The Hilbert Series of the Irreducible Quotient for the Polynomial Representation of the Rational Cherednik Algebra of Type A

Annie Wang
Mentor: Serina Hu

Acton Boxborough Regional High School

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Vector Space

Definition (Vector Space)

A vector space V is a set of elements equipped with addition and multiplication by scalars from a field.

Example

- $V = \mathbb{k}^n$
  - Addition: $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$
  - Scalar Multiplication: $\lambda \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{bmatrix}$ where $\lambda \in \mathbb{k}$
- $V = \mathbb{C}^n$

Vector Space Examples

Example

- $V = \Bbbk[x_1, x_2]$
  - Addition:

$$\sum_{i,j\geq 0} a_{ij}x_1^i x_2^j + \sum_{i,j\geq 0} b_{ij}x_1^i x_2^j = \sum_{i,j\geq 0} (a_{ij} + b_{ij})x_1^i x_2^j$$

  - Scalar Multiplication:

$$\lambda \sum_{i,j\geq 0} a_{ij}x_1^i x_2^j = \sum_{i,j\geq 0} \lambda a_{ij}x_1^i x_2^j$$

  for $\lambda \in \Bbbk$
- $\Bbbk[x_1, \ldots, x_n]$

Dimension

Definition (Dimension)

The dimension of a finite-dimensional vector space V is the least number of vectors needed to span V.

Example

- $V_1 = \mathbb{R}^n = \text{span}\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\right)$, so $\dim(V_1) = n$
- $V_2 = \Bbbk[x_1, x_2] = \text{span}\left(x_1^i x_2^j\right)$ for all $i, j \geq 0$, so $\dim(V_2)$ is infinite

Algebra

Definition (Algebra)

An algebra is a vector space A equipped with a bilinear product.

Example

We can multiply polynomials with each other in addition to adding them and multiplying by scalars.

- $\mathbb{k}[x_1, x_2]$
- $\mathbb{k}[x_1, \ldots, x_n]$

Graded Algebra

Definition (Graded Algebra)

A graded algebra A can be decomposed into the form

$$A = \oplus_{i \geq 0} A_i$$

for subspaces A_i and $A_j \cdot A_k \subset A_{j+k}$.

Example

- $A = \Bbbk[x_1, x_2]$, where the ith graded component is homogeneous polynomials of degree i
  - A_0 generator: 1
  - A_1 generators: x_1, x_2
  - A_2 generators: x_1^2, $x_1 x_2$, x_2^2
  - A_i generators: x_1^i, $x_1^{i-1} x_2$, $x_1^{i-2} x_2^2$, $\ldots$, x_2^i

Hilbert Series

Definition (Hilbert Series)

To a graded algebra A, we can associate a power series, the Hilbert series, given by

$$h_A(z) = \sum_{i \geq 0} \dim(A_i) z^i.$$

Hilbert Series Example

Hilbert series: $h_A(z) = \sum_{i \geq 0} \dim(A_i) z^i$.

Example

Consider $A = \Bbbk[x_1, x_2]$.

- A_0 generator: 1, so $\dim(A_0) = 1$
- A_1 generators: x_1, x_2, so $\dim(A_1) = 2$
- A_2 generators: x_1^2, $x_1 x_2$, x_2^2, so $\dim(A_2) = 3$
- A_i generators: x_1^i, $x_1^{i-1} x_2$, $x_1^{i-2} x_2^2$, $\ldots$, x_2^i, so $\dim(A_i) = i + 1$

We have

$$h_A(z) = \sum_{i \geq 0} (i+1) z^i = \frac{1}{(1-z)^2}.$$

Another Hilbert Series Example

Hilbert series: $h_A(z) = \sum_{i \geq 0} \dim(A_i) z^i$.

Example

Consider $A = \mathbb{k}[x_1, \ldots, x_n]$.

- A_0 generator: 1, so $\dim(A_0) = 1$
- A_1 generators: x_1, ..., x_n, so $\dim(A_1) = n$
- A_2 generators: x_j^2, $x_j x_k$, so 2 stars, $n-1$ bars gives $\dim(A_2) = \binom{n+1}{2}$
- A_i: we have i stars and $n-1$ bars, so $\dim(A_i) = \binom{n+i-1}{i}$

So we have

$$h_A(z) = \sum_{i \geq 0} \binom{n+i-1}{i} z^i = \frac{1}{(1-z)^n}.$$

Representation

Definition (Representation)

A representation of an algebra A is a vector space V equipped with a homomorphism $\rho : A \to \mathrm{End}(V)$.

The elements of A act on V.

Representation Example

Example

Group algebra $A = \Bbbk[S_3]$ where $V = \mathbb{C}^3$ and S_3 acts on elements of V by permutations:

- $123 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
- $231 : \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$
- $132 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ z \\ y \end{bmatrix}$
- $312 : \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$
- $213 : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ x \\ z \end{bmatrix}$
- $321 : \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ y \\ x \end{bmatrix}$

So $\mathbb{C}^3$ is a representation of this algebra.

Irreducible Representation

A representation V of A is irreducible if there does not exist any proper subspace $W \subset V$ closed under the action of A.

Example

$A = \Bbbk[S_3]$ where $V = \mathbb{C}^3$ and S_3 acts by permutations: span$\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$ is irreducible.

This is the trivial representation, which is also a subrepresentation of $\mathbb{C}^3$.

Polynomial Representation of the RCA

Polynomial representation with structure given by

$$\mathbb{k}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n).$$

Purpose: study the RCA's actions on the polynomial representation, study its irreducible representation, and compute its Hilbert polynomial.

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