MULTIDISPERSE RANDOM SEQUENTIAL ADSORPTION AND GENERALIZATIONS

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Abstract. In this paper, we present a unified study of the limiting density in one-dimensional random sequential adsorption (RSA) processes where segment lengths are drawn from a given distribution. In addition to generic bounds, we are also able to characterize specific cases, including multidisperse RSA, in which we draw from a finite set of lengths, and power-law RSA, in which we draw lengths from a power-law distribution.

1. Introduction

The field of random sequential adsorption (RSA) studies processes in which particles are sequentially adsorbed onto a substrate such that the particles do not overlap. Known also as *simple sequential inhibition*, *on-line packing*, and the *hard-core model*, RSA is a fundamental process that has been studied in mathematics and statistical physics. RSA also has many applications to biology and chemistry: for example, reactions on polymer chains have been modelled with RSA, along with various chemisorption (chemical adsorption) processes. See [1, 2] for surveys of RSA and its applications.

Meanwhile, the theoretical study of RSA has proven difficult. Higher dimensional RSA has scarcely been studied mathematically [3, 4], and even in one dimension, only a small number of RSA processes are rigorously understood, despite the large amount of interest in them. In statistical physics, many papers study RSA with empirical Monte-Carlo simulations [5, 6].

The mathematical study of RSA began with the *Rényi parking problem*, first proposed by Alfred Rényi [7] in 1958. On the interval $[0, L]$, we randomly park a length-1 segment by choosing its left endpoint uniformly at random from $[0, L-1]$. Then, we randomly park a second length-1 segment. If the second segment intersects the first, we discard it and randomly choose a new length-1 segment until it does not overlap with the first. In this way, we continue parking length-1 segments by randomly choosing them on $[0, L]$ and discarding them if they overlap with any previously parked segments. We repeat this process until the gaps between segments are too small for any more segments to be parked, at which point we say the process has reached *saturation*.

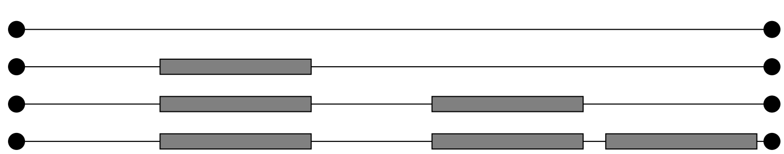


Figure 1. Rényi's parking problem.

Multidisperse RSA with lengths 1 and 3.

RSA with convergent ldf $\nu(\ell) = \ell^{-2}$.

RSA with divergent ldf $\nu(\ell) = 1$.

FIGURE 2. Different RSA processes.

Rényi studied the number of segments parked at saturation, given by the random variable N_L. He first derived an integral recurrence equation for $\mathbb{E}[N_L]$, the expected value of N_L:

$$\mathbb{E}[N_{L+1}] = 1 + 2\int_0^L \mathbb{E}[N_t]\,\mathrm{d}t.$$

Then, using Laplace transforms and a Tauberian theorem, he proved that $\mathbb{E}[N_L]$ grew linearly in L:

Theorem 1.1 (Rényi)**.** *Let N_L be defined as above. Then,*

$$\lim_{L\to\infty} \frac{\mathbb{E}[N_L]}{L} = \alpha,$$

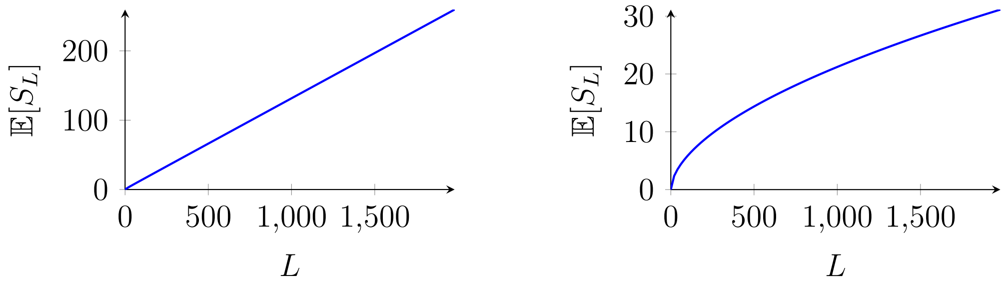
where α is the Rényi parking constant, given by

$$\alpha = \int_0^\infty \exp\left(-2\int_0^t \frac{1-e^{-u}}{u}\,\mathrm{d}u\right)\mathrm{d}t \approx 0.747598.$$

In this paper, we study generalizations of Rényi's parking problem in which we park segments of varying lengths. These processes are also known as *cooperative* and *competitive RSA*. One such process we study is known as *multidisperse RSA*, in which there are n different possible lengths, each with a different probability. In other words, when we randomly choose a segment to park, we first randomly choose its length from the n different lengths. Then, we choose a random location for the segment on the interval by choosing its left endpoint uniformly at random from $[0, L]$. If the segment is not fully contained within the interval $[0, L]$, or if it overlaps with another segment, we discard it and choose a new segment with a new random length and position. Multidisperse RSA has been widely studied in statistical physics [5, 6, 8, 9] and is motivated by many chemisorption processes in which a mixture of chemicals are absorbed onto a substrate [10, 11].

In this paper, we study the total number of each type of segments parked at saturation. In particular, let the total number of parked type-k segments be $N_{k,L}$. We first prove an integral recurrence equation for $\mathbb{E}[N_{k,L}]$, and similarly to Rényi's original analysis, we use Laplace transforms with a Tauberian theorem to prove that $\mathbb{E}[N_{k,L}]$ grows linearly in L. In Theorem 4.5, we derive an exact analytical expression for the limit $\lim_{L\to\infty} \frac{\mathbb{E}[N_{k,L}]}{L}$.

Moreover, we study another generalization of the Rényi parking problem given by choosing segment lengths from a continuous distribution of lengths. We describe the distribution of lengths with a *length distribution function* (ldf) $\nu : [1, \infty) \to \mathbb{R}$, which describes the relative probabilities of choosing each length. When $\int_1^\infty \nu(\ell)\,\mathrm{d}\ell < \infty$, we call ν *convergent*, and when the integral diverges, we call ν *divergent*. Intuitively, convergent ldfs tend to weight small segment lengths more, whereas divergent ldfs tend to weight large segment lengths more (c.f. Figure 2).



(A) Convergent ldf $\nu(\ell) = \ell^{-2}$.

(B) Divergent ldf $\nu(\ell) = 1$.

Figure 3. Growth of $\mathbb{E}[S_L]$ with various length distributions.

In our analysis, we study S_L, the empty space on the interval not covered by segments at saturation, and we find that convergent and divergent ldfs yield fundamentally different behavior. Define $Z_\nu(L) = \int_0^L \nu(\ell)\,\mathrm{d}\ell$ to be the *normalizing constant* of ν. We again derive a general integral recurrence equation for $\mathbb{E}[S_L]$:

$$\left(\int_0^L Z_\nu(t)\,\mathrm{d}t\right)\mathbb{E}[S_L] = 2\int_0^L \mathbb{E}[S_t]Z_\nu(L-t)\,\mathrm{d}t.$$

When ν is convergent, we then prove that $\mathbb{E}[S_L]$ grows linearly with L, under a general condition on ν (c.f. Theorem 5.3). In the proof, we first derive inequalities from the integral recurrence equation, which we use to derive differential inequalities on the Laplace transform of $\mathbb{E}[S_L]$. We then use a novel analysis to characterize the asymptotic behavior of the Laplace transform of $\mathbb{E}[S_L]$ around 0, which we use with a Tauberian theorem to prove our result.

Meanwhile, when ν is divergent, we use an inductive argument to prove, under a general condition on ν, that $\mathbb{E}[S_L]$ grows *sublinearly* with L (c.f. Theorem 5.5).

Different RSA processes in which lengths are drawn from a distribution have been previously considered by mathematicians [12, 13], which our work generalizes. Moreover, only distributions with a maximum segment length have been studied before, but we allow arbitrarily large segment lengths, which yields a different, more complicated analysis.

Finally, we consider a class of ldfs given by power-law functions. Various power-law size distributions have been considered in RSA processes before [14]. Using bounding techniques, we are able to prove tight bounds on $\mathbb{E}[S_L]$ to show that it grows roughly asymptotic to a specific power function (c.f. Theorem 6.3). We are also able to characterize the uniform length distribution as a special case of our work.

In this paper, we provide formal definitions of the processes we study in Section 2. In Section 4, we study multidisperse RSA processes. In Section 5, we consider RSA with a general length distribution, and in Section 6, we study power-law length distributions.

2. Preliminaries

2.1. **Notation.** Throughout this paper, we always park *segments* on an *interval.* We always use L to denote the length of interval, T to denote time in RSA processes, ν to denote length distribution functions (c.f. Definition 2.1), and Z_ν to denote the normalizing constant of ν (c.f. Definition 2.2). We use $[n]$ to denote the set $\{1, 2, \cdots, n\}$. To write that x is a real number drawn uniformly at random from an interval $[a, b]$, we write $x \xleftarrow{R} [a, b]$.

We use $\mathbb{R}^{>0}$ to denote the positive reals and $\mathbb{R}^{\geq 0}$ to denote the nonnegative reals. We define the function $\Gamma : \mathbb{R}^{>0} \to \mathbb{R}$ as the Gamma function, given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^t \, \mathrm{d}t.$$

We use $\mathrm{Ein} : \mathbb{R} \to \mathbb{R}$ to denote the modified exponential integral function, defined as

$$\mathrm{Ein}(z) = \int_0^z \frac{1 - e^{-t}}{t} \, \mathrm{d}t. \tag{2.1}$$

Finally, we use the following symbols for asymptotic notation (always taken as $L \to \infty$):

- $f = o(g)$ and $f \ll g$ denote $\lim_{L\to\infty} f(L)/g(L) = 0$.
- $f(L) \sim g(L)$ denotes $\lim_{L\to\infty} f(L)/g(L) = 1$.

2.2. **RSA Processes.** We study the ν-RSA process, in which segment lengths are drawn from a distribution given by a *length distribution function*. If the distribution of lengths contained arbitrarily small lengths, then the process would never terminate, as we would always be able to park more segments. Thus, without loss of generality, we require that the minimum segment length in the distribution be 1. A formal definition is as follows:

Definition 2.1. A *length distribution function* (or *ldf*) is a nonnegative integrable function $\nu : [0, \infty) \to \mathbb{R}^{>0}$ such that $\nu(\ell) = 0$ for $\ell \in [0, 1)$, and $\int_1^\ell \nu(t) \, \mathrm{d}t > 0$ for all $\ell > 1$, which is the condition that the minimum segment length is equal to 1.

We will construct distributions on $[0, L]$ induced by an ldf ν on $[0, L]$. To normalize the distribution, we define a *normalizing constant* for every ldf:

Definition 2.2. Given ldf $\nu(\ell)$, its *normalizing constant* is the function $Z_\nu : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ given by

$$Z_\nu(L) = \int_0^L \nu(\ell) \, \mathrm{d}\ell.$$

We also must make the following distinction between *convergent* and *divergent* ldfs:

Definition 2.3. We say an ldf $\nu(\ell)$, with normalizing constant Z_ν, is *convergent* if

$$\lim_{L\to\infty} Z_\nu(L) < \infty.$$

Otherwise, $\nu(\ell)$ is *divergent*.

Given an ldf, we describe an RSA process in which segment lengths are drawn from the ldf truncated below the interval length L. We call this the *ν-RSA process*, formally defined as follows:

Definition 2.4 (RSA Process)**.** Let $\nu(\ell)$ be an ldf. Then, let the *ν-RSA process* be the following stochastic process, in which we attempt to park segments on an interval of length L where the segment lengths drawn are from $\nu(\ell)$:

Initialize

- $I_0 = [0, L]$, the empty region not occupied by parked segments,
- $P_0 = \emptyset$, the set of parked segments.

Then, for $T = 1, 2, \ldots$

- Choose $b \overset{R}{\leftarrow} [0, L]$, and choose a length $\ell \in [1, L]$ according to the probability density function $p_L(\ell)$, where $p_L(\ell) \propto \nu(\ell)$, viz. $p_L(\ell) = \frac{\nu(\ell)}{Z_\nu(L)}$.
- If the segment $(b, b+\ell) \subseteq I_{T-1}$, let $I_T = I_{T-1} \setminus (b, b+\ell)$ and $P_T = P_{T-1} \cup \{(b, b+\ell)\}$. We say the segment $(b, b+\ell)$ has been *parked*. Otherwise, let $I_T = I_{T-1}$ and $P_T = P_{T-1}$, and we say that the segment $(b, b+\ell)$ has been *rejected*.
- If all connected intervals in I_T are of length less than 1, we say the process is *at saturation*.

We define the *empty space* at saturation to be the random variable S_L, defined as the total length not covered by parked segments at saturation, viz.

$$S_L = \lim_{T \to \infty} \lambda(I_T),$$

where $\lambda(I_T)$ denotes the Lesbegue measure of I_T.

Remark 2.5. Ney in [12] and Ananjevskii in [13] analyze similar processes to the one described above. However, in their processes, we first choose the segment length and *then* place the segment randomly on the interval, with no possibility of rejection. Because of this, their process tends to weight large segments more than ours. Moreover, they consider length distributions bounded above by a maximum segment length, whereas we allow arbitrarily large segments, which yields a different analysis.

A version of the ν-RSA process is the *multidisperse process* (c.f. Definition 4.1), in which we draw segment lengths from a discrete set of lengths $\ell_1 = 1, \ell_2, \cdots, \ell_n$, according to probabilities $q_1, \cdots, q_n$. Informally, the multidisperse process can be though of as the ν-RSA process with

$$\nu(\ell) = q_1 \delta(\ell - \ell_1) + \cdots + q_n \delta(\ell - \ell_n),$$

where δ is the Dirac delta function. We will study multidisperse processes in Section 4.

In Section 3, we study a related collection of processes, known as the "ghost" or "Matérn" processes, given by an appropriate thinning of RSA processes (see Definition 3.1 for a formal definition). In general, ghost processes are better understood than RSA processes.

2.3. **Analytic Tools.** We employ the Laplace transform throughout this paper.

Definition 2.6. The *Laplace transform* of a function $f : \mathbb{R} \to \mathbb{R}$ is the function $\mathcal{L}\{f\} : \mathbb{R}^{>0} \to \mathbb{R}$ given by

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(x) e^{-sx} \, \mathrm{d}x.$$

Proposition 2.7 (Laplace Transform Properties)**.** *Let $F(s), G(s)$ be the Laplace transforms of functions f, g, and let $\star$ be the convolution operator, defined as*

$$(f \star g)(x) = \int_0^x f(t) g(x-t) \, \mathrm{d}t = \int_0^x f(x-t) g(t) \, \mathrm{d}t.$$

Then, the following well-known properties hold:

1. *(Linearity) For $a, b \in \mathbb{R}$, $\mathcal{L}\{af(x) + bg(x)\} = aF(s) + bG(s)$.*
2. *(Differentiation) Let f be n-times differentiable. Moreover, let $f(0^+)$ denote the limit $\lim_{x \to 0^+} f(x)$, and let $f^{(n)}$ denote the n-th derivative of f. Then, $\mathcal{L}\{f^{(n)}(x)\} = s^n F(s) - \sum_{k=1}^n s^{n-k} f^{k-1}(0^+)$.*
3. *(Integration) $\mathcal{L}\left\{\int_0^x f(t) \, \mathrm{d}t\right\} = \frac{F(s)}{s}$.*

(4) *(Translation) For* $a \in \mathbb{R}$, $\mathcal{L}\{f(x+a)\} = e^{as}\left(F(s) - \int_0^a f(x)e^{-sx}\,\mathrm{d}x\right)$.
(5) *(Time-Multiplication) For* $n \in \mathbb{Z}^{>0}$, $\mathcal{L}\{x^n f(x)\} = F^{(n)}(s)$.
(6) *(Convolution)* $\mathcal{L}\{(f \star g)(x)\} = F(s)G(s)$.
(7) *(Abelian Final Value Theorem) If* f *is bounded and there exists a constant* C *for which* $\lim_{x\to\infty} f(x) = C$, *then* $\lim_{s\to 0^+} sF(s) = C$.

Our analysis relies on the Hardy-Littlewood Tauberian Theorem (c.f. [15] p. 30), which relates the behavior of a function's Laplace transform around 0 to the behavior of the function at infinity.

Theorem 2.8 (Hardy-Littlewood Tauberian Theorem)**.** *If* $f : \mathbb{R} \to \mathbb{R}$ *is positive and integrable,* $e^{-st}f(t)$ *is integrable, and as* $s \to 0$, *there exist constants* H, β *such that* $\mathcal{L}\{f\}(s) \sim \frac{H}{s^\beta}$, *then as* $x \to \infty$, $\int_0^x f(t)\,\mathrm{d}t \sim \frac{H}{\Gamma(\beta+1)}x^\beta$.

We also will require a couple technical propositions, which we will prove here. Analogous forms of Proposition 2.9 and Proposition 2.10 with the opposite inequality also hold.

Proposition 2.9. *Let* $y, v : [a,b) \to \mathbb{R}$ *be* k*-times differentiable functions such that for all* $x \in [a,b)$,

$$y^{(k)}(x) < C(x) + \sum_{i=0}^{k-1} c_i(x)y^{(i)}(x), \qquad v^{(k)}(x) = C(x) + \sum_{i=0}^{n-1} c_i(x)v^{(i)}(x),$$

where each $c_i(x)$ *is a nonnegative function on* $[a,b)$, *and* $C : [a,b) \to \mathbb{R}$. *Moreover, let* $y^{(i)}(a) = v^{(i)}(a)$ *for each* $0 \le i < k$. *Then, for all* $x \in [a,b)$, $y(x) \le v(x)$.

Proof. Consider $f(x) = v(x) - y(x)$. Note that $f^{(k)}(a) = 0$ for $k < n$, and

$$f^{(n)}(x) > \sum_{k=0}^{n-1} c_k(x)f^{(k)}(x). \tag{2.2}$$

Notably, $f^{(n)}(a) > 0$. We now claim there exists ϵ such that $f^{(k)}(x)$ is positive on $(a, a+\epsilon)$ for all $k \le n$. This is because $f^{(k)}(x)$ satisfies

$$\left(f^{(k)}\right)^{(1)}(a) = \left(f^{(k)}\right)^{(2)}(a) = \cdots = \left(f^{(k)}\right)^{(n-k-1)}(a) = 0, \qquad \left(f^{(k)}\right)^{(n-k)}(a) > 0,$$

where $\left(f^{(k)}\right)^{(i)}$ is the i-th derivative of $f^{(k)}$. The general derivative test on $f^{(k)}$ then implies that $f^{(k)}$ is strictly increasing in a small neighborhood $[a, a+\epsilon_k)$ around a. As $f^{(k)}(a) = 0$, this implies that $f^{(k)}(x)$ is positive on $(a, a+\epsilon_k)$. Letting ϵ be the minimum of the ϵ_k gives us our desired interval $(a, a+\epsilon)$.

Now, for contradiction, assume there exists x such that $f(x) < 0$. Then, the following infimum exists:

$$m := \inf\{x \in [a,b) : f^{(k)}(x) < 0 \text{ for some } k < n\}.$$

Note that $m \ge a + \epsilon$, as the $f^{(k)}$'s are nonnegative on $[a, a+\epsilon)$. We claim there exists some $K < n$ for which $f^{(K)}(m) \le 0$. Otherwise, $f^{(k)}(m) > 0$ for all $k < n$, and by continuity of the $f^{(k)}$'s, there exists a sufficiently small ϵ' for which all $f^{(k)}(x)$ are positive on $[m, m+\epsilon']$, which contradicts the definition of m.

Thus, let $f^{(K)}(m) \leq 0$, and let $x_K = m$. By the mean value theorem, there exists some x_{K+1} in $(0, m)$ such that

$$f^{(K+1)}(x_{K+1}) = \frac{f^{(K)}(x_K) - f^{(K)}(0)}{x_K - 0} = \frac{f^{(K)}(x_K)}{x_K} \leq 0.$$

We may repeatedly apply the mean value theorem in this fashion, showing that there exists $x_{K+2} \in (0, m)$ for which $f^{(K+2)}(x_{K+2}) \leq 0$, and so on, until we find $x_n \in (0, m)$ for which $f^{(n)}(x_n) \leq 0$. However, because $x_n < m$, we know $f^{(k)}(x_n) \geq 0$ for all $k < n$. By Equation 2.2, we have

$$0 \geq f^{(n)}(x_n) > \sum_{k=0}^{n-1} c_k(x_n) f^{(k)}(x_n) \geq 0.$$

Of course, it is impossible for $0 > 0$, so $f(x) = v(x) - y(x) \geq 0$ on $[a, b)$. This implies the lemma. □

Proposition 2.10. *Let $y, v : (a, b] \to \mathbb{R}$ be differentiable functions such that $y(b) = v(b)$ and for all $x \in (a, b]$,*

$$-y'(x) < C(x) + c_0(x)y,$$
$$-v'(x) = C(x) + c_0(x)v,$$

where $c_0(x) : (a, b] \to \mathbb{R}^{\geq 0}$ and $C : (a, b] \to \mathbb{R}$. Then, $y \leq v$ on (a, b).

Proof. Let $y_\star(x) = y(-x)$ and $v_\star(x) = v(-x)$. Using our given conditions, the substitution $x = -z$ yields

$$y_\star'(z) < C(-z) + c_0(-z)y_\star(-z),$$
$$v_\star'(z) = C(-z) + c_0(-z)y_\star(-z),$$

for all $z \in [-b, -a)$. This now satisfies Proposition 2.9, which implies that $y_\star(z) \leq v_\star(z)$ on $[-b, -a)$, and thus that $y(x) \leq v(x)$ on $(a, b]$. □

Proposition 2.11. *Let $f(x)$ be a function $f : (0, a) \to \mathbb{R}$. If for all $\epsilon > 0$, there exists $l_\epsilon \in \mathbb{R}$ and $\delta_\epsilon \in \mathbb{R}^{>0}$ such that for all $0 < x < \delta_\epsilon$, we have $l_\epsilon < f(x) < l_\epsilon + \epsilon$, then $\lim_{x\to 0^+} f(x)$ exists.*

Proof. Let $L = \sup_{\epsilon>0} l_\epsilon$, which must be finite: if there existed arbitrarily large l_ϵ, then $f(x)$ as $x \to 0^+$ would be unbounded, which contradicts our bound $f(x) < l_\epsilon + \epsilon$.

Fix arbitrary $\nu > 0$. We will find δ for which $x < \delta$ implies $L - \nu < f(x) < L + \nu$, proving that $\lim_{x\to 0^+} f(x) = L$.

First, choose some $\epsilon_1 > 0$ for which $l_{\epsilon_1} > L - \nu$. For $x < \delta_{\epsilon_1}$, we have $f(x) > l_{\epsilon_1} > L - \nu$, which completes one direction. Now let $\epsilon_2 = \nu$. For $x < \delta_{\epsilon_2}$, we have $f(x) < l_{\epsilon_2} + \epsilon_2 < L + \nu$. Thus, taking $\delta = \inf\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}$ completes the proof. □

3. Warm Up: Multidisperse Ghost Process

Here, we compute the behavior of the expected jamming length of multidisperse ghost processes, formally defined as follows (this is a slight modification of Definition 4.1):

Definition 3.1 (Multidisperse Ghost Process)**.** Fix lengths $\ell_1 = 1, \ldots, \ell_n \in \mathbb{R}^{>0}$ and rates $q_1, \ldots, q_n \in \mathbb{R}^{>0}$ such that $\ell_1 < \ell_2 < \cdots < \ell_n$ and $q_1 + \cdots + q_n = 1$. Then, let the $((\ell_1, q_1), \cdots, (\ell_n, q_n))$-*multidisperse ghost process* (abbreviated as $\mathcal{M}_{\mathrm{G}}((\ell_1, q_1), \cdots, (\ell_n, q_n))$) be the following:

Initialize

- $I_0 = [0, L]$, the empty region not occupied by parked or ghost segments,
- $P_0 = \emptyset$, the set of parked segments.

Then, at times $T = 1, 2, \cdots$

(1) Choose segment center $b \overset{R}{\leftarrow} \left[-\frac{1}{2}\ell_n, L + \frac{1}{2}\ell_n\right]$, and choose i from $[n]$ with the probability of choosing $i = k$ as q_k. Then, we say that segment $\left(b - \frac{1}{2}\ell_i, b + \frac{1}{2}\ell_i\right)$ is a *candidate segment* of *type* i.
(2) If $\left(b - \frac{1}{2}\ell_i, b + \frac{1}{2}\ell_i\right) \subseteq I_{T-1}$, let $P_T = P_{T-1} \cup \{\left(b - \frac{1}{2}\ell_i, b + \frac{1}{2}\ell_i\right)\}$. We say the candidate segment has been *parked*. Otherwise, let $P_T = P_{T-1}$, and we say that the candidate segment has become a *ghost*.
(3) Regardless of whether the segment has been parked, let $I_T = I_{T-1} \setminus \left(b - \frac{1}{2}\ell_i, b + \frac{1}{2}\ell_i\right)$.

We define the *type-i jamming number* as the total number of type i segments parked, viz.

$$N_{i,L} = \lim_{T\to\infty} |\{A \in P_T : \lambda(A) = \ell_i\}|,$$

where $\lambda(A)$ again denotes the Lesbegue measure of A. Moreover, let J_L be the random variable representing the total length of parked segments, viz.

$$J_L = \lim_{T\to\infty} \sum_{A\in P_T} \lambda(A).$$

Remark 3.2. This definition allows segment centers to be parked on $[-\frac{1}{2}\ell_n, 0]$ and $[L, L+\frac{1}{2}\ell_n]$ to avoid different behavior at the ends of the interval $[0, L]$. Moreover, we use segment centers in this definition for simplicity in the following proofs, but when we define RSA processes (c.f. Definition 2.4, Definition 4.1), we use the left endpoints.

For the remainder of the section, the ℓ_i's will always be positive reals representing the segment lengths in the multidisperse ghost process, and the q_i's will always be positive reals summing to 1 that represent the probabilities of choosing each segment. Moreover, we will always use σ to denote the *average segment length*, defined as follows:

Definition 3.3. In $\mathcal{M}_G((\ell_1, q_1), \cdots, (\ell_n, q_n))$, the *average segment length* σ is given by $\sigma = \sum_{i=1}^n q_i\ell_i$.

We are now able to derive a simple asymptotic for the multidisperse ghost process.

Proposition 3.4. *In $\mathcal{M}_{\mathrm{G}}((\ell_1, q_1), \cdots, (\ell_n, q_n))$, the type-$k$ jamming number $N_{k,L}$ (c.f. Definition 3.1) satisfies*

$$\lim_{L\to\infty} \frac{\mathbb{E}[N_{k,L}]}{L} = \frac{q_k}{\sigma + \ell_k}.$$

Proof. Fix $k \in [n]$. At time T, we choose a type-k candidate segment with probability q_k, and this segment is entirely contained within $[0, L]$ exactly when the segment center $b \in \left[\frac{1}{2}\ell_k, L - \frac{1}{2}\ell_k\right]$, which occurs with probability $\frac{L-\ell_k}{L+\ell_n}$.

Now, assume we have chosen candidate segment $\left(b - \frac{1}{2}\ell_k, b + \frac{1}{2}\ell_k\right) \subseteq [0, L]$, so b is fixed. Fix time $T' < T$. To compute the probability that the candidate segments at times T and T' intersect, we condition over the type of the candidate segment at time T'.

Assume that at time T', we have chosen a candidate segment of type-i with center b'. The candidate segments at times T and T' do not intersect when $|b - b'| > \frac{1}{2}(\ell_i + \ell_k)$. Since b' is uniformly distributed on $\left[-\frac{1}{2}\ell_n, L + \frac{1}{2}\ell_n\right]$, the candidate segments do not intersect with probability $\frac{L+\ell_n-(\ell_i+\ell_k)}{L+\ell_n}$.

The probability of choosing type-i at time T' is simply q_i, so the probability that the candidate segments at times T and T' do not intersect is

$$\sum_{i=1}^{n} q_i \cdot \frac{L+\ell_n-(\ell_i+\ell_k)}{L+\ell_n} = 1-\frac{\sigma+\ell_k}{L+\ell_n}.$$

The candidate segments before time T are independently chosen, so in fact, the probability that the candidate at time T intersects with no previous candidate is $\left(1-\frac{\sigma+\ell_k}{L+\ell_n}\right)^{T-1}$. Thus, the probability that at time T, we successfully park a type-k segment is

$$q_k \cdot \frac{L-\ell_k}{L+\ell_n} \cdot \left(1-\frac{\sigma+\ell_k}{L+\ell_n}\right)^{T-1}.$$

Summing over times T from 1 to ∞, the expected number of parked type-k segments is

$$\mathbb{E}[N_{k,L}] = \sum_{T=1}^{\infty} q_k \cdot \frac{L-\ell_k}{L+\ell_n} \cdot \left(1-\frac{\sigma+\ell_k}{L+\ell_n}\right)^{T-1} = \frac{q_k(L-\ell_k)}{\sigma+\ell_k}.$$

Thus, $\lim_{L\to\infty} \frac{\mathbb{E}[N_{k,L}]}{L} = \frac{q_k}{\sigma+\ell_k}$. □

Using Proposition 3.4, we may find that in $\mathcal{M}_{\mathrm{G}}((\ell_1,q_1),\cdots,(\ell_n,q_n))$, the total jamming length satisfies

$$\lim_{L\to\infty} \frac{\mathbb{E}[J_L]}{L} = \sum_{k=1}^{n} \frac{q_k\ell_k}{\sigma+\ell_k}, \tag{3.1}$$

as by definition, $J_L = \sum_{k=1}^{n} \ell_k N_{k,L}$. We now prove bounds on this expression.

Corollary 3.5. *In $\mathcal{M}_{\mathrm{G}}((\ell_1,q_1),(\ell_2,q_2))$, the jamming length satisfies*

$$\frac{2\sqrt{\ell_1\ell_2}}{(\sqrt{\ell_1}+\sqrt{\ell_2})^2} \le \lim_{L\to\infty}\frac{\mathbb{E}[J_L]}{L} \le \frac{1}{2}.$$

The maximum is achieved when either $q_1 = 0$ or $q_2 = 0$. The minimum is achieved when $\sigma = \sqrt{\ell_1\ell_2}$.

Proof. Fix the lengths $\ell_1 < \ell_2$, and let $F:\mathbb{R}^2 \to \mathbb{R}$ be given as

$$F(q_1,q_2) = \lim_{L\to\infty}\frac{\mathbb{E}[J_L]}{L} = \frac{q_1\ell_1}{\sigma+\ell_1} + \frac{q_2\ell_2}{\sigma+\ell_2},$$

where the last equality is given by Equation 3.1. We wish to bound F in the region $R \subset \mathbb{R}^2$ where $q_1+q_2 = 1$ and $q_1, q_2 \ge 0$. If any relative extrema occur in the interior of R, the method of Lagrange multipliers implies that $\nabla F = \lambda\langle 1,1\rangle$, or that $\frac{\mathrm{d}F}{\mathrm{d}q_1} = \frac{\mathrm{d}F}{\mathrm{d}q_2}$, viz.

$$\frac{\ell_1^2+q_2\ell_1\ell_2}{(\sigma+\ell_1)^2} - \frac{q_2\ell_1\ell_2}{(\sigma+\ell_2)^2} = -\frac{q_1\ell_1\ell_2}{(\sigma+\ell_1)^2} + \frac{q_1\ell_1\ell_2+\ell_2^2}{(\sigma+\ell_2)^2}.$$

Rearranging, this becomes $\frac{\ell_1}{(\sigma+\ell_1)^2} = \frac{\ell_2}{(\sigma+\ell_2)^2}$, which precisely holds when $\sigma = \sqrt{\ell_1\ell_2}$. This occurs when $q_1 = \frac{\sqrt{\ell_2}}{\sqrt{\ell_1}+\sqrt{\ell_2}}$ and $q_2 = \frac{\sqrt{\ell_1}}{\sqrt{\ell_1}+\sqrt{\ell_2}}$, which together yield $F(q_1,q_2) = \frac{2\sqrt{\ell_1\ell_2}}{(\sqrt{\ell_1}+\sqrt{\ell_2})^2}$. This value is also equal to $\frac{1}{2} - \left(\frac{\sqrt{\ell_1}-\sqrt{\ell_2}}{\sqrt{\ell_1}+\sqrt{\ell_2}}\right)^2$, so it is strictly less than $\frac{1}{2}$.

There are also relative extrema which lie on the boundary of R. In particular, if $q_1 = 0$ or $q_2 = 0$, we are immediately given a value of $F(q_1, q_2) = \frac{1}{2}$. We have now found all the relative extrema, which thus bound the value of F between $\frac{2\sqrt{\ell_1\ell_2}}{(\sqrt{\ell_1}+\sqrt{\ell_2})^2}$ and $\frac{1}{2}$. $\square$

We now extend our work with the process $\mathcal{M}_{\mathrm{G}}((\ell_1, q_1), (\ell_2, q_2))$ to the more general process $\mathcal{M}_{\mathrm{G}}((\ell_1, q_1), \ldots, (\ell_n, q_n))$.

Corollary 3.6. *In $\mathcal{M}_{\mathrm{G}}((\ell_1, q_1), \cdots, (\ell_n, q_n))$, the jamming length satisfies*

$$\frac{2\sqrt{\ell_1\ell_n}}{(\sqrt{\ell_1}+\sqrt{\ell_n})^2} \le \lim_{L\to\infty} \frac{\mathbb{E}[J_L]}{L} \le \frac{1}{2}.$$

Proof. Fix the lengths $\ell_1 < \cdots < \ell_n$. We wish to bound the function $F : \mathbb{R}^n \to \mathbb{R}$, given by

$$F(q_1, \ldots, q_n) = \lim_{L\to\infty} \frac{\mathbb{E}[J_L]}{L} = \sum_{k=1}^{n} \frac{q_k\ell_k}{\sigma + \ell_k}.$$

Again, we only consider F in the region $R \subset \mathbb{R}^n$ where $q_1 + \cdots + q_n = 1$ and $q_1, \ldots, q_n \ge 0$. We will induct on the value of n, where the base case with $n = 2$ is proved in Corollary 3.5 (and the case where $n = 1$ is trivial).

Assume now that Corollary 3.6 holds for $n = M - 1$. We will show it holds for $n = M$, where $M \ge 3$. We first investigate relative extrema within the region R using Lagrange multipliers. If such an extrema existed, we must have $\nabla F = \lambda\langle 1, \cdots, 1\rangle$, or equivalently, $\frac{\mathrm{d}F}{\mathrm{d}q_1} = \cdots = \frac{\mathrm{d}F}{\mathrm{d}q_M}$.

Fix arbitrary $i, j \in [M]$. Then, $\frac{\mathrm{d}F}{\mathrm{d}q_i} = \frac{\mathrm{d}F}{\mathrm{d}q_j}$ becomes

$$\ell_i \left(\frac{1}{\sigma + \ell_i} - \sum_{k\in[M]} \frac{q_k\ell_k}{(\sigma+\ell_k)^2} \right) = \ell_j \left(\frac{1}{\sigma + \ell_j} - \sum_{k\in[M]} \frac{q_k\ell_k}{(\sigma+\ell_k)^2} \right),$$

which simplifies to

$$\frac{\sigma}{(\sigma+\ell_i)(\sigma+\ell_j)} = \sum_{k\in[M]} \frac{q_k\ell_k}{(\sigma+\ell_k)^2}.$$

The right hand side is a constant expression not depending on i and j. Thus, $\frac{\sigma}{(\sigma+\ell_i)(\sigma+\ell_j)}$ too must be constant for any choice of i and j, which only can occur when all the lengths are equal, viz. $\ell_1 = \cdots = \ell_M$. This cannot happen, as our lengths are all distinct. Thus, F has no relative extrema in the interior of the region R.

Both the minimum and maximum of F must then lie on the boundary of R, so fix an arbitrary $q_i = 0$. This now reduces our problem to the case with $n = M - 1$, and by the inductive hypothesis, the maximum is always $\frac{1}{2}$.

Meanwhile, if $i = 1$, the minimum is $\frac{2\sqrt{\ell_2\ell_M}}{(\sqrt{\ell_2}+\sqrt{\ell_M})^2}$. If $i = M$, the minimum is $\frac{2\sqrt{\ell_1\ell_{M-1}}}{(\sqrt{\ell_1}+\sqrt{\ell_{M-1}})^2}$. Otherwise, the minimum is $\frac{2\sqrt{\ell_1\ell_M}}{(\sqrt{\ell_1}+\sqrt{\ell_M})^2}$. By writing $\frac{2\sqrt{\ell_1\ell_M}}{(\sqrt{\ell_1}+\sqrt{\ell_M})^2}$ as $\frac{1}{2} - \left(\frac{\sqrt{\ell_1}-\sqrt{\ell_M}}{\sqrt{\ell_1}+\sqrt{\ell_M}}\right)^2$ and the others analogously, it is straightforward to show that $\frac{2\sqrt{\ell_1\ell_M}}{(\sqrt{\ell_1}+\sqrt{\ell_M})^2}$ is the least of the three minima, which completes the proof. $\square$

4. Multidisperse Process

Here, we will analyze the behavior of multidisperse processes, defined formally as follows:

Definition 4.1 (Multidisperse Process)**.** Fix lengths $\ell_1 = 1, \ldots, \ell_n \in \mathbb{R}^{>0}$ and rates $q_1, \ldots, q_n \in \mathbb{R}^{>0}$ such that $\ell_1 < \ell_2 < \cdots < \ell_n$ and $q_1 + \cdots + q_n = 1$. Then, let the $((\ell_1, q_1), \cdots, (\ell_n, q_n))$-multidisperse process, abbreviated as $\mathcal{M}((\ell_1, q_1), \cdots, (\ell_n, q_n))$, be the following process:

Initialize

- $I_0 = [0, L]$, the empty region not occupied by parked segments,
- $P_0 = \emptyset$, the set of parked segments.

Then, for $T = 1, 2, \ldots$

(1) Choose left endpoint $b \xleftarrow{R} [0, L]$, and choose i from $[n]$ with the probability of choosing $i = k$ as q_k. Then, we say that segment $(b, b + \ell_i)$ is a *candidate segment* of *type* i.
(2) If $(b, b + \ell_i) \subseteq I_{T-1}$, let $I_t = I_{T-1} \setminus (b, b + \ell_i)$ and $P_T = P_{T-1} \cup \{(b, b + \ell_i)\}$. We say the segment $(b, b + \ell_i)$ has been *parked*. Otherwise, let $I_T = I_{T-1}$ and $P_T = P_{T-1}$, and we say that the segment $(b, b + \ell_i)$ has been *rejected*.

We define the *type* i *jamming number* to be the random variable $N_{i,L}$, defined as the total number of type i segments parked at saturation, viz.

$$N_{i,L} = \lim_{T\to\infty} |\{A \in P_T : \lambda(A) = \ell_i\}|,$$

where $\lambda(A)$ denotes the Lesbegue measure of A.

Like in the multidisperse ghost process, for the remainder of the section, the ℓ_i's will always be positive reals representing the segment lengths, and the q_i's will always be positive reals summing to 1 that represent the probabilities of choosing each segment. The shortest segment length, ℓ_1, is always implicitly taken to be 1. Moreover, we will always take σ to be the *average segment length*, defined exactly as it was for the multidisperse ghost process:

Definition 4.2. In $\mathcal{M}((\ell_1, q_1), \cdots, (\ell_n, q_n))$, the *average segment length* σ is given by $\sigma = \sum_{i=1}^n q_i \ell_i$.

We will study the expected number of type-k segments placed. We begin with the following integral recurrence formula:

Proposition 4.3. *In* $\mathcal{M}((\ell_1, q_1), \cdots, (\ell_n, q_n))$*, for* $L \geq \ell_n$*,*

$$\mathbb{E}[N_{k,L}] = \frac{1}{L - \sigma}\left(q_k(L - \ell_k) + 2\sum_{i=1}^n q_i \int_0^{L-\ell_i} \mathbb{E}[N_{k,t}]\,\mathrm{d}t\right).$$

Proof. On an interval $[0, L]$ with length $L \geq \ell_n$, our first attempt to park a segment results in parking a type i segment with probability $q_i \cdot \frac{L-\ell_i}{L}$, as we must first choose a type i segment with probability q_i, then successfully park it by choosing its left endpoint to be in $[0, L - \ell_i]$, which occurs with probability $\frac{L-\ell_i}{L}$.

Let A_i be the event that the first segment eventually parked is type i. If no segment is parked on the first attempt, we repeatedly make new, independent attempts, so $\mathbb{P}[A_i] \propto q_i \cdot \frac{L-\ell_i}{L}$. Renormalizing the probabilities yields $\mathbb{P}[A_i] = \frac{q_i(L-\ell_i)}{L-\sigma}$.

Given that A_i occurs, let the position of the first segment's left endpoint be t, which is a random variable uniformly distributed on $[0, L - \ell_i]$. The interval $[0, L]$ is then broken into two subintervals of length t and $L - l_i - t$, on which the same multidisperse process continues. The expected numbers of type-k intervals placed on the two subintervals are then $\mathbb{E}[N_{k,t}]$ and $\mathbb{E}[N_{k,L-\ell_i-t}]$, respectively. Thus, integrating over the random variable t, we have that when $i \neq k$,

$$\mathbb{E}[N_{k,L} \mid A_i] = \frac{1}{L-\ell_i}\int_0^{L-\ell_i} (\mathbb{E}[N_{k,t}] + \mathbb{E}[N_{k,L-\ell_i-t}])\,\mathrm{d}t = \frac{2}{L-\ell_i}\int_0^{L-\ell_i} \mathbb{E}[N_{k,t}]\,\mathrm{d}t.$$

When $i = k$, we have just placed a type-k segment and must accordingly add 1 to $\mathbb{E}[N_{k,L}]$:

$$\mathbb{E}[N_{k,L} \mid A_k] = 1 + \frac{2}{L-\ell_k}\int_0^{L-\ell_k} \mathbb{E}[N_{k,t}]\,\mathrm{d}t.$$

Finally, conditioning over the A_i and simplifying, we have

$$\mathbb{E}[N_{k,L}] = \sum_{i=1}^{n} \mathbb{P}[A_i]\,\mathbb{E}[N_{k,L} \mid A_i] = \frac{1}{L-\sigma}\left(q_k(L-\ell_k) + 2\sum_{i=1}^{n} q_k \int_0^{L-\ell_k} \mathbb{E}[N_{k,t}]\,\mathrm{d}t\right).$$

□

Using this recurrence, we will use Laplace transforms to derive precise asymptotics for $\mathbb{E}[N_{k,L}]$ as $L \to \infty$.

Remark 4.4. The *bidisperse process*, i.e. the multidisperse process with $n = 2$ different segment lengths, has been investigated before by various authors. In particular, Subashiev and Luryi derive in [8] an exact expression for $\lim_{L\to\infty} \frac{\mathbb{E}[N_{k,L}]}{L}$ in the bidisperse process. Our work here extends their work to the multidisperse process, and by choosing $n = 2$, one recovers their formula.

Theorem 4.5. *Consider $\mathcal{M}((\ell_1, q_1), \cdots, (\ell_n, q_n))$, and fix $k \in [n]$. Define functions $P_{i;k} : \mathbb{R}^{>0} \to \mathbb{R}$ as*

$$P_{i;k}(s) := \int_{\ell_n-\ell_i}^{\ell_n} \mathbb{E}[N_{k,L}]e^{-sL}\,\mathrm{d}L, \tag{4.1}$$

and define $G_k : \mathbb{R}^{>0} \to \mathbb{R}$ as

$$G_k(s) := e^{-(\ell_n-\sigma)s}\Big(q_k + s(\ell_n - \sigma)\,\mathbb{E}[N_{k,\ell_n}]\Big) + 2se^{\sigma s}\sum_{i=1}^{n} q_i e^{-\ell_i s} P_{i;k}(s). \tag{4.2}$$

Then,

$$\lim_{L\to\infty} \frac{\mathbb{E}[N_{k,L}]}{L} = \int_0^\infty G_k(t)\exp\left(-2\sum_{i=1}^{n} q_i\,\mathrm{Ein}(\ell_i t)\right)\mathrm{d}t. \tag{4.3}$$

Example 4.6. Consider $\mathbb{E}[N_{1,L}]$ in the multidisperse process $\mathcal{M}((1, .5), (1.3, .3), (1.5, .2))$, which has 3 possible segment lengths of 1, 1.3, and 1.5. Note that because $\mathbb{E}[N_{1,L}] = 0$ when $L < 1$, each $P_{i;1}$ is equal to $\int_1^{1.5} \mathbb{E}[N_{1,L}]e^{-sL}\,\mathrm{d}L$ in this case. To compute $P_{i;1}$, we compute that $\mathbb{E}[N_{1,L}]$ is 1 when $1 \le L < 1.3$ and $\frac{.5(L-1)}{.5(L-1)+.3(L-1.3)}$ when $1.3 \le L < 1.5$. Then, noting $\sigma = .31$, we have

$$G_1(s) := e^{-.31s}(.5 + .25s) + 2s(.5e^{.19s} + .3e^{-.11s} + .2e^{-.31s})\int_1^{1.5} \mathbb{E}[N_{1,L}]e^{-sL}\,\mathrm{d}L,$$

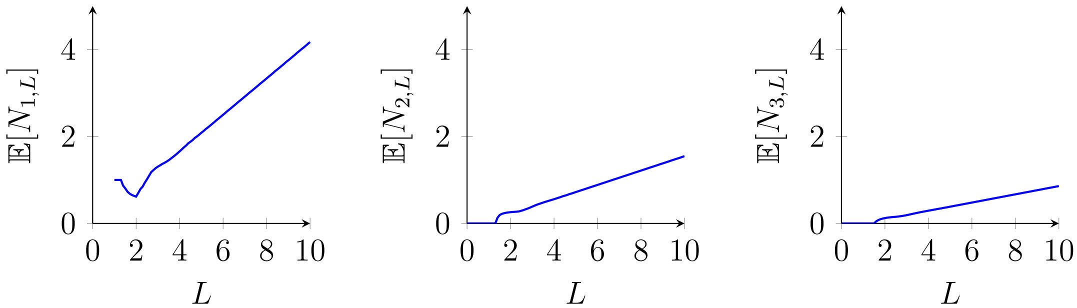


FIGURE 4. Plots of $\mathbb{E}[N_{k,L}]$ in $\mathcal{M}((1, .5), (1.3, .3), (1.5, .2))$.

and numerically integrating Equation 4.3 yields

$$\lim_{L\to\infty} \frac{\mathbb{E}[N_{1,L}]}{L} = \int_0^\infty G_1(t) \exp\Big(-2\Big(.5\,\mathrm{Ein}(t) + .3\,\mathrm{Ein}(1.3t) + .2\,\mathrm{Ein}(1.5t)\Big)\Big)\,\mathrm{d}t \approx .4204.$$

That is, $\mathbb{E}[N_{1,L}] \sim .4204L$. We may similarly compute that $\mathbb{E}[N_{2,L}] \sim .1655L$ and $\mathbb{E}[N_{3,L}] \sim 0.0949L$ (c.f. Figure 4). With these values, we see that the total length covered by all segments grows asymptotically equal to $.7778L$.

Proof of Theorem 4.5. Fix $k \in [n]$. We first derive a formula for the Laplace transform of $\mathbb{E}[N_{k,L}]$, which we then use to solve for the behavior of the Laplace transform around 0. Finally, we use this with Theorem 2.8 to determine the behavior of $\mathbb{E}[N_{k,L}]$ as $L \to \infty$.

For brevity, define constants

$$\rho := \ell_n - \sigma \qquad \text{and} \qquad \rho_i := \ell_n - \ell_i \tag{4.4}$$

for $i \in [n]$. The ρ_i's are the differences in length between the largest segment and the other segments, and ρ is the difference between the largest segment length and the mean. Because ℓ_n is the largest segment length, ρ and ρ_i are all nonnegative.

Finally, let

$$\varphi(s) := \int_{\ell_n}^\infty \mathbb{E}[N_{k,L}]e^{-sL}\,\mathrm{d}L. \tag{4.5}$$

Note $\varphi(s)$ is not exactly the Laplace transform of $\mathbb{E}[N_{k,L}]$. Notably, we do not include $\mathbb{E}[N_{k,L}]$ when $L < \ell_n$ because the multidisperse process has a fundamentally different behavior then, as segments of length ℓ_n are never parked.

With these definitions, we are able to formulate a differential equation for $\varphi(s)$:

Lemma 4.6.1. *Let $w(s) = e^{\sigma s}\varphi(s)$. Then, with $G_k(s)$ as defined in Theorem 4.5,*

$$w'(s) + \frac{2w(s)}{s}\sum_{i=1}^n q_i e^{-\ell_i s} + \frac{G_k(s)}{s^2} = 0.$$

Proof. By Proposition 4.3 with $L + \ell_n$ as the interval length, we have

$$\mathbb{E}[N_{k,L+\ell_n}] = \frac{1}{L+\rho}\left(q_k(L+\rho_k) + 2\sum_{i=1}^n q_i \int_0^{L+\rho_i} \mathbb{E}[N_{k,t}]\,\mathrm{d}t\right).$$

This equation holds for all $L \geq 0$. Rearrange and differentiate with respect to L to get

$$\frac{\mathrm{d}}{\mathrm{d}L}\Big[(L+\rho)\,\mathbb{E}[N_{k,L+\ell_n}]\Big] = q_k + 2\sum_{i=1}^{n} q_i\,\mathbb{E}[N_{k,L+\rho_i}],$$

which has Laplace transform

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}L}\Big[(L+\rho)\,\mathbb{E}[N_{k,L+\ell_n}]\Big]e^{-sL}\,\mathrm{d}L = q_k\int_0^\infty e^{-sL}\,\mathrm{d}L + 2\sum_{i=1}^{n} q_i\int_0^\infty \mathbb{E}[N_{k,L+\rho_i}]e^{-sL}\,\mathrm{d}L. \tag{4.6}$$

We integrate the right and left sides separately. Note first

$$\int_0^\infty \mathbb{E}[N_{k,L+\ell_n}]e^{-sL}\,\mathrm{d}L = e^{\ell_n s}\int_0^\infty \mathbb{E}[N_{k,L+\ell_n}]e^{-s(L+\ell_n)}\,\mathrm{d}L = e^{\ell_n s}\varphi(s). \tag{4.7}$$

Then, by integration by parts, the left side of Equation 4.6 is equivalent to

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}L}\Big[(L+\rho)\,\mathbb{E}[N_{k,L+\ell_n}]\Big]e^{-sL}\,\mathrm{d}L &= s\int_0^\infty (L+\rho)\,\mathbb{E}[N_{k,L+\ell_n}]e^{-sL}\,\mathrm{d}L - \rho\,\mathbb{E}[N_{k,\ell_n}]\\
&= -s\cdot\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_0^\infty \mathbb{E}[N_{k,L+\ell_n}]e^{-sL}\,\mathrm{d}L\right]\\
&\quad + s\rho\int_0^\infty \mathbb{E}[N_{k,L+\ell_n}]e^{-sL}\,\mathrm{d}L - \rho\,\mathbb{E}[N_{k,\ell_n}]\\
\text{(By Equation 4.7)}\qquad &= -s\cdot\frac{\mathrm{d}}{\mathrm{d}s}[e^{\ell_n s}\varphi(s)] + s\rho e^{\ell_n s}\varphi(s) - \rho\,\mathbb{E}[N_{k,\ell_n}]\\
&= -se^{\ell_n s}(\varphi'(s)+\sigma\varphi(s)) - \rho\,\mathbb{E}[N_{k,\ell_n}]. \qquad (4.8)
\end{aligned}$$

The integral on the right side of Equation 4.6 evaluates to

$$\begin{aligned}
\int_0^\infty \mathbb{E}[N_{k,L+\rho_i}]e^{-sL}\,\mathrm{d}L &= e^{\rho_i s}\int_0^\infty \mathbb{E}[N_{k,L+\rho_i}]e^{-s(L+\rho_i)}\,\mathrm{d}L\\
&= e^{\rho_i s}\left(\int_{\rho_i}^{\ell_n} \mathbb{E}[N_{k,L}]e^{-sL}\,\mathrm{d}L + \int_{\ell_n}^\infty \mathbb{E}[N_{k,L}]e^{-sL}\,\mathrm{d}L\right)\\
&= e^{\rho_i s}\left(P_{i;k}(s)+\varphi(s)\right),
\end{aligned}$$

where $P_{i;k}(s)$ is as defined in Equation 4.1. The right side of Equation 4.6 then simplifies to

$$\frac{q_k}{s} + 2\sum_{i=1}^{n} q_i e^{\rho_i s}(P_{i;k}(s)+\varphi(s)). \tag{4.9}$$

Equating Equation 4.8 and Equation 4.9 yields

$$-se^{\ell_n s}(\varphi'(s)+\sigma\varphi(s)) - \rho\,\mathbb{E}[N_{k,\ell_n}] = \frac{q_k}{s} + 2\sum_{i=1}^{n} q_i e^{\rho_i s}(P_{i;k}(s)+\varphi(s)).$$

Rearranging and multiplying the equation by $\frac{e^{-\rho s}}{s}$ yields

$$e^{\sigma s}(\varphi'(s)+\sigma\varphi(s)) + \frac{2e^{\sigma s}\varphi(s)}{s}\sum_{i=1}^{n} q_i e^{-\ell_i s} + \frac{e^{-\rho s}}{s^2}\left(q_k + s\rho\,\mathbb{E}[N_{k,\ell_n}]\right) + \frac{2e^{\sigma s}}{s}\sum_{i=1}^{n} q_i e^{-\ell_i s}P_{i:k}(s) = 0.$$

Define $G_k(s)$ as in Equation 4.2, and note that $\frac{G_k(s)}{s^2}$ is exactly the constant term in this first order differential equation for $\varphi(s)$. That is,

$$e^{\sigma s}(\varphi'(s) + \sigma\varphi(s)) + \frac{2e^{\sigma s}\varphi(s)}{s}\sum_{i=1}^{n} q_i e^{-\ell_i s} + \frac{G_k(s)}{s^2} = 0.$$

Finally, substituting in $w(s) = e^{\sigma s}\varphi(s)$ proves our lemma. □

Having derived a differential equation for the Laplace transform of $\mathbb{E}[N_{k,L}]$, we must show that the integral in Equation 4.3 actually exists.

Lemma 4.6.2. *The following integral is finite:*

$$\alpha_k := \int_0^\infty G_k(t)\exp\left(-2\sum_{i=1}^{n} q_i \operatorname{Ein}(\ell_i t)\right) \mathrm{d}t.$$

We omit the proof of this lemma, which is a routine calculus exercise. We now analyze the behavior of the Laplace transform of $\mathbb{E}[N_{k,L}]$ around $s = 0$:

Lemma 4.6.3. *As $s \to 0^+$, we have that*

$$\int_0^\infty \mathbb{E}[N_{k,L}]e^{-sL}\,\mathrm{d}L \sim \frac{\alpha_k}{s^2}.$$

Proof. Let $w(s) = e^{\sigma s}\varphi(s)$, as in lemma 4.6.1. The maximum number of type-k segments we can place on an interval of length L is $\frac{L}{\ell_k}$, so $\mathbb{E}[N_{k,L}] \le \frac{L}{\ell_k}$, and

$$w(s) = e^{\sigma s}\varphi(s) \le e^{\sigma s}\int_{\ell_n}^\infty \frac{L}{\ell_k}\cdot e^{-sL}\,\mathrm{d}L = \frac{e^{-\rho s}}{\ell_k}\left(\frac{\ell_n}{s} + \frac{1}{s^2}\right).$$

This implies the initial condition that $\lim_{s\to\infty} w(s) = 0$.

By Lemma 4.6.1, we have the differential equation $w'(s) + \frac{2w(s)}{s}\sum_{i=1}^{n} q_i e^{-\ell_i s} + \frac{G_k(s)}{s^2} = 0$. Using our initial condition and the method of integrating factors, we may solve this first order linear differential equation, which yields

$$w(s) = \frac{1}{s^2}\int_s^\infty G_k(t)\exp\left(-2\sum_{i=1}^{n} q_i \int_s^t \frac{1 - e^{-\ell_i u}}{u}\,\mathrm{d}u\right)\mathrm{d}t.$$

Thus, as $s \to 0^+$, by dominated convergence we have

$$w(s) \sim \frac{1}{s^2}\int_0^\infty G_k(t)\exp\left(-2\sum_{i=1}^{n} q_i \int_0^t \frac{1 - e^{-\ell_i u}}{u}\,\mathrm{d}u\right)\mathrm{d}t = \frac{a_k}{s^2},$$

with α_k defined as the expression in Lemma 4.6.2. Finally, we have the simple bound $0 \le \mathbb{E}[N_{k,L}] \le \frac{L}{\ell_k}$, which implies that as $s \to 0^+$, $\varphi(s) \le \int_0^\infty \mathbb{E}[N_{k,L}]e^{-sL}\,\mathrm{d}L \le \frac{\ell_n^2}{2\ell_k} + \varphi(s)$, and so $\int_0^\infty \mathbb{E}[N_{k,L}]e^{-sL}\,\mathrm{d}L \sim \frac{\alpha_k}{s^2}$. □

Using Theorem 2.8, we may convert the behavior of the Laplace transform of $\mathbb{E}[N_{k,L}]$ into the behavior of $\mathbb{E}[N_{k,L}]$ as $L \to \infty$. In particular, the theorem implies $\int_0^L \mathbb{E}[N_{k,t}]\,\mathrm{d}t \sim \frac{\alpha_k}{2}\cdot L^2$.

Recall that by Proposition 4.3,

$$\mathbb{E}[N_{k,L}] = \frac{1}{L-\sigma}\left(q_k(L-\ell_k) + 2\sum_{i=1}^{n} q_i \int_0^{L-\ell_i}\mathbb{E}[N_{k,t}]\,\mathrm{d}t\right).$$

Substituting in our asymptotic for the integral of $\mathbb{E}[N_{k,L}]$ into the right side of this equation completes the proof that $\mathbb{E}[N_{k,L}] \sim \alpha_k L$. ☐

5. General Length Distribution

In this section, we consider general ν-RSA processes (c.f. Definition 2.4). In particular, we study the behavior of $\mathbb{E}[S_L]$ as $L \to \infty$, where S_L is the random variable representing the empty space at saturation. Similar to our investigation of multidisperse process, we first derive a general integral recurrence formula for $\mathbb{E}[S_L]$.

Remark 5.1. In [16], Burridge and Mao derive a similar recurrence equation for length distributions with finite support (i.e. there exists C for which $\ell > C$ implies $\nu(\ell) = 0$). They then use it to consider a specific case of bidisperse RSA (c.f. Remark 4.4). Here, we extend their recurrence to general length distributions, and we provide a proof for completeness.

Proposition 5.2. *Let $\nu(\ell)$ be an ldf. Then, in the ν-RSA process, for all $L \geq 0$,*

$$\left(\int_0^L Z_\nu(t)\,\mathrm{d}t\right)\mathbb{E}[S_L] = 2\int_0^L \mathbb{E}[S_t]Z_\nu(L-t)\,\mathrm{d}t.$$

Proof. We must have $Z_\nu(t) = 0$ for $t < 1$, so when $L \leq 1$, the proposition reduces to $0 = 0$. Thus fix the interval length as $L > 1$.

Let $\mu_L(\ell)$ be the probability density function for the length of the segment that will be parked first. In the ν-RSA process, segment lengths are chosen according to a distribution proportional to $\nu(\ell)$. Moreover, the segment is then parked successfully on the interval with probability $\frac{L-\ell}{L}$. Thus, $\mu_L(\ell) \propto \nu(\ell)(L-\ell)$. Normalizing this distribution then yields

$$\mu_L(\ell) = \frac{(L-\ell)\nu(\ell)}{\int_0^L (L-t)\nu(t)\,\mathrm{d}t}.$$

We now compute $\mathbb{E}[S_L]$. Given the length ℓ of the first segment parked, we use an identical argument to the proof of Proposition 4.3 to show that $\mathbb{E}[S_L]$ is $\frac{2}{L-\ell}\int_0^{L-\ell}\mathbb{E}[S_t]\,\mathrm{d}t$.

Using our probability function $\mu_L(\ell)$, we now integrate over the possible values of ℓ:

$$\mathbb{E}[S_L] = \int_0^L \left(\frac{(L-\ell)\nu(\ell)}{\int_0^L (L-t)\nu(t)\,\mathrm{d}t}\cdot\frac{2}{L-\ell}\int_0^{L-\ell}\mathbb{E}[S_t]\,\mathrm{d}t\right)\mathrm{d}\ell,$$

which simplifies to

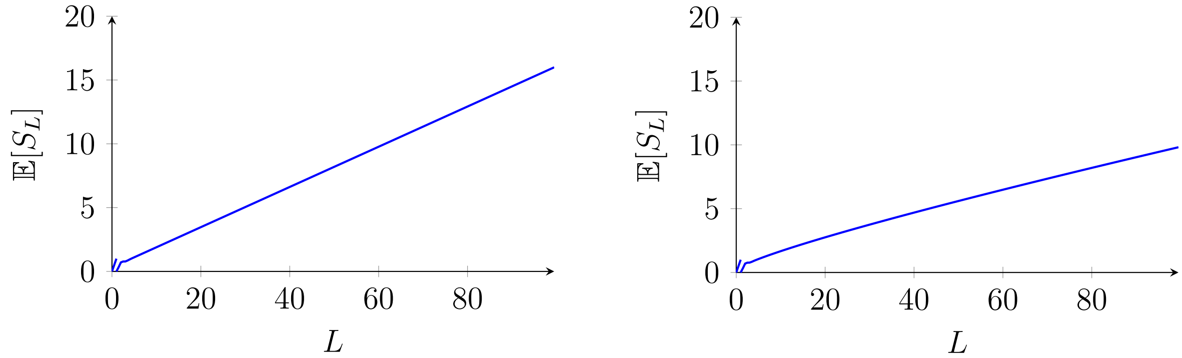
$$\left(\int_0^L (L-t)\nu(t)\,\mathrm{d}t\right)\mathbb{E}[S_L] = 2\int_0^L\int_0^{L-\ell}\nu(\ell)\,\mathbb{E}[S_t]\,\mathrm{d}t\,\mathrm{d}\ell. \tag{5.1}$$

Switching the order of integration on the right side yields

$$2\int_0^L\int_0^{L-\ell}\nu(\ell)\,\mathbb{E}[S_t]\,\mathrm{d}t\,\mathrm{d}\ell = 2\int_0^L\mathbb{E}[S_t]\int_0^{L-t}\nu(\ell)\,\mathrm{d}\ell\,\mathrm{d}t = 2\int_0^L\mathbb{E}[S_t]Z_\nu(L-t)\,\mathrm{d}t.$$

Meanwhile, by integration by parts, $\int_0^L (L-t)\nu(t)\,\mathrm{d}t = \int_0^L Z_\nu(t)\,\mathrm{d}t$. Substituting this into Equation 5.1 then proves the proposition. ☐

Using Laplace transforms with the recurrence equation in Proposition 5.2, we will later find that $\mathbb{E}[S_L]$ grows linearly for a general class of convergent ldfs (c.f. Definition 2.1), whereas it grows sublinearly for a general class of divergent ldfs.



(A) $\nu(\ell) = e^{-\ell}$.

(B) $\nu(\ell) = \ell^{-1.1}$.

FIGURE 5. $\mathbb{E}[S_L]$ grows linearly under various convergent ldfs.

5.1. **Convergent Length Distributions.** We first consider the ν-RSA process for convergent ldfs $\nu(\ell)$. When considering such ldfs, we always assume $\int_1^\infty \nu(\ell)\,\mathrm{d}\ell = 1$ without any loss of generality.

Note as $L \to \infty$, we must have $Z_\nu(L) \to 1$. If $Z_\nu(L) = 1 - o(L^{-\epsilon})$ for any $\epsilon > 0$, the improper integral $\int_1^\infty \frac{1-Z_\nu(L)}{L}\,\mathrm{d}L$ must have a finite value. Our analysis will rely on the weak condition that this integral is finite, which holds for many natural length distributions (c.f. Example 5.4).

Theorem 5.3. *If $\nu(\ell)$ is a convergent ldf such that $\int_1^\infty \frac{1-Z_\nu(L)}{L}\,\mathrm{d}L < \infty$, then in the ν-RSA process, there exists positive constant α_ν such that as $L \to \infty$,*

$$\mathbb{E}[S_L] \sim \alpha_\nu L.$$

Example 5.4. In the following cases, Theorem 5.3 applies and $\mathbb{E}[S_L]$ grows linearly with L:

(1) The ldf $\nu(\ell)$ represents a finite distribution, i.e. there exists some C for which $\ell > C$ implies $\nu(\ell) = 0$.
(2) When $\ell > 1$, the ldf $\nu(\ell) \propto \ell^p$ with $p < -1$.
(3) When $\ell > 1$, the ldf $\nu(\ell) \propto e^{-a\ell}$ for any $a > 0$.

Proof of Theorem 5.3: Let $\varphi := \mathcal{L}\{\mathbb{E}[S_L]\}$, viz.

$$\varphi(s) := \int_0^\infty \mathbb{E}[S_L]e^{-sL}\,\mathrm{d}L.$$

Define $V : \mathbb{R}^{>0} \to \mathbb{R}$ as the Laplace transform of $1 - Z_\nu(L)$, and define the function $V_\star : \mathbb{R}^{>0} \to \mathbb{R}$ as a modified version of V, viz.

$$V(s) := \int_0^\infty (1 - Z_\nu(L))e^{-sL}\,\mathrm{d}L, \qquad V_\star(s) := \int_1^\infty (1 - Z_\nu(L))e^{-sL}\,\mathrm{d}L. \tag{5.2}$$

Since $Z_\nu(L) = 0$ for $L < 1$, we have $V(s) = \frac{1-e^{-s}}{s} + V_\star(s)$. We will first use these functions to derive differential inequalities on $\varphi(s)$.

Lemma 5.4.1. *The following differential inequalities hold when $s > 0$:*

$$-\varphi'(s) > \frac{2}{s}(e^{-s} - sV_\star(s))\varphi(s), \qquad -\varphi'(s) < -\left(\frac{V(s)}{s}\right)' + \frac{2}{s}(e^{-s} - sV_\star(s))\varphi(s).$$

Proof. By proposition 5.2,

$$\left(\int_0^L Z_\nu(t)\,\mathrm{d}t\right)\mathbb{E}[S_L] = 2\int_0^L \mathbb{E}[S_t]Z_\nu(L-t)\,\mathrm{d}t. \tag{5.3}$$

Because $Z_\nu(t) \leq 1$, we have $\int_0^L Z_\nu(t)\,\mathrm{d}t < L$, and

$$L\,\mathbb{E}[S_L] > 2\int_0^L \mathbb{E}[S_t]Z_\nu(L-t)\,\mathrm{d}t = 2\int_0^L \mathbb{E}[S_t]\,\mathrm{d}t - 2\int_0^L \mathbb{E}[S_t](1-Z_\nu(L-t))\,\mathrm{d}t, \tag{5.4}$$

Taking Laplace transforms, we have that by the time-multiplication property (c.f. Proposition 2.7), $\mathcal{L}\{L\,\mathbb{E}[S_L]\} = -\varphi'(s)$. Moreover, the integration property implies $\mathcal{L}\left\{2\int_0^L \mathbb{E}[S_t]\,\mathrm{d}t\right\} = \frac{2\varphi(s)}{s}$, and the convolution property implies

$$\mathcal{L}\left\{-2\int_0^L \mathbb{E}[S_t](1-Z_\nu(L-t))\,\mathrm{d}t\right\} = -2\varphi(s)V(s).$$

Taking the Laplace transform of Equation 5.4 now yields $-\varphi'(s) > \frac{2\varphi(s)}{s}(1-sV(s))$, and substituting $V_\star(s)$ for $V(s)$ yields the first differential inequality.

To derive the second inequality, we reuse Equation 5.3. This time, we note $\int_0^L Z_\nu(t)\,\mathrm{d}t = L - \int_0^L (1-Z_\nu(t))\,\mathrm{d}t$, so

$$L\,\mathbb{E}[S_L] = \left(\int_0^L (1-Z_\nu(t))\,\mathrm{d}t\right)\mathbb{E}[S_L] + 2\int_0^L \mathbb{E}[S_t]Z_\nu(L-t)\,\mathrm{d}t.$$

By definition, $S_L \leq L$, so $L\,\mathbb{E}[S_L] \leq L\int_0^L (1-Z_\nu(t))\,\mathrm{d}t + 2\int_0^L \mathbb{E}[S_t]Z_\nu(L-t)\,\mathrm{d}t$, or equivalently,

$$L\,\mathbb{E}[S_L] \leq L\int_0^L (1-Z_\nu(t))\,\mathrm{d}t + 2\int_0^L \mathbb{E}[S_t]\,\mathrm{d}t - 2\int_0^L \mathbb{E}[S_t](1-Z_\nu(L-t))\,\mathrm{d}t. \tag{5.5}$$

Taking Laplace transforms, we note that

$$\mathcal{L}\left\{L\int_0^L (1-Z_\nu(t))\,\mathrm{d}t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left\{\int_0^L (1-Z_\nu(t))\,\mathrm{d}t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{V(s)}{s}\right).$$

Taking the Laplace transform of Equation 5.5 then yields

$$-\varphi'(s) \leq -\left(\frac{V(s)}{s}\right)' + \frac{2\varphi(s)}{s} - 2\varphi(s)V(s),$$

and substituting in $V_\star(s)$ as before gives the second differential inequality. ☐

We now find functions that use the differential inequalities to directly bound $\varphi(s)$. If we treat the first inequality in Lemma 5.4.1 as an equality, then the solution to the differential equation is the function $b : \mathbb{R}^{>0} \to \mathbb{R}$, given by

$$b(s) := \frac{r(s)}{s^2}, \qquad \text{where } r(s) := \exp\left(2\int_1^s \frac{1-e^{-t}}{t}\,\mathrm{d}t + 2\int_1^s V_\star(t)\,\mathrm{d}t\right). \tag{5.6}$$

We will use $b(s)$ as a lower bound of $\varphi(s)$.

Lemma 5.4.2. *Fix sufficiently small $\eta > 0$. Then, for $s \in (0, \eta)$,*

$$\varphi(s) \geq \frac{\varphi(\eta)}{b(\eta)} \cdot b(s).$$

Proof. Let $\varphi_\star(s) = \frac{\varphi(\eta)}{b(\eta)} \cdot b(s)$. We may check that $\varphi_\star(\eta) = \varphi(\eta)$ and that $\varphi_\star$ is a solution to the differential equation

$$-\varphi_\star'(s) = \frac{2}{s}(e^{-s} - sV_\star(s))\varphi_\star(s).$$

By Lemma 5.4.1, $\varphi(s)$ satisfies $-\varphi'(s) > \frac{2}{s}(e^{-s} - sV_\star(s))\varphi(s)$, and we quickly note $V_\star(s) \leq \int_1^\infty 1 \cdot e^{-sL}\, \mathrm{d}L = \frac{e^{-s}}{s}$, which implies $e^{-s} - sV_\star(s) > 0$. Therefore, Proposition 2.10 holds here and implies that when $s \in (0, \eta)$, $\varphi(s) \geq \varphi_\star(s)$. □

Define function $h_\eta(s)$ as

$$h_\eta(s) := \int_s^\eta -\left(\frac{V(t)}{t}\right)' \cdot \frac{b(s)}{b(t)}\, \mathrm{d}t. \tag{5.7}$$

The function $h_\eta(s)$ is the error when approximating $\varphi(s)$ with $b(s)$. By adding $h_\eta(s)$ to $b(s)$, we will derive an upper bound for $\varphi(s)$.

Lemma 5.4.3. *Fix sufficiently small $\eta > 0$. Then, for $s \in (0, \eta)$,*

$$\varphi(s) \leq \frac{\varphi(\eta)}{b(\eta)} \cdot b(s) + h_\eta(s).$$

Proof. Let $\varphi^\star(s) = \frac{\varphi(\eta)}{b(\eta)} \cdot b(s) + h_\eta(s)$. Because $h_\eta(\eta) = 0$, we have $\varphi^\star(\eta) = \varphi(\eta)$. Moreover, $\varphi^\star(s)$ satisfies the differential equation

$$-\varphi^{\star\prime}(s) = -\left(\frac{V(s)}{s}\right)' + \frac{2}{s}(e^{-s} - sV_\star(s))\varphi^\star(s).$$

Lemma 5.4.1 implies $-\varphi'(s) < -\left(\frac{V(s)}{s}\right)' + \frac{2}{s}(e^{-s} - sV_\star(s))\varphi(s)$. Thus, Lemma 2.10 again proves that $\varphi(s) < \varphi^\star(s)$ when $s \in (0, \eta)$, completing the lemma. □

We now prove useful results about the functions $r(s)$ and $h_\eta(s)$, which will allow us to bound $\varphi(s)$.

Lemma 5.4.4. *The function $r(s)$ is positive, continuous, and increasing when $s \in (0, \infty)$. It also satisfies*

$$\lim_{s \to 0^+} r(s) = r(0) > 0.$$

Proof. Recall

$$r(s) := \exp\left(2\int_1^s \frac{1 - e^{-t}}{t}\, \mathrm{d}t + 2\int_1^s V_\star(t)\, \mathrm{d}t\right).$$

Note $r(s)$ is continuous (increasing) because the integrals are continuous (increasing), and it is positive by inspection. Moreover, $\frac{1-e^{-t}}{t}$ is bounded on $(0, 1]$, so $\int_0^1 \frac{1-e^{-t}}{t}\, \mathrm{d}t$ exists.

Meanwhile, we have assumed that $\int_1^\infty \frac{1-Z_\nu(L)}{L}\, \mathrm{d}L < \infty$, and

$$\int_1^\infty \frac{1 - Z_\nu(L)}{L}\, \mathrm{d}L = \int_1^\infty \int_0^\infty (1 - Z_\nu(L))e^{-tL}\, \mathrm{d}t\, \mathrm{d}L = \int_0^\infty V_\star(t)\, \mathrm{d}t, \tag{5.8}$$

where the last integral interchange is allowable due to Fubini's Theorem. Since the limit $\lim_{s\to 0^+} \int_s^1 V_\star(t)\, \mathrm{d}t$ is strictly less than the above value, it is finite as well, which proves that $\lim_{s\to 0^+} r(s)$ exists and is equal to $r(0)$, which by definition of r must be positive. □

This lemma will allow us to bound the approximation error $h_\eta(s)$ by a function that converges to 0 as η becomes small.

Lemma 5.4.5. *There exist continuous functions $H_\eta : [0, \eta] \to \mathbb{R}$ for each $\eta > 0$ that satisfy $\lim_{\eta\to 0^+} H_\eta(0) = 0$ and $s^2 h_\eta(s) \le H_\eta(s)$ when $s \in (0, \eta)$.*

Proof. Fix $\eta > 0$. First, note $-\left(\frac{V(s)}{s}\right)'$ is positive, as $V(s)$ is a decreasing function (c.f. Equation 5.2). Then, for all $0 < s < \eta$,

$$h_\eta(s) = \frac{r(s)}{s^2}\int_s^\eta -\left(\frac{V(t)}{t}\right)' \cdot \frac{t^2}{r(t)}\,\mathrm{d}t \qquad \text{(Equations 5.6, 5.7)}$$

$$\le \frac{r(s)}{s^2}\cdot\int_s^\eta -\left(\frac{V(t)}{t}\right)' \cdot \frac{t^2}{r(0)}\,\mathrm{d}t, \qquad (r(s) \text{ is increasing})$$

$$h_\eta(s) \le \frac{1}{s^2}\cdot\frac{r(s)}{r(0)}\cdot\int_s^\eta -\left(\frac{V(t)}{t}\right)' \cdot t^2\,\mathrm{d}t. \tag{5.9}$$

We may use integration by parts on the integral, which yields

$$\int_s^\eta -\left(\frac{V(t)}{t}\right)' \cdot t^2\,\mathrm{d}t = sV(s) - \eta V(\eta) + 2\int_s^\eta V(t)\,\mathrm{d}t \le sV(s) + 2\int_s^\eta V(t)\,\mathrm{d}t.$$

Recall $V(t) = \frac{1-e^{-t}}{t} + V_\star(t) \le 1 + V_\star(t)$. Substitution yields

$$sV(s) + 2\int_s^\eta V(t)\,\mathrm{d}t \le sV(s) + 2(\eta - s) + 2\int_s^\eta V_\star(t)\,\mathrm{d}t.$$

Substituting this upper bound back into Equation 5.9 yields the estimation $h_\eta(s) \le \frac{H_\eta(s)}{s^2}$, with $H_\eta : (0, \eta] \to \mathbb{R}$ given by

$$H_\eta(s) := \frac{r(s)}{r(0)}\cdot\left(sV(s) + 2(\eta - s) + 2\int_s^\eta V_\star(t)\,\mathrm{d}t\right).$$

The continuity of $H_\eta(s)$ follows by the continuity of $r(s)$ and $V(s)$. Note $H_\eta(s)$ is not yet defined at 0 since V is not defined at 0. However, because as $L \to \infty$, we have $1 - Z_\nu(L) \to 0$. The Abelian Final Value Theorem (c.f. Proposition 2.7) then implies $\lim_{s\to 0^+} sV(s) = 0$. Moreover, we have previously shown $\int_0^\eta V_\star(t)\,\mathrm{d}t$ is finite (c.f. Equation 5.8). It follows that

$$\lim_{s\to 0^+} H_\eta(s) = 2\eta + 2\int_0^\eta V_\star(t)\,\mathrm{d}t.$$

Define $H_\eta(0)$ to be this value. By inspection of this formula, $\lim_{\eta\to 0^+} H_\eta(0) = 0$. □

Finally, we analyze the behavior of $\varphi(s)$ as $s \to 0$.

Lemma 5.4.6. *As $s \to 0^+$, there exists a positive constant α_ν such that*

$$\varphi(s) \sim \frac{\alpha_\nu}{s^2}.$$

Proof. Fix any $\epsilon > 0$. We will now find $l_\epsilon, \delta_\epsilon$ that satisfy the conditions of Proposition 2.11 for our choice of ϵ, i.e. that for all $0 < s < \delta_\epsilon$, we have $l_\epsilon < s^2\varphi(s) < l_\epsilon + \epsilon$.

We use the functions H_η as defined in Lemma 5.4.5. Choose sufficiently small η such that $H_\eta(0) < \frac{\epsilon}{3}$. Then, choose sufficiently small δ_1 such that for all $0 < s < \delta_1$, we have

$$H_\eta(s) < H_\eta(0) + \frac{\epsilon}{3} < \frac{2\epsilon}{3}. \tag{5.10}$$

Finally, because $r(s)$ is increasing, we may choose δ_2 such that for all $0 < s < \delta_2$,

$$r(0) < r(s) < r(0) + \frac{b(\eta)}{\varphi(\eta)} \cdot \frac{\epsilon}{3}. \tag{5.11}$$

Let $\delta = \inf\{\eta, \delta_1, \delta_2\}$. By Lemma 5.4.2, for $0 < s < \delta$, we have $\varphi(s) \geq \frac{\varphi(\eta)}{b(\eta)} \cdot b(s)$, so

$$s^2\varphi(s) \geq \frac{\varphi(\eta)}{b(\eta)} \cdot r(s) > \frac{\varphi(\eta)}{b(\eta)} \cdot r(0).$$

Meanwhile, by Lemma 5.4.3, for $0 < s < \delta$, we have $\varphi(s) \leq \frac{\varphi(\eta)}{b(\eta)} \cdot b(s) + h_\eta(s)$, so

$$s^2\varphi(s) \leq \frac{\varphi(\eta)}{b(\eta)} \cdot r(s) + s^2 h_\eta(s) < \frac{\varphi(\eta)}{b(\eta)} \cdot r(0) + \epsilon,$$

where the second inequality follows from Equation 5.11 and from $s^2 h_\eta(s) \leq H_\eta(s) < \frac{2\epsilon}{3}$ (c.f. Lemma 5.4.5 and Equation 5.10).

Define $\delta_\epsilon = \delta$ and $l_\epsilon = \frac{\varphi(\eta)}{b(\eta)} \cdot r(0)$. We have shown that for $0 < s < \delta_\epsilon$, we have $l_\epsilon < s^2\varphi(s) < l_\epsilon + \epsilon$, which satisfies the conditions of Proposition 2.11 and proves that $\lim_{s\to 0^+} s^2\varphi(s)$ exists. Denote this limit α_ν. Notably, all the lower bounds l_ϵ are positive, so α_ν must also be positive, concluding the lemma. ☐

By the Hardy-Littlewood Tauberian Theorem (c.f. Theorem 2.8), $\varphi(s) \sim \frac{\alpha_\nu}{s^2}$ implies that as $L \to \infty$,

$$\int_0^L \mathbb{E}[S_t]\,\mathrm{d}t \sim \frac{\alpha_\nu}{2} L^2.$$

From here, an analytical argument will complete the proof.

Pick arbitrary $\epsilon > 0$. We will first show there exists L_ℓ such that $L > L_\ell$ implies $\frac{\mathbb{E}[S_L]}{L} > \alpha_\nu(1-\epsilon)$, and we will later show there exists L_u such that $L > L_u$ implies $\frac{\mathbb{E}[S_L]}{L} < \alpha_\nu(1+\epsilon)$, which will prove that $\lim_{L\to\infty} \frac{\mathbb{E}[S_L]}{L} = \alpha_\nu$.

Begin by choosing L_1 for which $L > L_1$ implies $\int_0^L \mathbb{E}[S_t]\,\mathrm{d}t > (1-\epsilon)^{1/4} \cdot \frac{\alpha_\nu}{2} L^2$. Moreover, choose L_2 for which $L > L_2$ implies $Z_\nu(L) \geq (1-\epsilon)^{1/4}$. By Proposition 5.2,

$$\left(\int_0^L Z_\nu(t)\,\mathrm{d}t\right) \mathbb{E}[S_L] = 2\int_0^L \mathbb{E}[S_t] Z_\nu(L-t)\,\mathrm{d}t.$$

We always have $Z_\nu(L) < 1$. Therefore,

$$L\,\mathbb{E}[S_L] \geq \left(\int_0^L Z_\nu(t)\,\mathrm{d}t\right) \mathbb{E}[S_L] = 2\int_0^L \mathbb{E}[S_t] Z_\nu(L-t)\,\mathrm{d}t \geq 2(1-\epsilon)^{1/4} \int_0^{L-L_2} \mathbb{E}[S_t]\,\mathrm{d}t,$$

where the last equality follows by the definition of L_2. Then, when $L - L_2 > L_1$, we have

$$2(1-\epsilon)^{1/4} \int_0^{L-L_2} \mathbb{E}[S_t]\,\mathrm{d}t > (1-\epsilon)^{1/2} \alpha_\nu (L-L_2)^2.$$

Choose L_ℓ sufficiently large so that $L > L_\ell$ implies $(L - L_2) > (1-\epsilon)^{1/4} L$, and so that $L_\ell > L_1 + L_2$. Then, for $L > L_\ell$,

$$(1-\epsilon)^{1/2} \alpha_\nu (L-L_2)^2 > (1-\epsilon)\alpha_\nu L^2.$$

Thus, for $L > L_\ell$, we have $L\,\mathbb{E}[S_L] > (1-\epsilon)\alpha_\nu L^2$, or $\mathbb{E}[S_L] > (1-\epsilon)\alpha_\nu L$.

We now show there exists L_u such that $L > L_u$ implies $\frac{\mathbb{E}[S_L]}{L} < \alpha_\nu(1+\epsilon)$. Choose L_3 for which $L > L_3$ implies $\int_0^L \mathbb{E}[S_t]\,\mathrm{d}t < (1+\epsilon)^{1/3} \cdot \frac{\alpha_\nu}{2} L^2$, and choose L_4 for which $L > L_4$ implies $Z_\nu(L) > (1+\epsilon)^{-1/3}$. For $L > L_3 + L_4$, we have similarly to before that

$$\begin{aligned}\alpha_\nu(1+\epsilon)^{1/3}L^2 > 2\int_0^L \mathbb{E}[S_t]\,\mathrm{d}t \geq 2\int_0^L \mathbb{E}[S_t]Z_\nu(L-t)\,\mathrm{d}t = \left(\int_0^L Z_\nu(t)\,\mathrm{d}t\right)\mathbb{E}[S_L] \\ \geq \left(\int_{L_4}^L Z_\nu(t)\,\mathrm{d}t\right)\mathbb{E}[S_L] > \frac{L-L_4}{(1+\epsilon)^{1/3}}\,\mathbb{E}[S_L].\end{aligned}$$

Choose L_u sufficiently large so that $L > L_u$ implies $(L - L_4) > \frac{L}{(1+\epsilon)^{1/3}}$, and so that $L_u > L_3 + L_4$. Then, $\frac{L-L_4}{(1+\epsilon)^{1/3}}\,\mathbb{E}[S_L] > \frac{L\,\mathbb{E}[S_L]}{(1+\epsilon)^{2/3}}$. Thus, for $L > L_u$, we have $\alpha_\nu(1+\epsilon)^{1/3}L^2 > \frac{L\,\mathbb{E}[S_L]}{(1+\epsilon)^{2/3}}$, or $\mathbb{E}[S_L] < (1+\epsilon)\alpha_\nu L$. This completes the proof of the theorem. □

5.2. **Divergent Length Distributions.** In this section we consider the ν-RSA process for a divergent ldf $\nu(\ell)$ (c.f. Definition 2.1). For any divergent ldf ν, consider its normalizing constant Z_ν. We must have

$$\frac{\int_0^L tZ_\nu(t)\,\mathrm{d}t}{\int_0^L Z_\nu(t)\,\mathrm{d}t} \geq \frac{L}{2},$$

because if we treat Z_ν as an non-normalized probability distribution on $[0, L]$, and we sample random variable X from the distribution, the left hand side is simply $\mathbb{E}[X]$. But Z_ν is a strictly increasing function, and so $\mathbb{E}[X]$ is skewed to above $\frac{L}{2}$. The difference between the left hand side and the right hand side is intuitively a measure of how quickly Z_ν grows, as when Z_ν grows very quickly, $\mathbb{E}[X]$ is skewed higher. Our next theorem relies on a condition related to this fact:

Theorem 5.5. *Let ν be a divergent ldf. If there exists $\epsilon > 0$ such that for sufficiently large L, we have $\frac{\int_0^L tZ_\nu(t)\,\mathrm{d}t}{\int_0^L Z_\nu(t)\,\mathrm{d}t} \geq (1+\epsilon)\cdot\frac{L}{2}$, then in the ν-RSA process,*

$$\mathbb{E}[S_L] = o(L).$$

We prove the theorem by induction. The inductive step is given by the following lemma:

Lemma 5.5.1. *If there exist a lower bound b and a factor $r > 0$ for which $L > b$ implies $\mathbb{E}[S_L] < rL$, then there exists a lower bound B for which $L > B$ implies $\mathbb{E}[S_L] < r\left(1 - \frac{\epsilon}{2}\right)L$.*

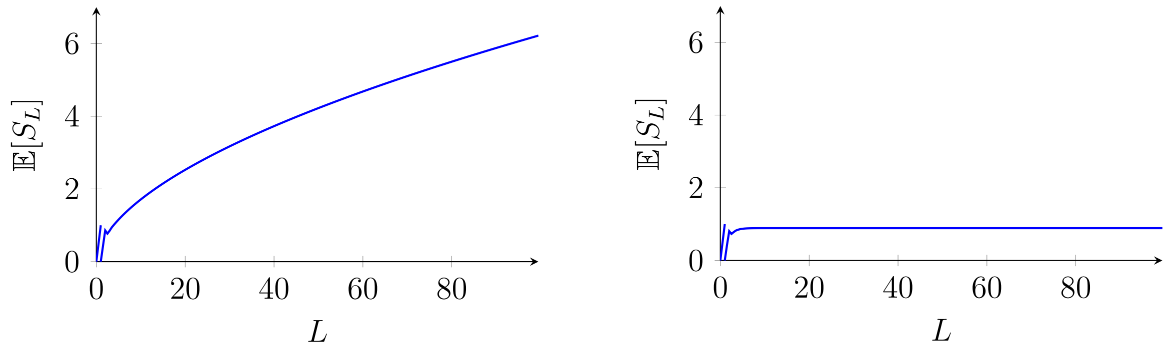
Proof. When $L > b$, we have

$$\text{(By Prop. 5.2)}\quad \left(\int_0^L Z_\nu(t)\,\mathrm{d}t\right)\mathbb{E}[S_L] = 2\int_0^b \mathbb{E}[S_t]Z_\nu(L-t)\,\mathrm{d}t + 2\int_b^L \mathbb{E}[S_t]Z_\nu(L-t)\,\mathrm{d}t$$

$$< 2r\int_0^L tZ_\nu(L-t)\,\mathrm{d}t + 2(1-r)\int_0^b tZ_\nu(L-t)\,\mathrm{d}t. \tag{5.12}$$

Note that $\int_0^L tZ_\nu(L-t)\,\mathrm{d}t = L\int_0^L Z_\nu(t)\,\mathrm{d}t - \int_0^L tZ_\nu(t)\,\mathrm{d}t$. Let $L^\star$ be sufficiently large so that $L > L^\star$ implies $\frac{\int_0^L tZ_\nu(t)\,\mathrm{d}t}{\int_0^L Z_\nu(t)\,\mathrm{d}t} \geq (1+\epsilon)\cdot\frac{L}{2}$, as described by the theorem statement. Then, when $L > L^\star$, we may rearrange Equation 5.12 into

$$\mathbb{E}[S_L] < 2rL - r(1+\epsilon)L + 2(1-r)\cdot\frac{\int_0^b tZ_\nu(L-t)\,\mathrm{d}t}{\int_0^L Z_\nu(t)\,\mathrm{d}t} < r(1-\epsilon)L + 2(1-r)b.$$



(A) $\nu(\ell) = 1$. (B) $\nu(\ell) = e^{\ell}$.

FIGURE 6. $\mathbb{E}[S_L]$ grows sublinearly under various divergent ldfs.

It then follows that $\mathbb{E}[S_L] < r\left(1 - \frac{\epsilon}{2}\right) L$ for sufficiently large L. □

Proof of Theorem 5.5: Let $b_0 = 1$. For $L > b_0$, we must place at least one segment of length at least 1 in the ν-RSA process, so $S_L \leq L - 1$. In particular, $\mathbb{E}[S_L] < L$. Then, by induction, for all integers $n > 0$, we may construct b_n such that $L > b_n$ implies $\mathbb{E}[S_L] < \left(1 - \frac{\epsilon}{2}\right)^n L$. The theorem follows. □

Example 5.6. In the following cases, Theorem 5.5 applies and $\mathbb{E}[S_L]$ grows sublinearly with L:

(1) When $\ell > 1$, the ldf $\nu(\ell) = \ell^p$ with $p > -1$. (We further investigate this case in Section 6.)
(2) When $\ell > 1$, the ldf $\nu(\ell) = e^{a\ell}$ for any $a > 0$.

6. POWER FUNCTION DISTRIBUTION

In this section, we consider the case when the ldf is given by a power-law function, viz. $\nu(\ell) = (\ell - 1)^{\beta - 1}$ with $\beta > 0$. The convolution of two power functions is again a power function, which allows us to derive precise bounds on $\mathbb{E}[S_L]$. We begin with this property:

Proposition 6.1. *If $\beta, \theta \in \mathbb{R}^{>0}$, then*

$$\int_0^x t^{\beta}(x - t)^{\theta} \, \mathrm{d}t = \frac{\Gamma(\beta + 1)\Gamma(\theta + 1)}{\Gamma(\beta + \theta + 2)} x^{\beta + \theta + 1}.$$

This can be directly proven from the fact that $\mathcal{L}\{x^{\beta}\}(s) = \frac{\Gamma(\beta+1)}{s^{\beta+1}}$.

Corollary 6.2. *Let $\Theta : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ be a function of $\beta > 0$, defined as the positive solution in θ to*

$$\Gamma(\beta + \theta + 2) = 2\Gamma(\beta + 2)\Gamma(\theta + 1).$$

Then, the function Θ is well-defined and at most 1.

Proof. By Proposition 6.1, $\int_0^1 t^{\beta}(1-t)^{\theta} \, \mathrm{d}t = \frac{\Gamma(\beta+1)\Gamma(\theta+1)}{\Gamma(\beta+\theta+2)}$. The left hand side is strictly decreasing in θ, so $\frac{\Gamma(\beta+1)\Gamma(\theta+1)}{\Gamma(\beta+\theta+2)}$ must also be decreasing in θ. When $\theta = 0$, this expression is equal to $\frac{1}{\beta+1} > \frac{1}{2(\beta+1)}$. Meanwhile, when $\theta = 1$, the expression is equal to $\frac{1}{(\beta+1)(\beta+2)} < \frac{1}{2(\beta+1)}$. By the continuity of Γ, there must exist θ with $0 < \theta < 1$ for which $\frac{\Gamma(\beta+1)\Gamma(\theta+1)}{\Gamma(\beta+\theta+2)} = \frac{1}{2(\beta+1)}$. This equation is equivalent to the condition in the corollary statement. Moreover, because $\frac{\Gamma(\beta+1)\Gamma(\theta+1)}{\Gamma(\beta+\theta+2)}$ is strictly decreasing in θ, our value of θ is unique, proving that Θ is well-defined. □

Given that $\nu(\ell) = (\ell-1)^{\beta-1}$, we will prove $\mathbb{E}[S_L]$ is bounded below by power functions of the form $L^{\Theta(\beta)-\epsilon}$, and above by $L^{\Theta(\beta)}$.

Theorem 6.3. *Fix $\beta > 0$ and consider ldf $\nu(\ell) = (\ell-1)^{\beta-1}$. Then, in the ν-RSA process, for all $\epsilon > 0$,*

$$L^{\Theta(\beta)-\epsilon} \ll \mathbb{E}[S_L] \leq L^{\Theta(\beta)}.$$

Example 6.4. Consider the case when $\beta = 1$, with ldf $\nu(\ell) = 1$. Then, $\Gamma(\beta+\theta+2) = 2\Gamma(\beta+2)\Gamma(\theta+1)$ is simply the polynomial equation

$$(\theta+1)(\theta+2) = 4,$$

which has unique positive solution $\theta = \Theta(1) = \frac{\sqrt{17}-3}{2} \approx 0.562$. Theorem 6.3 now implies that for all $\epsilon > 0$,

$$L^{(\sqrt{17}-3)/2-\epsilon} \ll \mathbb{E}[S_L] \leq L^{(\sqrt{17}-3)/2}.$$

In general, when β is an integer, the equation $\Gamma(\beta+\theta+2) = 2\Gamma(\beta+2)\Gamma(\theta+1)$ reduces to the degree-$(\beta+1)$ polynomial equation $\prod_{i=1}^{\beta+1}(\theta+i) = 2(\beta+1)!$.

Remark 6.5. Similar processes to the ν-RSA process with the uniform length distribution $\nu(\ell) = 1$ have been studied before. In particular, Coffman et. al. in [17] analyze a process in which they park segments with lengths drawn uniformly at random from $[0, L]$. Instead of studying the empty space left at saturation (arbitrarily small segments may be parked in their model, so it never in fact reaches saturation), as we do here, they study the expected number of parked segments after n attempts to park segments. Using methods different from ours, they derive that the number of parked segments grows as $n^{(\sqrt{17}-3)/2}$. Notably, this exponent is exactly the exponent that we derive for the empty space at saturation in Example 6.4, providing an interesting connection between two distinct processes.

Proof of Theorem 6.3. For brevity in this proof, we take $\nu(\ell) = \frac{1}{\beta}(\ell-1)^{\beta-1}$ so that $Z_\nu(L) = (L-1)^\beta$. Define $f_\theta(L) := \frac{\mathbb{E}[S_L]}{L^\theta}$. We will prove $f_{\Theta(\beta)}$ is bounded above by 1, and that $f_\theta(L)$ is bounded below by a positive constant for all $\theta < \Theta(\beta)$. First, define the following function $\mu : [0,1] \to \mathbb{R}$, viz.

$$\mu(t) := \frac{\Gamma(\theta+\beta+2)}{\Gamma(\theta+1)\Gamma(\beta+1)} \cdot t^\theta(1-t)^\beta.$$

By Proposition 6.1, we note $\int_0^1 \mu(t)\,\mathrm{d}t = 1$. We now use μ to derive an integral recurrence relation on f_θ.

Lemma 6.5.1. *For $\theta, L > 0$,*

$$f_\theta(L+1) = \left(\frac{2\Gamma(\theta+1)\Gamma(\beta+2)}{\Gamma(\theta+\beta+2)}\right) \cdot \left(\frac{L}{L+1}\right)^{\theta+1} \cdot \int_0^1 f_\theta(Lt)\mu(t)\,\mathrm{d}t.$$

Proof. By Proposition 5.2 with $L+1$ as the length of the interval,

$$\left(\int_0^{L+1} Z_\nu(t)\,\mathrm{d}t\right)\mathbb{E}[S_{L+1}] = 2\int_0^{L+1} \mathbb{E}[S_t]Z_\nu(L+1-t)\,\mathrm{d}t.$$

Substituting in $Z_\nu(L) = (L-1)^\beta$ and $\mathbb{E}[S_L] = L^\theta f_\theta(L)$ yields

$$f_\theta(L+1) = \frac{2(\beta+1)L^{\theta+1}}{(L+1)^{\theta+1}} \cdot \int_0^L \frac{t^\theta (L-t)^\beta}{L^{\theta+\beta+1}} f_\theta(t)\,\mathrm{d}t.$$

Substituting Lt for t and rearranging then proves the lemma. □

We may now prove the first part of the Theorem. By Lemma 6.5.1 with $\theta = \Theta(\beta)$ (and noting $\frac{2\Gamma(\Theta(\beta)+1)\Gamma(\beta+2)}{\Gamma(\Theta(\beta)+\beta+2)} = 1$ by the definition of Θ), we have

$$f_{\Theta(\beta)}(L+1) = \left(\frac{L}{L+1}\right)^{\Theta(\beta)+1} \int_0^1 f_{\Theta(\beta)}(Lt)\mu(t)\,\mathrm{d}t < \sup_{0\le t\le L} f_{\Theta(\beta)}(t). \tag{6.1}$$

But $\mathbb{E}[S_L] = L$ for $L < 1$. Because $\Theta(\beta) < 1$, we have $f_{\Theta(\beta)}(L) = L^{1-\Theta(\beta)} < 1$ when $L < 1$. Equation 6.1 then implies that $f_{\Theta(\beta)}(L) < 1$ for all L, and thus that $\mathbb{E}[S_L] \le L^{\Theta(\beta)}$.

Now, fix any $\theta < \Theta(\beta)$. Recall $\frac{2\Gamma(\theta+1)\Gamma(\beta+2)}{\Gamma(\beta+\theta+2)}$ is decreasing in θ (c.f. Proposition 6.1), so $\frac{2\Gamma(\theta+1)\Gamma(\beta+2)}{\Gamma(\beta+\theta+2)} > 1$. Then, there exists some $\epsilon > 0$ and some lower bound $L_1 > 0$ such that $L > L_1$ implies

$$\left(\frac{2\Gamma(\theta+1)\Gamma(\beta+2)}{\Gamma(\theta+\beta+2)}\right) \cdot \left(\frac{L}{L+1}\right)^{\theta+1} > 1+\epsilon,$$

and thus that $f_\theta(L+1) > (1+\epsilon)\int_0^1 f_\theta(Lt)\mu(t)\,\mathrm{d}t$.

Choose sufficiently large L_2 for which $\int_{2/L}^1 \mu(t)\,\mathrm{d}t > \frac{1}{1+\epsilon}$ for all $L > L_2$. Let $L^\star = \sup\{L_1, L_2\}$, so that for $L > L^\star$,

$$f_\theta(L+1) > (1+\epsilon)\int_{2/L}^1 f_\theta(Lt)\mu(t)\,\mathrm{d}t \ge \inf_{2\le t\le L} f_\theta(t). \tag{6.2}$$

Note that Lemma 6.5.1 implies $f_\theta(L) > 0$ for $L > 1$. Then, by Equation 6.2, for all $L > L^\star$, we have $f_\theta(L) > \sup_{2\le t\le L^\star} f_\theta(t)$, where $\sup_{2\le t\le L^\star} f_\theta(t)$ is positive.

Thus, for every $\theta < \Theta(\beta)$, we have shown there is a constant $c > 0$ such that $f_\theta(L) > c$ for sufficiently large L, or equivalently that $\mathbb{E}[S_L] > cL^\theta$. This in fact implies $\mathbb{E}[S_L] \gg L^\theta$ for all $\theta < \Theta(\beta)$, completing the proof of the theorem. □

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