Algorithmically Generated Pants Decompositions of Combinatorial Surfaces

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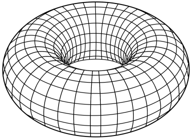
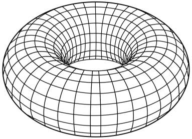


Cutting up Surfaces

Overarching Question: How can you cut up a surface?

- What do we mean by surface?
  - *Riemannian 2-Manifolds.*
  - 2-Manifold: A surface that looks "2-dimensional" around each point.
  - Riemannian: The surface is smooth and has a geometry: we can define length, angles, and area.



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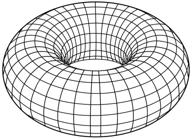

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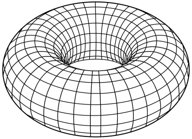
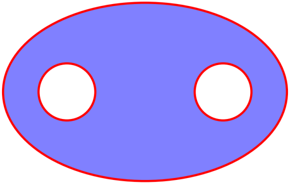
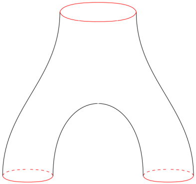


- Just like polygons can be cut up into triangles, Riemannian 2-Manifolds can be cut up into 3-holed spheres (called pairs of pants).

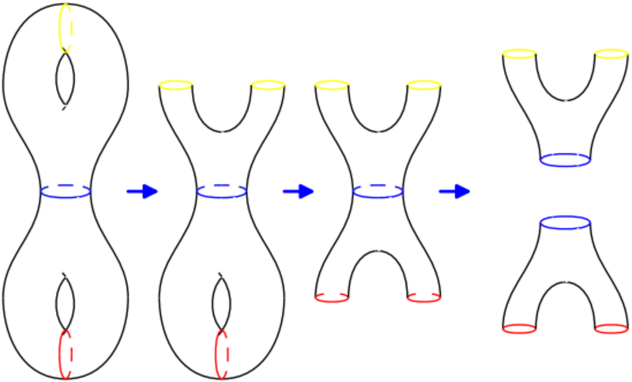




Pants Decompositions

Definition

A *pants decomposition* of a topological surface is a set of disjoint, closed, and non-contractible curves that decompose the surface into three-holed spheres.



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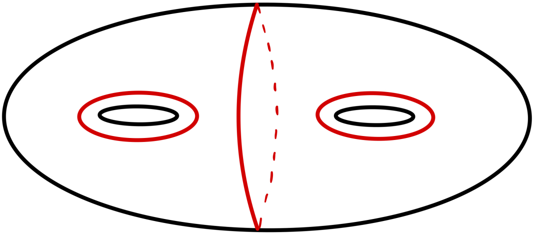
Fact

Any $3g - 3$ curves on a genus g surface that are disjoint, closed, non-contractible, and not homotopic give a pants decomposition.

Length of Decompositions

Definition

The *length* of a pants decomposition is the maximum length of a curve in the pants decomposition.



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- Describes how difficult it is to cut a surface S into simpler surfaces.
- Understanding $\mathfrak{B}_S$ is one the largest open problems in the geometry of surfaces.

Prior Results

Theorem (Buser, 1981)

A genus $g \geq 2$ hyperbolic surface S with no boundary components satisfies: $g^{1/2} \lesssim \mathfrak{B}_S \lesssim g \log(g)$.

$a(S) \lesssim b(S) \implies$ there exists universal constant C such that $a(S) \leq b(S)C$.

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Theorem (Buser, 1981)

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Theorem (Buser, 1992)

A genus $g \geq 2$ closed Riemann surface S with no boundary components satisfies: $\mathfrak{B}_S \lesssim (g\text{Area}(S))^{1/2}$.

- Uses theoretical algorithm.
- Unknown optimal behavior.

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Motivating Questions

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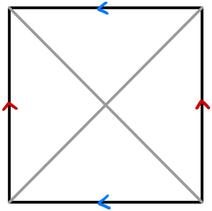
Question #3

What's the length of the nth cut in the decomposition?

Combinatorial Surfaces

How do we make a “discrete” surfaces?



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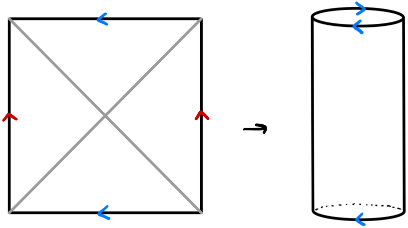

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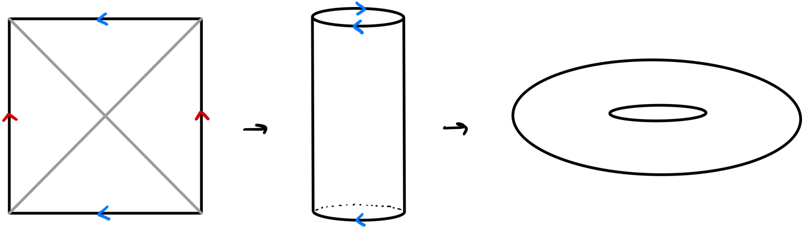

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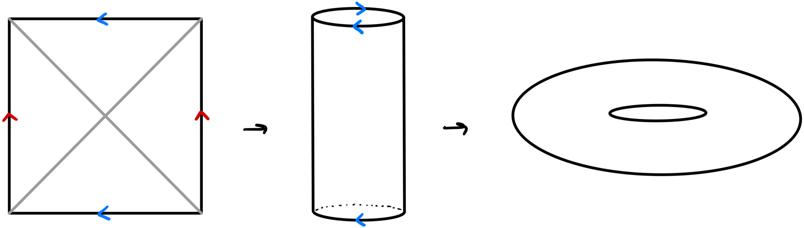

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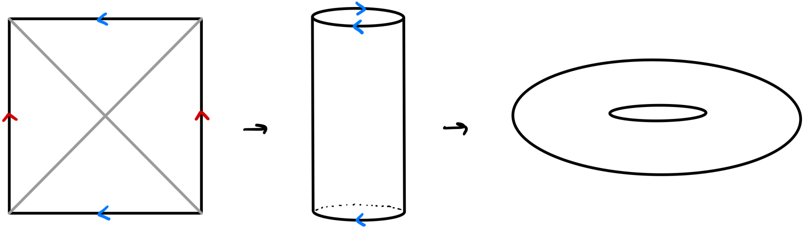

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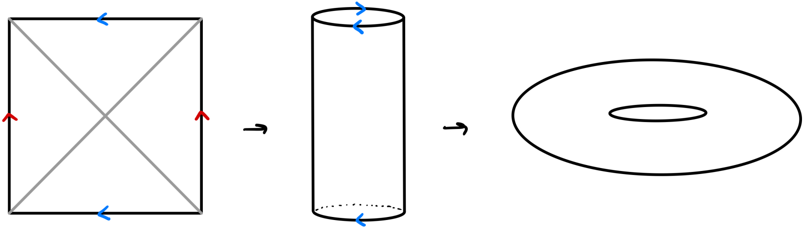

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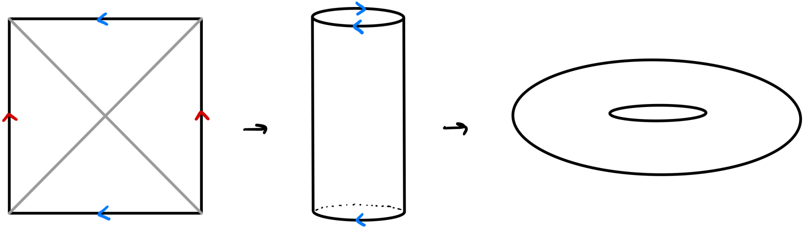
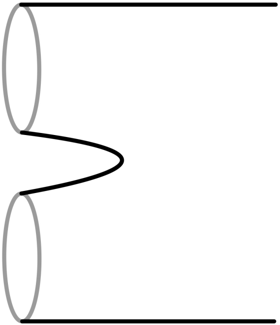


- Glue together n triangles with side length one into an n-gon.
- Identify edges of the polygon.
- A combinatorial surface is a type of Riemannian 2-manifold that is amenable to computation.
- Gives rise to random surfaces.

Finding Short Curves: Algorithm #1

Two main ideas:

(1) Add together homotopically distinct curves.



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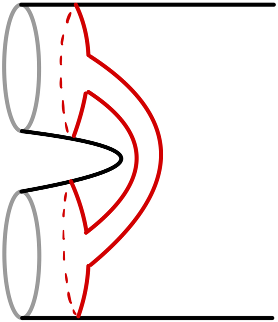

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Theorem (H. 2023)

Let S be a genus g combinatorial surface. Algorithm #1 finds a length $\lesssim (g\text{Area}(S))^{1/2}$ pants decomposition of S in $\mathcal{O}(g^3)$ time.

Question #2

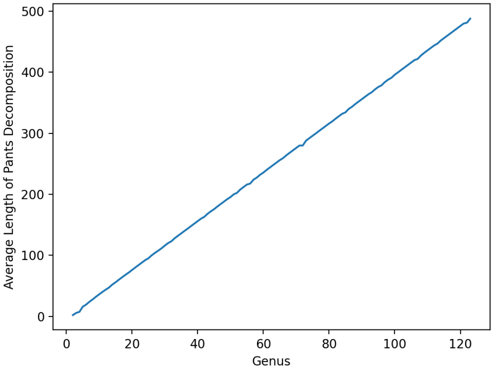
Does Buser's algorithm give shorter pants decompositions for "average" surfaces?

Results of Algorithm #1

Question #2

Does Buser's algorithm give shorter pants decompositions for "average" surfaces?

No!



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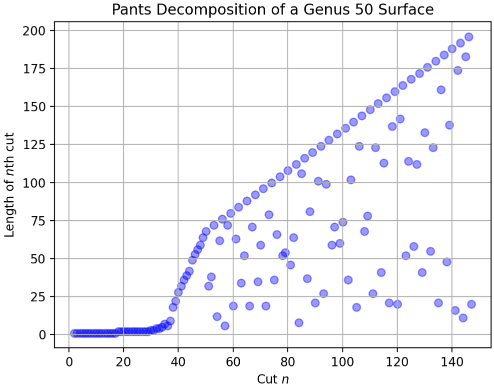
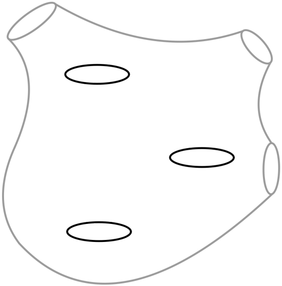
After a certain point, every third cut has length $\frac{4}{3}n$.



Finding Short Curves: Algorithm #2

New idea:

- Grow a ball around a random point until we find a loop that is in a different homotopy class than previous loops.



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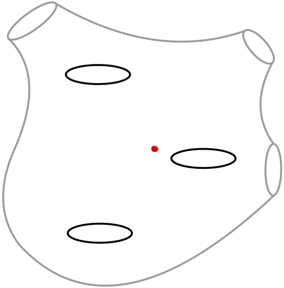

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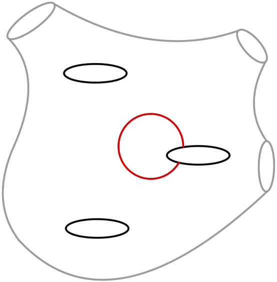

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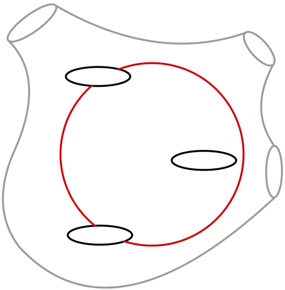

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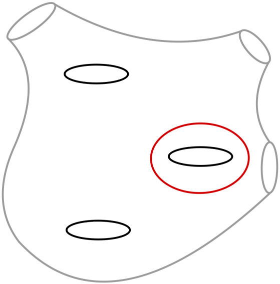

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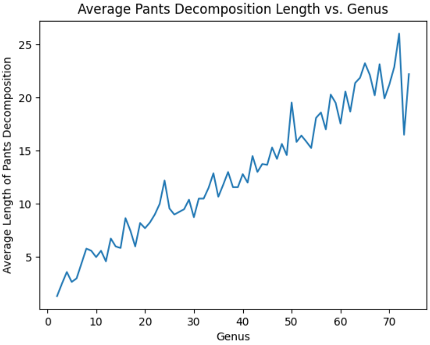

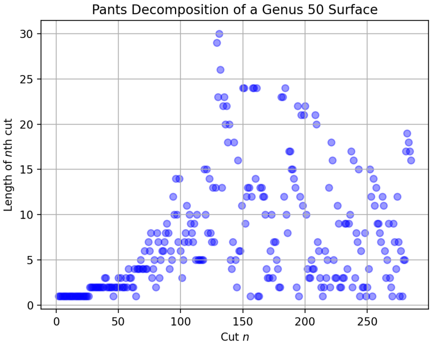

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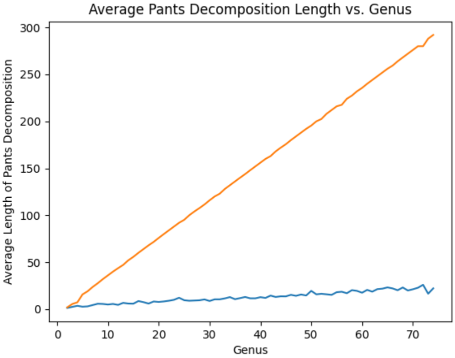

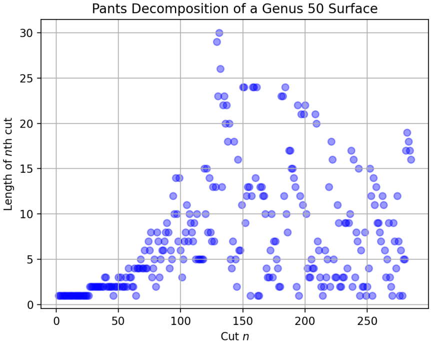

Acknowledgments

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- Dr. Tanya Khovanova, Dr. Slava Gerovitch, Prof. Pavel Etingof, and the MIT PRIMES-USA Program for making this research possible.
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