatorname{span}\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\right)$$
, so dim $(V_1) = n$
• $V_2 = \mathbb{k}[x_1, x_2] = \operatorname{span}\left(x_1^i x_2^j\right)$ for all $i, j \ge 0$, so dim (V_2) is infinite

Algebra

Definition (Algebra)

An algebra is a vector space A equipped with a bilinear product.

Example

We can multiply polynomials with each other in addition to adding them and multiplying by scalars.

•
$$\Bbbk[x_1, x_2]$$

•
$$\Bbbk[x_1,\ldots,x_n]$$

Graded Algebra

Definition (Graded Algebra)

A graded algebra A can be decomposed into the form

$$A = \oplus_{i \ge 0} A_i$$

for subspaces A_i and $A_j \cdot A_k \subset A_{j+k}$.

Example

- $A = k[x_1, x_2]$, where the *i*th graded component is homogeneous polynomials of degree *i*
 - A_0 generator: 1
 - A₁ generators: x₁, x₂
 - A_2 generators: x_1^2 , x_1x_2 , x_2^2
 - A_i generators: x_1^i , $x_1^{i-1}x_2$, $x_1^{i-2}x_2^2$, ..., x_2^i

Definition (Hilbert Series)

To a graded algebra A, we can associate a power series, the Hilbert series, given by

$$h_A(z) = \sum_{i\geq 0} \dim (A_i) z^i.$$

Hilbert Series Example

Hilbert series:
$$h_A(z) = \sum_{i\geq 0} \dim(A_i) z^i$$
.

Example

Consider $A = \Bbbk[x_1, x_2]$.

- A_0 generator: 1, so dim $(A_0) = 1$
- A_1 generators: x_1 , x_2 , so dim $(A_1) = 2$
- A_2 generators: x_1^2 , x_1x_2 , x_2^2 , so dim $(A_2) = 3$
- A_i generators: x_1^i , $x_1^{i-1}x_2$, $x_1^{i-2}x_2^2$, ..., x_2^i , so dim $(A_i) = i + 1$

We have

$$h_A(z) = \sum_{i \ge 0} (i+1) z^i = rac{1}{(1-z)^2}.$$

Another Hilbert Series Example

Hilbert series:
$$h_A(z) = \sum_{i\geq 0} \dim (A_i) z^i$$
.

Example

Consider $A = \Bbbk[x_1, \ldots, x_n]$.

- A_0 generator: 1, so dim $(A_0) = 1$
- A_1 generators: x_1, \ldots, x_n , so dim $(A_1) = n$
- A_2 generators: x_i^2 , $x_j x_k$, so 2 stars, n-1 bars gives dim $(A_2) = \binom{n+1}{2}$
- A_i : we have *i* stars and n-1 bars, so dim $(A_i) = \binom{n+i-1}{i}$

So we have

$$h_A(z) = \sum_{i\geq 0} \binom{n+i-1}{i} z^i = \frac{1}{(1-z)^n}.$$

Definition (Representation)

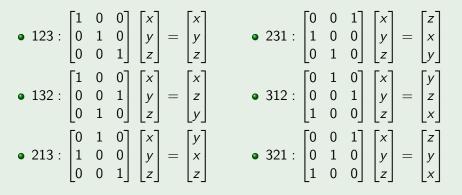
A representation of an algebra A is a vector space V equipped with a homomorphism $\rho: A \to End(V)$.

The elements of A act on V.

Representation Example

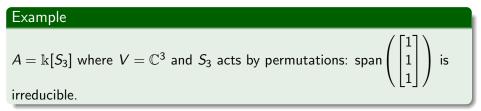
Example

Group algebra $A = \Bbbk[S_3]$ where $V = \mathbb{C}^3$ and S_3 acts on elements of V by permutations:



So \mathbb{C}^3 is a representation of this algebra.

A representation V of A is irreducible if there does not exist any proper subspace $W \subset V$ closed under the action of A.



This is the trivial representation, which is also a subrepresentation of \mathbb{C}^3 .

Polynomial representation with structure given by

$$\Bbbk[x_1,\ldots,x_n]/(x_1+\cdots+x_n).$$

Purpose: study the RCA's actions on the polynomial representation, study its irreducible representation, and compute its Hilbert polynomial.

A huge thank you to

- My mentor Serina Hu
- My parents
- MIT PRIMES Program
- Professor Etingof, Dr. Gerovitch, and Dr. Khovanova
- The MIT Math Department

References

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