Further Bounds on the Helly Numbers of Product Sets

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Abstract

The Helly number $h(S)$ of a set $S \subseteq \mathbb{R}^d$ is defined as the smallest positive integer h, if it exists, such that the following statement is true: for any finite family of convex sets in $\mathbb{R}^d$, if every subfamily of h sets intersects, then all sets in the family intersect. We study Helly numbers of product sets of the form A^d for some one-dimensional set A.

Inspired by Dillon's research on the Helly numbers of product sets, Ambrus, Balko, Frankl, Jung, and Naszódi recently obtained the first bounds for Helly numbers of exponential lattices in two dimensions, which are sets of the form $S = \{\alpha^n : n \in \mathbb{N}\}^2$ for some $\alpha > 1$. We develop a different, simpler method to obtain better upper bounds for exponential lattices. In addition, we generalize the lower bounds of Ambrus et al. to higher dimensions.

We additionally investigate sets $A \in \mathbb{Z}$ whose consecutive elements differ by at most 2 such that $h(A^2) = \infty$. We slightly strengthen a theorem of Dillon that such sets exist while also providing a shorter proof.

We obtain Helly number bounds for certain sets defined by arithmetic congruences.

Finally, we introduce a generalization of the notion of an empty polygon, and show that in one case, it is equivalent to the original definition.

1 Introduction

An active area in discrete geometry is studying how convex sets intersect. A foundational result in this topic is Helly's Theorem.

Theorem 1.1 (Helly [9], 1923)**.** *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If every $d+1$ or fewer sets in $\mathcal{F}$ have a point in common, then all sets in $\mathcal{F}$ have a point in common.*

Doignon [7] proved a similar theorem for integer points, which was also discovered independently by Bell [3] and Scarf [11].

Theorem 1.2 (Doignon's Theorem)**.** *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If every 2^d or fewer sets in $\mathcal{F}$ have an integer point in common, then all sets in $\mathcal{F}$ have a lattice point in common.*

Example. The 2^d in the Doignon's Theorem cannot be reduced. To see this, take $\mathcal{F}$ to be the family of convex hulls of $2^d - 1$ vertices of a hypercube in $\mathbb{R}^d$. Every $2^d - 1$ of these sets intersect at a vertex of the hypercube, but the intersection of all these sets does not contain a lattice point.

As this example shows, Doignon's Theorem is tight; that is, we cannot replace 2^d with anything smaller. In fact, Helly's Theorem is also tight, as the family of facets of a simplex shows.

Thus, the integer lattice also has a "Helly-type" property. We can make precise the strength of the necessary assumption for a Helly-type property of *any* set by introducing the Helly number.

Definition. Given a set $S \subseteq \mathbb{R}^d$, the *Helly number* of S, denoted $h(S)$, is the smallest h such that the following *Helly-type theorem* holds:

> Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If every h or fewer sets in $\mathcal{F}$ intersect at a point in S, then the intersection of all sets in $\mathcal{F}$ contains a point in S.

If no such h exists, we say $h(S) = \infty$.

Helly-type properties have several applications in optimization algorithms, as outlined by Amenta, De Loera, and Soberón [2].

Example. We have $h(\mathbb{R}^d) = d + 1$ and $h(\mathbb{Z}^d) = 2^d$.

Hoffman[10] proved a powerful characterization of Helly numbers of discrete sets.

Definition. We say $T \subseteq S$ is *empty* if $\text{conv}(T) \cap S$ only contains vertices of $\text{conv}(T)$. (Here, conv refers to the convex hull)

Theorem 1.3 (Hoffman [10], 1979)**.** *If $S \in \mathbb{R}^d$ and S is discrete, then $h(S)$ is equal to the maximum size of an empty subset of S.*

Example. In $\mathbb{Z}^d$, consider a $2^d + 1$-vertex convex set. The Pigeonhole Principle implies that some two vertices have coordinates with the same modulo-2 residues, so their midpoint is a lattice point. In particular, no $2^d + 1$-vertex convex set can be empty, so this theorem shows that the Helly number is at most 2^d.

In practice, most Helly numbers are found using Hoffman's Theorem.

The following result immediately follows from the definition of the Helly number:

Proposition 1.4. *For all $S_1, S_2 \in \mathbb{R}^d$,*

$$h(S_1 \cup S_2) \leq h(S_1) + h(S_2).$$

Similarly, it is not hard to show that the product of two convex empty sets is convex and empty. The following theorem, based on this fact, gives us a way to look at the Helly number across dimensions.

Proposition 1.5. *For all discrete S_1, S_2,*

$$h(S_1 \times S_2) \geq h(S_1)h(S_2).$$

Many authors, such as DeLoera, La Haye, Oliveros, and Roldán-Pensado [5], have investigated $h(S)$ for specific choices of S and d. Product sets, or sets of the form $S = A^k$ for some 1-dimensional set A are especially interesting, because Proposition 1.5 immediately gives a lower bound for such sets. For example, if $h(A^2) = \infty$, then Proposition 1.5 implies that $h(A^d) = \infty$ for all $d \geq 2$.

Dillon [6] proved a general criterion which implies that if p is a polynomial with degree at least 2 and $A = \{p(n) : n \in \mathbb{Z}\}$, then $h(A^2) = \infty$ for all $d \geq 2$. Based on this result, it is natural to conjecture that whether or not the Helly number of A^2 is finite is related to the sparseness of the set A. However, Dillon ruled this out by constructing $A \subseteq \mathbb{Z}$ whose consecutive elements differ by at most 2 (a "2-syndetic" set) such that the A^2 has arbitrarily large empty polygons, and thus arbitrarily large Helly number.

In Section 3, we show that in fact, there is a 2-syndetic set such that A^2 has an empty polygon with *infinitely many* vertices.

Dillon's proof for sets defined by polynomials analyzes the ratios of successive differences between consecutive terms in A. This method gives no information for exponential lattices, sets of the form $L_d(\alpha) = \{\alpha^n : n \in \mathbb{N}_0\}^d$. Ambrus, Balko, Frankl, Jung, and Naszódi [1] studied these sets in two dimensions, and they obtained lower bounds for all $\alpha > 1$ and exact values for $\alpha \in [\frac{1+\sqrt{5}}{2}, \infty)$.

Theorem 1.6 (Ambrus, Balko, Frankl, Jung, Naszódi [1], 2023)**.** *Let $\alpha > 1$.*

- *If $\alpha \geq 2$, then $h(L_2(\alpha)) = 5$.*
- *If $\alpha \in [\frac{1+\sqrt{5}}{2}, 2)$, then $h(L_2(\alpha)) = 7$.*
- *If $\alpha \in (1, \frac{1+\sqrt{5}}{2})$, then $h(L_2(\alpha)) \leq 3\lceil \log_\alpha(\frac{\alpha}{\alpha-1})\rceil + 3$.*

In Section 2.1, we obtain a stronger bound, with an approach that is shorter, more geometric, and less computational than that of Ambrus et al. [1]:

Theorem 1.7. *For $\alpha > 1$,*

$$h(L_2(\alpha)) \leq 2\left\lceil \log_\alpha\left(\frac{\alpha}{\alpha-1}\right)\right\rceil + 3.$$

This method recovers the exact values of $h(L_2(\alpha))$ for $\alpha > \frac{1+\sqrt{5}}{2}$. It also allows us to fully characterize all maximal empty polygons when $\alpha \geq 2$.

Ambrus et al. [1] proved a general lower bound for $h(L_2(\alpha))$ by constructing an empty polygon with vertices on a hyperbola. They showed that for $\alpha > 1$, we have $h(L_2(\alpha)) \geq \sqrt{\frac{1}{\alpha-1}}$.

In Section 2.2, we consider the analog of a hyperbola in higher dimensions to generalize this bound:

Theorem 1.8. *For $\alpha > 1$ and $d > 1$,*

$$h(L_d(\alpha)) \geq \binom{k+d-1}{d-1},$$

where $k = \left\lceil \sqrt{\frac{1}{\alpha-1}} \right\rceil$.

This bound gives $h(L_d(\alpha)) \geq \Omega_d((\alpha-1)^{-(d-1)/2})$, whereas using $h(L_2(\alpha)) \geq \sqrt{\frac{1}{\alpha-1}}$ in conjunction with Proposition 1.5 gives $h(L_d(\alpha)) \geq \Omega_d((\alpha-1)^{-d/4})$. Hence, it is asymptotically stronger than the previous best-known result.

In Section 4, we investigate Helly numbers of powers of arithmetic congruence sets, which are sets of the form

$$A = \{n | n \pmod m \in \{a_1, \ldots, a_k\}\}$$

where $0 \leq a_1 < a_2 < \cdots < m$. Since A^d consists of k^d scaled translates of the integer lattice $m\mathbb{Z}^d$, Proposition 1.4 immediately gives $h(A^d) \leq 2^d k^d$. Garber [8] improved this bound in two dimensions, showing that $h(A^2) \leq k^2 + 6$. However, he worked with arbitrary unions of integer lattices rather than the more specific case of an arithmetic congruence set. We prove better bounds for certain arithmetic congruence sets.

We algebraically show the following.

Theorem 1.9. *If $A = \{n \in \mathbb{Z} : n \equiv 0, 1 \pmod 3\}$, then $h(A \times A) = 8$.*

We execute the same method with the help of a computer program to obtain similar results for different moduli.

Theorem 1.10. *If $A = \{n \in \mathbb{Z} : n \equiv 0, 1 \pmod k\}$ for $k = 4, 5, 6$, then $h(A \times A) = 8$.*

By analyzing segments between vertices of empty polygons, we obtain a more general bound.

Theorem 1.11. *Let m be a positive integer with smallest prime factor p and let k and d be positive integers such that $d < \frac{p(m-1)}{m(m-k)}$. Choose k residues modulo m, and let A be the set of all integers with one of these residues. Then we have*

$$h(A^d) \leq k^d.$$

Finally, in Section 5, we propose a variation to the definition of an empty polytope and prove a theorem similar to Doignon's Theorem.

2 Exponential Lattices

2.1 Upper Bounds

Ambrus et al. prove Theorem 1.6 by considering an empty polygon and dividing its edges into four quadrants. In each quadrant, they bound the number of edges by considering various points in the lattice which must not lie inside the polygon.

We also consider an empty polygon in an exponential lattice, and in one case (where all the edge slopes are nonnegative), we repeat the analysis in [1]. However, in all other cases, we analyze a different set of points that cannot lie inside the polygon, and our approach requires significantly less computation.

Proof of Theorem 1.7. Consider an empty polygon $\mathcal{P}$ in $L(\alpha)$. We have two cases.

If all edges of $\mathcal{P}$ have nonnegative slope, then Corollary 11 of Ambrus et al. [1] shows that $\mathcal{P}$ has at most $2\lceil \log_\alpha(\frac{\alpha}{\alpha-1})\rceil + 2$ edges, implying the conclusion.

Otherwise, $\mathcal{P}$ contains vertices $A = (\alpha^p, \alpha^q)$ and $B = (\alpha^r, \alpha^s)$ such that $p < r$ and $q > s$. We may choose A and B so that no vertices are above and to the left of A or below and to the right of B. A diagram is shown in Figure 1.

Let $C = (\alpha^{r-1}, \alpha^{q-1})$ and let ℓ_1, ℓ_2 be the horizontal and vertical lines passing through C respectively. Also let $D = (\alpha^{r-1-d}, \alpha^{q-1})$ be the rightmost point on ℓ_1 strictly to the left of $\overline{AB}$, and let $E = (\alpha^{r-1}, \alpha^{q-1-e})$ be the topmost point on ℓ_2 strictly below $\overline{AB}$. It is possible for C to lie below the line $\overline{AB}$, in which case the points C, D, E coincide; this will not affect our proof. Finally, let X and Y be the intersections of $\overline{AB}$ with ℓ_1 and ℓ_2 respectively.

To bound d and e, let $A' = (0, \alpha^q)$ and $B' = (\alpha^r, 0)$. The line $\overline{A'B'}$ and ℓ_1 intersect at $X' = (\alpha^r - \alpha^{r-1}, \alpha^{q-1})$. Since X' is strictly to the left of X, we have

$$\alpha^r - \alpha^{r-1} < \alpha^{r-d} \implies d < \log_\alpha\left(\frac{\alpha}{\alpha-1}\right).$$

The same bound holds for e by considering the intersection of $\overline{A'B'}$ and ℓ_2.

Let F be the point such that $CDEF$ is a rectangle. Let G be the lattice point immediately to the right of A, and let H be the lattice point immediately above B. Consider a point P below $\overline{AB}$ lying outside pentagon $XYEFD$.

- If P lies below and to the left of D, then triangle PAB contains D.
- If P lies below and to the left of E, then triangle PAB contains E.
- Otherwise, P lies below and to the right of B or above and to the left of A, violating our assumption.

Therefore, all vertices of $\mathcal{P}$ below $\overline{AB}$ must lie within $XYEFD$.

Similarly, consider a point P above $\overline{AB}$ lying outside triangle CXY.

- If P lies on or above ℓ_1, then triangle PAB contains G.
- If P lies on or to the the right of ℓ_2, then triangle PAB contains H.

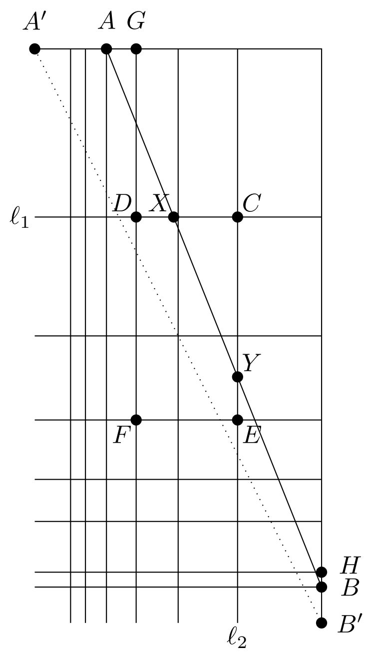


Figure 1: An illustration of the proof of Theorem 1.7

Thus, all vertices of $\mathcal{P}$ above $\overline{AB}$ must either be G or H, or lie within CXY.

To finish, we bound the number of vertices in rectangle $CDFE$. For any given x-coordinate, no two vertices with that x-coordinate can lie on the same side of $\overline{AB}$ (and similarly for y-coordinates). This implies that the number of vertices in rectangle $CDFE$ is at most

$$\min(d, e) + \min(d + 1, e + 1) \leq 2 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil - 1.$$

Adding the vertices A, B, G, H yields the desired bound. ∎

Based on this proof, we can easily characterize all maximal empty polygons for $\alpha \geq 2$.

Corollary 2.1. *If $\alpha \geq 2$, then all empty pentagons in $L_2(\alpha)$ have vertices of the form $(\alpha^p, \alpha^q), (\alpha^r, \alpha^s), (\alpha^{r-1}, \alpha^{q-1}), (\alpha^{p+1}, \alpha^q), (\alpha^r, \alpha^{s+1})$, where $p < r$ and $q > s$.*

Proof. We neglect the case where all edges have positive slope, because Corollary 11 of Ambrus et al. [1] already implies that equality cannot be reached in this case.

If $\alpha \geq 2$, then $\log_\alpha(\frac{\alpha}{\alpha-1}) \leq 1$. This means both d and e in the above proof are 0, which means $C = D = E$. So the only possible vertices in an empty polygon are A, B, C, G, H, which are exactly the points stated in the corollary.

In fact, our proof already shows that $ABCGH$ is, indeed, empty. Any point in the interior must lie within the rectangle $CDEF$, which consists of only the point C. However, C is a vertex, so it is not in the interior. ■

2.2 Lower Bounds

Below is the proof of the lower bound mentioned in the introduction.

Proof of Theorem 1.8. Consider the set of points in $\mathbb{Z}^d$ which lie on the convex curve $x_1 \dots x_d = \alpha^k$. We claim that this set is empty in $L_d(\alpha)$. To prove this, it suffices to check that the closest point to the curve, with coordinates $(\alpha^{(k+1)/d}, \dots, \alpha^{(k+1)/d})$, lies above the facet of the polytope that is furthest from the origin, which has vertices $(1, 1, \dots, \alpha^k), \dots, (\alpha^k, 1, \dots, 1)$. The equation of this facet is $x_1 + \cdots + x_d = \alpha^k + n - 1$, so it suffices to show

$$n\alpha^{(k+1)/d} \geq \alpha^k + n - 1.$$

Let $\alpha = 1 + s^2$. Note that $(1+s^2)^{(k+1)/d} \geq 1 + \frac{k+1}{d}s^2$ by the Binomial Theorem and $(1+s^2)^k < e^{s^2 k}$ by Taylor approximation, so it suffices to prove

$$n\left(1 + \frac{k+1}{d}s^2\right) \geq e^{s^2 k} + n - 1.$$

By plugging in our choice of k, this simplifies to

$$s^2 + s + 1 \geq e^s,$$

which follows from the Taylor expansion of e^x. Thus we have found an empty polytope.

The number of vertices in this polytope is the number of ordered d-tuples of nonnegative integers with sum k, which (say, by stars and bars) is well-known to be $\binom{k+d-1}{d-1}$. ■

3 Infinite Empty Polygons in Integer Sets

To construct a 2-syndetic set whose square has infinite Helly number, Dillon considered a sequence of rational approximations for an irrational number and used these to construct a sequence of successively larger empty polygons. We use a similar approach to construct an infinite empty polygon.

The polygon we construct will actually be a scaled, translated version of the one in Section 5 of Amrbus et al.'s paper [1].

Construction of 2-syndetic set with infinite Helly number. Start by letting $A = \mathbb{Z}$. We will modify A to satisfy the desired property.

Define the Fibonacci numbers by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Let $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. Let $\mathcal{P}$ be the infinite polygon with vertices

$$\{P_i = (-F_{2i}, 2F_{2i+1} - 1)\}_{i \geq 1}.$$

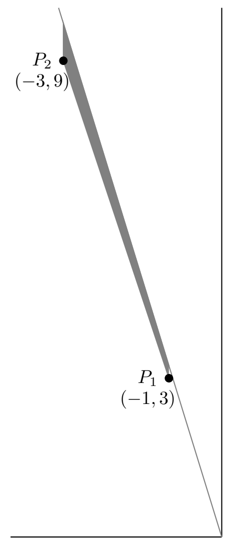


Figure 2: One strip of our empty polygon construction

Let ℓ be the line $y = -2\phi x$. By Binet's Formula, the vertical distance between P_i and ℓ is

$$2\phi\left(\frac{\phi^{2i} - \psi^{2i}}{\phi - \psi}\right) - \left(\frac{2\phi^{2i+1} - 2\psi^{2i+1}}{\phi - \psi} - 1\right) = 1 - 2\psi^{2i}.$$

In particular, as i increases, the distance between P_i and ℓ increases and approaches 1.

Consider the strip of $\mathcal{P}$ lying between the lines $x = -F_{2k+2}$ and $x = -F_{2k}$. Let L be the set of lattice points in this strip which are not vertices of $\mathcal{P}$. Since the vertical width of the strip is always less than 1 and the slope of ℓ is less than -2, no two points in L have y-coordinates differing by 1. Therefore, by simply removing the y-coordinates of points in L from A, we can ensure that the strip is empty while enforcing the 2-syndetic condition.

Repeating for all strips, we have constructed a valid A, so we are done. ■

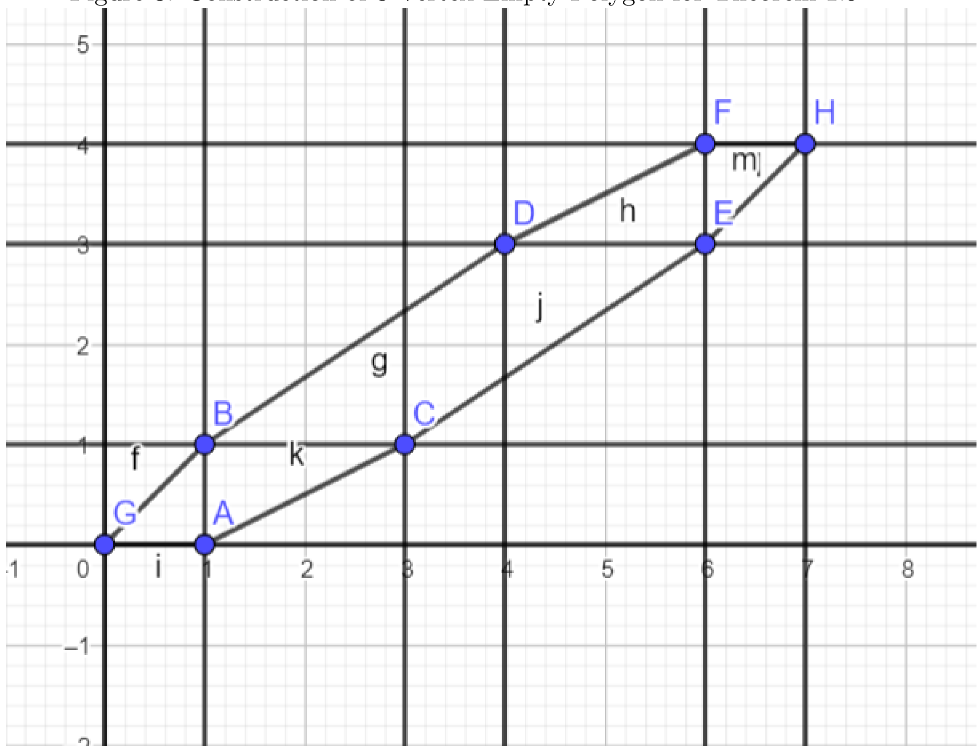
4 Arithmetic Congruence Sets

We begin with the simplest nontrivial arithmetic congruence set, consisting of numbers congruent to 0 or 1 modulo 3.

Proof of Theorem 1.9. A construction of 8 points is shown in Figure 3.

We claim that any empty polygon has at most two vertices whose modulo-3 residues are $(0, 0)$. Then, by symmetry, we will obtain a bound of $2 \cdot 4 = 8$, as desired.

Figure 3: Construction of 8-Vertex Empty Polygon for Theorem 1.9



Indeed, suppose for the sake of contradiction that $(3x_1, 3y_1), (3x_2, 3y_2), (3x_3, 3y_3)$ is a triangle whose sides and interior contain no points in $A \times A$. Then the centroid, $(x_1 + x_2 + x_3, y_1 + y_2 + y_3)$, must have a coordinate that is $2 \pmod 3$. We may assume $x_1 + x_2 + x_3 \equiv 2 \pmod 3$. Clearly x_1, x_2, x_3 cannot be distinct modulo 3, so assume $x_1 \equiv x_2 \pmod 3$. Now consider the following points on the sides of the triangle:

$$(2x_1 + x_2, 2y_1 + y_2),$$

$$(2x_2 + x_1, 2y_2 + y_1).$$

The x-coordinates are $0 \pmod 3$, so both y-coordinates must be $2 \pmod 3$. However, the sum of the y-coordinates is $0 \pmod 3$, contradiction. ■

In fact, a similar proof works for modulo $4, 5,$ and 6 as well. However, we do not have a simple algebraic argument; we used code to check all cases by brute force.

Proof of Theorem 1.10. A similar construction achieves 8 vertices, and the first code in the Appendix proves the bound. ■

Note that the first code in the Appendix checks moduli up to 10, and finds that the proof only works for $3, 4$, and 5. This suggests that our method of only considering the most obvious points in the convex hull (those whose coefficients are rational numbers with denominator the modulus) is unviable for large moduli.

In fact, for some higher moduli, it is not even true that there are at most 2 points of each residue pair in an empty polygon. The second code in the Appendix finds a specific set of three points of the same residue which is empty in several moduli.

However, the method of bounding the number of points of each residue tuple is useful for a more general class of congruence sets.

Proof of Theorem 1.11. Consider an empty polygon $\mathcal{P}$ with vertices in A^d. Suppose for the sake of contradiction that $\mathcal{P}$ has two vertices A and B whose coordinates in all dimensions have the same remainder modulo m. Construct points $C_1, \ldots, C_{m-1}$ which partition segment $\overline{AB}$ into m equal parts (assume $A, C_1, \ldots, C_{m-1}, B$ appear in that order). Clearly, in any given dimension, the coordinates of $C_1, \ldots, C_{m-1}$ form an arithmetic progression modulo m. Therefore, in any given dimension, there are at most $\frac{m}{p}(m-k)$ coordinates among $C_1, \ldots, C_{m-1}$ which are not in A. However, we know $\frac{m}{p}(m-k)d < m-1$. So by the Pigeonhole Principle, there is some C_i with all of its coordinates in A. This violates the emptiness condition, as desired.

Thus, there is at most one vertex of any given sequence of remainders modulo m, so there are at most k^d total vertices in $\mathcal{P}$. ∎

This theorem is strongest when A contains all but one residue modulo m, and m is prime. In this case, the bound holds for any $d < m-1$. The bound is better than that obtained by Garber[8] in two dimensions, and furthermore applies to higher dimensions.

5 Variations on Emptiness

Authors such as Steinitz[12] and Carathéodory[4] proved theorems where a property of a large set of points implies the property for a subset of those points. This motivates generalizing the emptiness condition, which allows lattice points only at vertices, to related emptiness conditions involving more points. In particular, the following theorem is a natural extension of Doignon's Theorem:

Theorem 5.1. *Suppose a convex polytope $\mathcal{P}$ is in $\mathbb{Z}^d$ such that the only lattice points in $\mathcal{P}$ are on its boundary. Suppose further that all facets of $\mathcal{P}$ are $d-1$-simplices. Then $\mathcal{P}$ has at most 2^d vertices.*

Proof. We will show that for any polytope satisfying the property, we can construct an empty polytope with the same number of vertices.

Assume that $\mathcal{P}$ contains exactly B lattice point and has the minimum number of non-vertex lattice points among all polytopes with B lattice points. Suppose for the sake of contradiction that there is a lattice point X lying on some facet. Let $\{P_1, \ldots, P_k\}$ be the minimal set of vertices whose convex hull contains X. Let $\mathcal{Q}$ be the polytope formed by replacing P_1 with X in $\mathcal{P}$. Clearly, $\mathcal{Q}$ has strictly fewer lattice points on its boundary than $\mathcal{P}$. Thus, $\mathcal{P}$ must in fact have no non-vertex lattice points, which means it satisfies the Doignon bound of 2^d. ■

Note that the bound in Theorem 5.1 is not necessarily sharp for $d > 2$, because the facets of parallelepipeds (the equality cases of Doignon's Theorem) are not all simplices.

However, if we remove the condition that all facets must be simplices, the question becomes less interesting due to the following construction.

Theorem 5.2. *There exist convex polytopes $\mathcal{P}$ in $\mathbb{Z}^d$ with arbitrarily many vertices such that the only lattice points in $\mathcal{P}$ are at its vertices, edges, or two-dimensional faces, and $\mathcal{P}$ has nonempty interior.*

Proof. Let $n \geq 3$. We will construct $\mathcal{P}$ with n vertices satisfying the desired property.

Choose $(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1})$ to be the vertices of a convex $(n-1)$-gon in two dimensions which contains $(0, 0)$ in its interior. Let the vertices of $\mathcal{P}$ be

$$\begin{gathered}(x_1, y_1, 0, 0, \ldots, 0),\\ (x_2, y_2, 0, 0, \ldots, 0),\\ \vdots\\ (x_{n-1}, y_{n-1}, 0, 0, \ldots, 0),\\ (0, 0, 1, 1, \ldots, 1).\end{gathered}$$

Any convex combination of the last vertex with any other vertices will have a non-integral coordinate. Moreover, any convex combination of the first $n - 1$ vertices lies in a two-dimensional facet.

Thus, $\mathcal{P}$ is a valid construction, so we are done. ■

6 Open Problems

The main unsolved problem in this paper is bounding the number of vertices of empty polygons in high-dimensional exponential lattices. In particular, it is still unknown whether any three-dimensional exponential lattice can have empty polygons with infinitely many vertices. The proof of Corollary 11 of Ambrus et al. [1] relies on ordering the edge slopes of a polygon. Since facets in higher dimensions do not have ordered slopes, this approach cannot be easily generalized, so a new method is required.

The growth of $h(L_d(\alpha))$ is also unknown. In the two-dimensional case, Ambrus et al. [1] note that if $\alpha = 1 + \frac{1}{x}$, then our best bounds currently are

$$\lfloor \sqrt{x} \rfloor \leq h(L_2(\alpha)) \leq O(x \log_2(x)).$$

In higher dimensions, as $x \to \infty$, our methods give

$$\Omega(x^{(d-1)/2}) \leq h(L_d(\alpha)).$$

In both cases, more work is needed to determine the exact growth in terms of α.

Further bounds on arithmetic congruence sets, in general or in specific cases, would also be fascinating.

Finally, while Theorem 5.1 provides one upper bound for our generalized notion of emptiness, the precise value remains unknown.

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8 Appendix

The following is the C++ code used to show Theorem 1.10. We assume that three points whose coordinates are both divisible by the modulus are vertices of our polygon. Then, through brute force, we find a point in our set in the interior of their convex hull. This shows that there are at most 2 vertices of any given residue pair in an empty polygon, which by symmetry implies that there are at most 8 vertices.

```
#include <bits/stdc++.h>
using namespace std;

int main(){
    for (int mod = 3; mod <= 10; mod++){
        /* Suppose (mod*x1,mod*y1),(mod*x2,mod*y2),(mod*
            x3,mod*y3) all
        lie in the polygon. Loop through all possible
            residues of
        x1,y1,x2,y2,x3,y3 modulo mod. */
        bool proofFails = false;
        for (int x1 = 0; x1 < mod; x1++){
            for (int y1 = 0; y1 < mod; y1++){
                for (int x2 = 0; x2 < mod; x2++){
                    for (int y2 = 0; y2 < mod; y2++){
                        for (int x3 = 0; x3 < mod; x3++){
                            for (int y3 = 0; y3 < mod; y3
                                ++){
                                /* loop though all convex
                                     combinations
                                with coefficients a/mod,
                                    b/mod, c/mod
                                where a+b+c = mod */
```

```
                  bool foundPoint = false;
                  for (int a = 0; a <= mod;
                      a++){
                      for (int b = 0; b <=
                          mod-a; b++){
                          int c = mod-a-b;
                          if (a!=mod&&b!=
                              mod&&c!=mod){
                              if ((a*x1+b*
                                  x2+c*x3)%
                                  mod==0||(a
                                  *x1+b*x2+c
                                  *x3)%mod
                                  ==1){
                                  if ((a*y1
                                      +b*y2+
                                      c*y3)%
                                      mod
                                      ==0||(
                                      a*y1+b
                                      *y2+c*
                                      y3)%
                                      mod
                                      ==1){
                                      foundPoint
                                          =

                                          true
                                          ;
                                  }
                              }
                          }
                      }
                  }
                  if (!foundPoint){
                      proofFails=true;
                  }
              }
          }
      }
  }
}
if (!proofFails){
    cout << mod << "\n";
}
```

```
    }
}
```

The code prints $3, 4$, and 5, implying Theorem 1.10.

Next, we show that it is possible to have three vertices of the same residue pair in empty polygons in higher moduli. To do this, we scale the plane down by a factor of the modulus and exhibit an empty triangle with vertices at lattice points. Scaling back up, this corresponds to an empty triangle whose coordinates are divisible by the modulus, as desired.

```
#include <bits/stdc++.h>
using namespace std;

struct Point{
    double x,y;
};
bool inConv(Point d, Point a, Point b, Point c){
    /* sign of m1 represents which side of AB that D lies
        in,
    sign of n1 represents which side of AB that C lies in
       , and etc.*/
    double m1 = ((b.x-a.x)*(d.y)-(b.y-a.y)*(d.x)-(b.x*a.y
      -a.x*b.y));
    double n1 = ((b.x-a.x)*(c.y)-(b.y-a.y)*(c.x)-(b.x*a.y
      -a.x*b.y));
    double m2 = ((c.x-b.x)*(d.y)-(c.y-b.y)*(d.x)-(c.x*b.y
      -b.x*c.y));
    double n2 = ((c.x-b.x)*(a.y)-(c.y-b.y)*(a.x)-(c.x*b.y
      -b.x*c.y));
    double m3 = ((a.x-c.x)*(d.y)-(a.y-c.y)*(d.x)-(a.x*c.y
      -c.x*a.y));
    double n3 = ((a.x-c.x)*(b.y)-(a.y-c.y)*(b.x)-(a.x*c.y
      -c.x*a.y));
    /* check if D lies on same side of AB as C, etc. */
    if (m1*n1>=0&&m2*n2>=0&&m3*n3>=0){
        return true;
    }
    else{
        return false;
    }
}

bool equals(Point a, Point b){
    return (a.x==b.x&&a.y==b.y);
}

int main(){
```

```
ifstream cin;
ofstream cout;
cin.open("input.in");
cout.open("output.out");

Point a = {0.0,0.0};
Point b = {43.0,3.0};
Point c = {100.0,7.0};
double bound = 110.0;
for (double mod = 3.0; mod <= 50.0; mod+=1.0){
    /* Assume that we have scaled down our problem by
        a factor of mod;
    so S consists of all points whose coordinates are
        0 or 1/mod modulo 1*/
    bool empty = true;
    // Check mods points that are (1/mod,1/mod)
       modulo 1
    for (double i = 1.0/mod; i <= bound; i+=1.0){
        for (double j = 1.0/mod; j <= bound; j +=
            1.0){
            Point p = {i,j};
            if (!equals(a,p)&&!equals(b,p)&&!equals(c
                ,p)){
                if (inConv(p,a,b,c)){
                    empty = false;
                }
            }
        }
    }
    // Check mods points that are (0,1/mod) modulo 1
    for (double i = 0; i <= bound; i+=1.0){
        for (double j = 1.0/mod; j <= bound; j +=
            1.0){
            Point p = {i,j};
            if (!equals(a,p)&&!equals(b,p)&&!equals(c
                ,p)){
                if (inConv(p,a,b,c)){
                    empty = false;
                }
            }
        }
    }
    // Check mods points that are (1/mod,0) modulo 1
    for (double i = 1.0/mod; i <= bound; i+=1.0){
        for (double j = 0; j <= bound; j += 1.0){
            Point p = {i,j};
```

```
                if (!equals(a,p)&&!equals(b,p)&&!equals(c
                    ,p)){
                    if (inConv(p,a,b,c)){
                        empty = false;
                    }
                }
            }
        }
        // Check mods points that are (0,0) modulo 1
        for (double i = 0; i <= bound; i+=1.0){
            for (double j = 0; j <= bound; j += 1.0){
                Point p = {i,j};
                if (!equals(a,p)&&!equals(b,p)&&!equals(c
                    ,p)){
                    if (inConv(p,a,b,c)){
                        empty = false;
                    }
                }
            }
        }
        if (empty){
            cout << mod << ",";
        }
    }

}
```

The code prints the set

$$\{9, 10, 11, 15, 20, 22, 23, 29, 30, 31, 44, 45, 46\}$$

implying that for all of these moduli (and possibly more), the Helly number cannot be bounded in the same way as we bounded it for moduli $3, 4$, and 5.