PRIMES Math Problem Set

PRIMES 2022

Due November 30, 2021

Dear PRIMES applicant:

This is the PRIMES 2022 Math Problem Set. Please send us your solutions as part of your PRIMES application by **November 30, 2021**. For complete rules, see `http: //math.mit.edu/research/highschool/primes/apply.php`

- Note that this set contains two parts: "General Math problems" and "Advanced Math." Please solve as many problems as you can in both parts.
- You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, "`etingof-solutions.pdf`" or similar. Include your full name in the heading of the file.
- Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.
- Submissions in LaTeX are preferred, but handwritten submissions are also accepted.
- You are allowed to use any resources to solve these problems, *except other people's help*. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.
- **Note that posting these problems on problem-solving websites before the application deadline is strictly forbidden!** Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days. We encourage you to apply if you can solve at least 50% of the problems.

Enjoy!

Why it makes no sense to cheat

PRIMES expects its participants to adhere to MIT rules and standards for honesty and integrity in academic studies. As a result, **any cases of plagiarism, unauthorized collaboration, cheating, or facilitating academic dishonesty during the application process or during the work at PRIMES may result in immediate disqualification from the program, at the sole discretion of PRIMES.** In addition, PRIMES reserves the right to notify a participant's parents, schools, and/or recommenders in the event it determines that a participant did not adhere to these expectations. For explanation of these expectations, see *What is Academic Integrity?*, `integrity.mit.edu`.

Moreover, even if someone gets into PRIMES by cheating, it would immediately become apparent that their background is weaker than expected, and they are not ready for research. This would prompt an additional investigation with serious consequences. By trying to get into PRIMES by cheating, students run very serious risks of exposing their weak background and damaging their college admissions prospects.

The PRIMES 2022 problem set

Updated October 18, 2021. Problem G3 was updated to add that the finite sums were allowed.

Updated October 27, 2021. Problem M3 was updated to add the definition of conjugate group homomorphisms. Problem M6 was updated to add that the finite sums were allowed.

Updated November 12, 2021. Problem G4 was updated to add that the finite sums were allowed.

General math problems

G1. Consider a plane passing through the midpoints of two opposite edges of a regular tetrahedron. The projection of the tetrahedron to this plane is a quadrilateral of area A with one of the angles $60°$. Find the surface area of the tetrahedron.

G2. For an m-digit number A and $(n-m)$-digit number B let $A \circ B$ be the n-digit number obtained by concatenation of A and B (where we allow the leftmost digit to be zero). For example, if $m = 2, n = 5$, $A = 23$, $B = 045$, then $A \circ B = 23045$ and $B \circ A = 04523$.

From now on assume that $m = 2$. Let k be a 2-digit number, and consider the equation

$$\frac{B \circ A}{A \circ B} = k$$

with $A > 0$ and any $n \geq 3$. It is clear that if $X := A \circ B$ is a solution of this equation then so is $X \circ X$, $X \circ X \circ X$, etc. We say that a solution X is *primitive* if it is not obtained in this way, by concatenating a smaller solution with itself several times.

(i) Find all primitive solutions for $k = 09$ and $k = 15$.

(ii) Describe all primitive solutions for general k. Are there finitely many?

G3. Let m be a fixed positive integer, and consider the following game. At each move, you pick uniformly at random an integer $0 \leq k \leq m$. Then you score k points, but only if k does not exceed the smallest previously picked number (otherwise you don't score any points on that move). For example, if $m = 3$ and your random numbers are $2, 3, 1, 2, 1, 0, 3, \ldots$ then you score only on the 1st, 3rd and 5th move and don't score anything after the 5th move, so you total score is $2 + 1 + 1 = 4$.

(i) How much will you score on average if you play indefinitely?

(ii) Let $a(n, m)$ be the average amount you score in n steps. Find an explicit formula for $a(n, 1)$ and $a(n, 2)$. Finite sums are allowed.

(iii) Find an explicit formula for $a(n,m)$. Finite sums are allowed.

G4. A street is lit by n street lights arranged in a row ($n \geq 2$). If one of them burns out but its neighbors are still working[1], the Department of Public Works (DPW) does not do anything. However, once two consecutive lights are out of order, the DPW immediately replaces the light bulbs in all broken lights. For example:

$$\circ \quad \circ \quad \circ \quad \circ \quad \circ$$

$$\circ \quad \circ \quad \bullet \quad \circ \quad \circ$$

do nothing

$$\circ \quad \circ \quad \bullet \quad \circ \quad \bullet$$

still do nothing

$$\circ \quad \bullet \quad \bullet \quad \circ \quad \bullet$$

replace all broken lights

(i) What is the chance that the DPW will have to replace k lights, if lights break independently and with equal probability?

(ii) What is the average number of lights that they have to replace in each repair? Finite sums are allowed.

Compute the answers for general n, k, then compute them for $n = 9, k = 4$ with two digits precision after the decimal point.

G5. (i) Describe an algorithm to find the closed ball (disk) of smallest radius containing a given finite set of points (x_i, y_i), $i = 1, ..., n$, in $\mathbb{R}^2$.

(ii) Do the same for points (x_i, y_i, z_i), $i = 1, ..., n$, in $\mathbb{R}^3$.

(iii) Show that the ball in (i),(ii) is unique.

[1] Both neighbors, or only one if it is the first or the last light.

Advanced math problems

M1. Suppose one picks uniformly at random an n-by-n matrix (a_{ij}) of zeros and ones with odd determinant. This means that the entries a_{ij} are picked independently (each taking value 0 with probability 1/2 or 1 with probability 1/2), but then matrices with even determinant are discarded. What is the probability p that $a_{11} = 0$?

(i) Compute the answer for $n = 2, 3$.

(ii) Compute the answer for general n.

M2. Let $t > 0$ and b_n be the sequence defined by the recursion

$$b_0 = 1, \quad b_n = t^{-1}(b_{n-1} + \tfrac{1}{2}b_{n-2} + ... + \tfrac{1}{n}b_0).$$

(i) Show that there exists

$$b = \lim_{n\to\infty} b_n^{1/n}$$

and compute b.

(ii) Show that there exists

$$C = \lim_{n\to\infty} \frac{b_n}{b^n}$$

and compute C.

(iii) Compute $\limsup_{n\to\infty} |b_n - Cb^n|^{1/n}$.

(iv) Do (i)-(iii) for the recursion

$$b_0 = 1, \quad b_n = t^{-1} \sum_{k=1}^{n} \frac{k^{k-1}}{k!} b_{n-k}$$

with $0 < t < 1$.

(v) Compute b for the recursion of (iv) if $t \geq 1$.

M3. Let S_n be the symmetric group on n elements (we agree that $S_0 = \{1\}$). For a finite group G, let $a_n(G)$ be the number of conjugacy classes (under S_n) of homomorphisms $\phi : G \to S_n$. [2] Compute

$$h_G(z) := \sum_{n=0}^{\infty} a_n(G) z^n,$$

where

(i) $G = \mathbb{Z}/2 \times \mathbb{Z}/2$;

(ii) $G = S_3$.

[2]Two group homomorphisms $f_1, f_2 : G \to H$ are said to be conjugate if there exists $h \in H$ such that for each $g \in G$ we have $f_2(g) = hf_1(g)h^{-1}$.

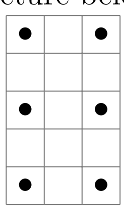
(iii) Let $a_n^*(G)$ be the number of conjugacy classes of *injective* homomorphisms $\phi : G \to S_n$. Compute

$$h_G^*(z) := \sum_{n=0}^{\infty} a_n^*(G) z^n$$

for the groups G as in (i),(ii).

(iv) How many conjugacy classes of subgroups isomorphic to S_3 are there in S_{10} ? Can you describe all of them? How many are there in S_{100}?

M4. Cubicles in a software company are arranged in n adjacent rows, 3 cubicles in each or, equivalently, 3 columns with n cubicles in each (so they look like an n by 3 chessboard). Two cubicles are adjacent if they share at least one corner. The covid-19 social distancing protocol prohibits placing people in adjacent cubicles. If the company has k employees, let $a(n,k)$ be the number of allowable arrangements of k cubicles to be occupied by employees. For example, $a(1,2) = 1$, $a(2,2) = 4$, $a(5,6) = 1$ (the only allowable arrangement is on the picture below), etc.



(i) Find $a(5,3)$.

(ii) Compute the generating function

$$A(x,y) := \sum_{k,n \geq 0} a(n,k) x^n y^k.$$

(iii) Find $a(10,5)$.

(iv) What is the largest number of employees you can seat and how many different ways to do so are there[3] (for each n)?

M5. An MIT class meets 31 times. In each meeting the professor divides the students into working groups of 5 so that every two students are in the same group exactly once.

(i) How many students are there in the class?

(ii) How to make an arrangement as in (i)?

(iii) Another MIT class has 72 students. Each week, the professor divides the students into working groups of exactly 8 people on some

[3]Here, if two employees switch cubicles with each other, this counts as the same way of seating.

weeks or exactly 9 people on other weeks, so that every two students are in the same group exactly once. For how many weeks will the class meet?

(iv) How to make an arrangement as in (iii)?

M6. An even number n of identical metal rods are connected into a chain of length L by hinges (so the length of each rod is L/n). The ends of the chain are pinned to a wall at points $(x_0, y_0) = (-D/2, 0)$ and $(x_n, y_n) = (D/2, 0)$, where $0 < D < L$; otherwise the chain is hanging freely. Denote by (x_i, y_i) the coordinates of the i-th hinge, $1 \leq i \leq n-1$. The potential energy of the chain is then

$$E = \sum_{i=1}^{n-1} y_i.$$

The chain settles in the equilibrium position where E is minimal.

(i) Suppose that $y_1 = -L/cn$ for some $c > 1$. Find D and (x_i, y_i) of the equilibrium position for all i. Finite sums are allowed.

(ii) Explain what happens when $n \to \infty$ when D is fixed.