Oscillating Near Circles at Intermediate Reynolds Numbers

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Mentored by Dr. Nick Derr

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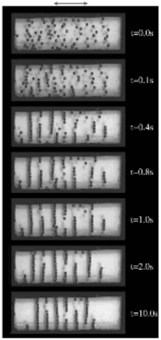
Outline

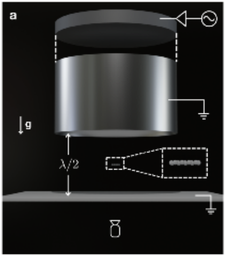
Numerics and Fluid Dynamics

1. Numerical Analysis is the application of computers to numerically create approximate solutions to complex problems.
2. Fluid Dynamics is the branch of physics that models fluid motion using differential equations in multiple variables.

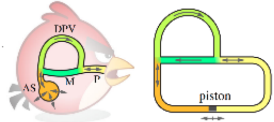
Steady Action Driven by Oscillation



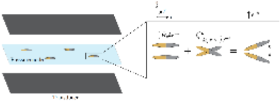
Klotsa et al., PRE 2009



Wu et al., arXiv (2211.02750)

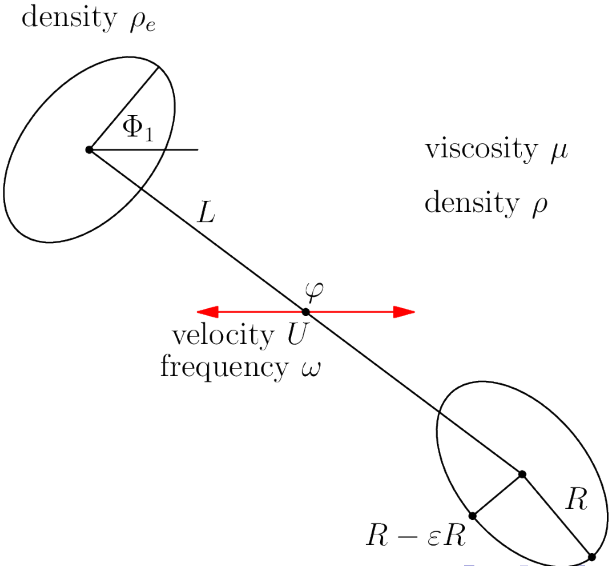


Nguyen et al., PRL 2021

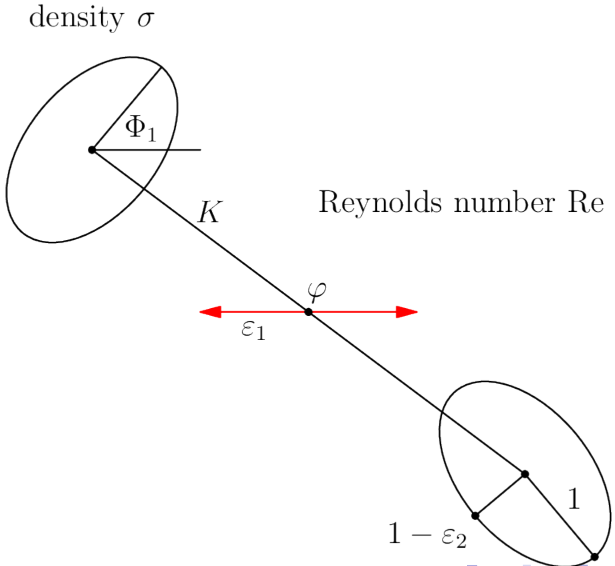


Collis et al., JFM 2017

Problem Geometry



Non-Dimensionalized Problem Geometry



Perturbation Method

If one parameter in a problem is known to be small, we can write the solution as a Taylor expansion in that parameter, and drop high-degree terms.

Example (Hinch)

Solve $x^2 - \varepsilon x - 1 = 0$ for the positive solution.
Expanding $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ yields

$$x_0^2 - 1 + \varepsilon(2x_0x_1 - x_0) + \varepsilon^2(2x_0x_2 + x_1^2 - x_1) + \dots = 0.$$

Setting each coefficient to 0, and solve the resulting equations.
Then $x = 1 + \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \dots$, agreeing with $\frac{1}{2}\varepsilon + \sqrt{1 + \frac{1}{4}\varepsilon}$.

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Finite Element Method

Example (Hughes)

Find the solution u to $u'' = -f$ on $[0, 1]$ where $u(0) = u(1) = 0$.

In this problem, $u'' + f = 0$ is the field equation and $u(0) = u(1) = 0$ is the boundary condition.

Equivalently, $\int u''w + fw\,dx = 0$ for all w. We call w the test function.

If $w(0) = w(1) = 0$, integrating by parts yields the weak form

$$\int fw\,dx = \int u'w'\,dx.$$

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Finite Element Method 2

Restrict w and u to piecewise linear functions.

Using ϕ_i to denote basis functions, $u = \sum_i u_i \phi_i$ and $w = \sum_i w_i \phi_i$.

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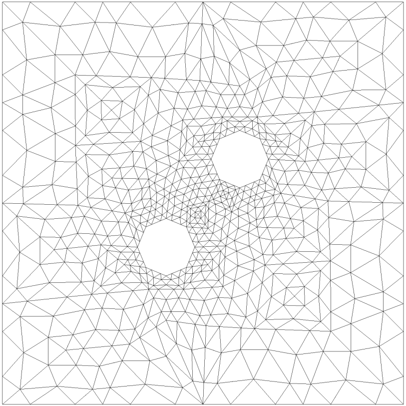

Using ϕ_i to denote basis functions, $u = \sum_i u_i \phi_i$ and $w = \sum_i w_i \phi_i$.

Finite Element Method 3

Then the coefficients u_i form a vector.

Recall

$$\int fw\,dx = \int u'w'\,dx.$$

If u satisfies this equation for w equalling each basis function ϕ_i then u is a solution.

$$\forall i, \int f\phi_i\,dx = \sum_j u_j \left(\int \phi_j'\phi_i'\,dx \right).$$

This now becomes a linear matrix problem.

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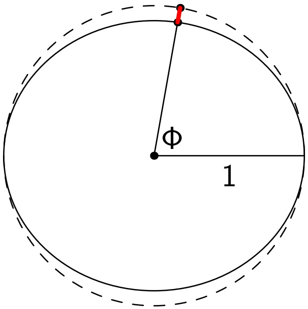
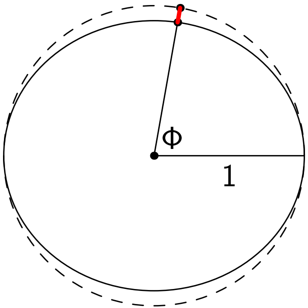
Boundary Perturbation



The length of the red line is $\varepsilon_2 \sin^2 \Phi$, to first order in ε_2.

Boundary Perturbation



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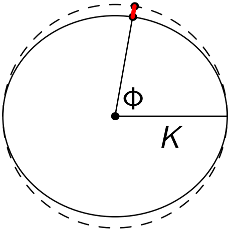
Boundary Perturbation 2

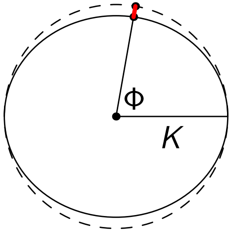


There is no velocity on the ellipse boundary; Taylor expand this condition from the circle.

On the circle boundary,

$$\vec{u} - \varepsilon_2 \sin^2(\Phi)(\vec{r} \cdot \nabla)\vec{u} = 0.$$

Boundary Perturbation 2



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Expansion Motivation

We will use the expansion

$$\vec{u} = \varepsilon_1 e^{it}(\hat{u}_0 + \varepsilon_2 \hat{u}_1) + \varepsilon_1^2(\bar{u}_0 + \varepsilon_2 \bar{u}_1).$$

1. Solving for time-independent velocity fields $\hat{u}_0, \hat{u}_1, \bar{u}_0, \bar{u}_1$.
2. At order ε_1 the velocity is driven by oscillation, necessitating the e^{it} term.
3. At order ε_1^2 the velocity self-interferes, creating a steady flow.
4. We have additional expansions in ε_2 at both orders to examine the contribution from eccentricity.

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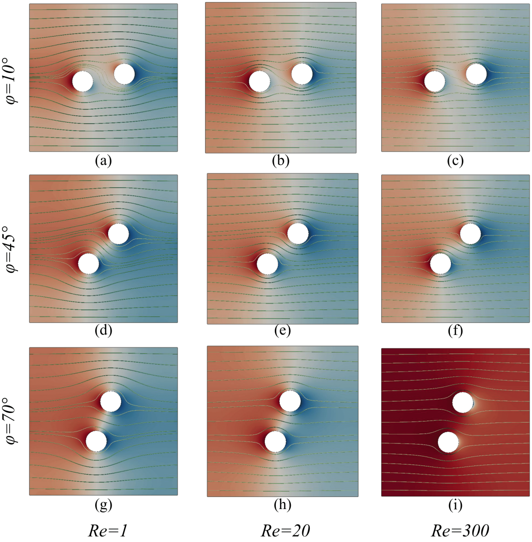
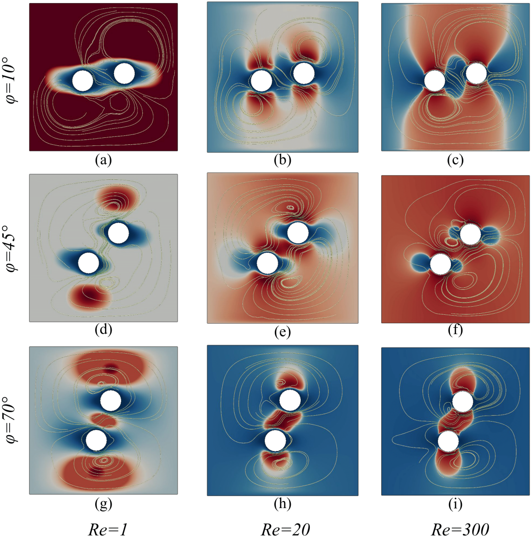
Leading Oscillatory Flow Plots



Leading Steady Flow Plots



Conclusion

In this talk, we

1. Discussed concepts in numerical fluid dynamics such as the perturbation method and the finite element method
2. Introduced geometry and equations for our research problem
3. Presented pressure and velocity streamline plots

Many thanks to

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References I

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