Hyperoctahedral Schur Algebra and the Hyperoctahedral Web Algebra

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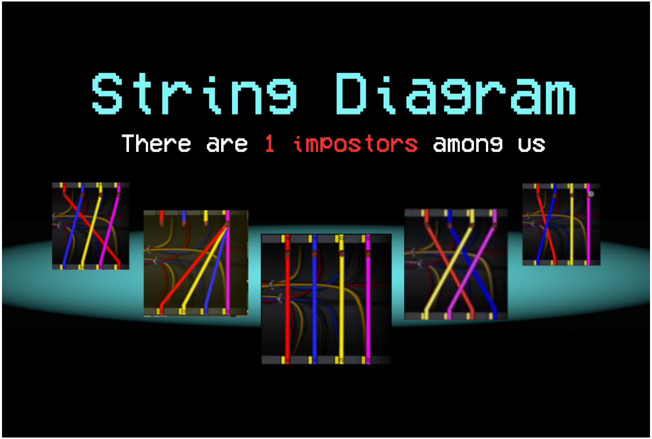


Question

What does the hit video game *Among Us* have to do with my project?

Symmetric Group

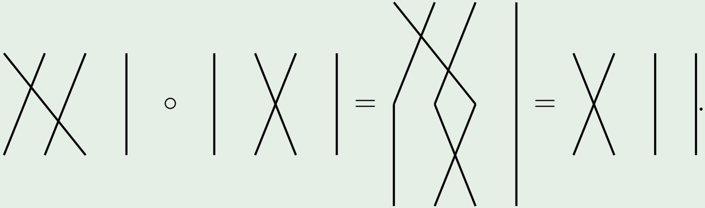
Definition

The symmetric group of degree n, denoted as S_n, is the set of bijective functions on $\{1, \ldots, n\}$ with function composition as the group operation.

Example (Read Bottom-To-Top)

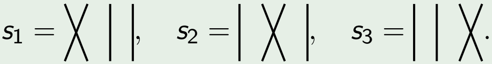
$$\begin{pmatrix} 2 & 3 & 1 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 3 & 2 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Example (Read Bottom-To-Top)

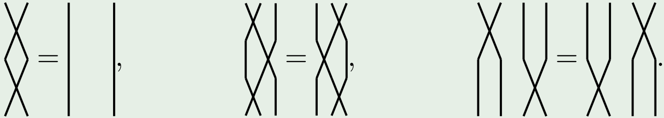


Topological intuition for the Symmetric Group

Example (Generators for S_4)



Example (Relations in S_2, S_3, and S_4)



Coxeter Presentation of Symmetric Group

Theorem (Coxeter)

$$S_n \cong \left\langle \left\langle s_1, s_2, \ldots, s_{n-1} \,\middle|\, \begin{array}{c} s_i^2=1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i, |i-j|>1 \end{array} \right\rangle \right\rangle.$$

Key Idea

Coxeter used visual observations to deduce structure.

Remark

It's non-trivial to see that these relations are sufficient.

Subgroups of S_n

Definition

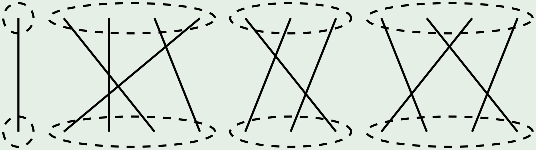
We say $\lambda = (\lambda_1, \ldots, \lambda_d)$ is a composition of S_n if $\sum \lambda_i = n$.

Definition

For composition λ, define $S_\lambda \subset S_n$ by

$$S_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_d}.$$

Example (of $S_\lambda \subset S_n$)

 $\in S_{(1,4,3,4)}$.

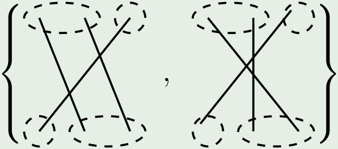
Double Cosets of S_n

Definition

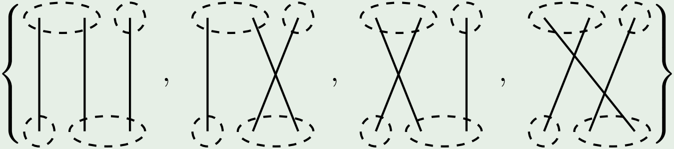
For subgroups H, K of G, $g \in G$, define a double coset

$$HgK := \{hgk | h \in H, k \in K\} \in H \backslash G / K.$$

Example (of $S_\mu \backslash S_n / S_\lambda$)



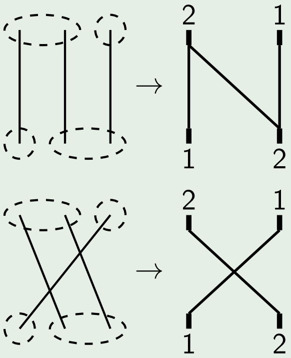
$\in S_{(2,1)} \backslash S_3 / S_{(1,2)}.$



$\in S_{(2,1)} \backslash S_3 / S_{(1,2)}.$

Thick String Diagrams

Example



Homomorphisms and Permutation Modules

Definition

$\mathbb{F}[S_n/S_\lambda]$ has basis $\{e_{gS_\lambda} | gS_\lambda \in S_n/S_\lambda\}$ and group action $g \cdot e_x = e_{g \cdot x}$.

Definition

$\mathrm{Hom}_{S_n}(\mathbb{F}[S_n/S_\lambda], \mathbb{F}[S_n/S_\mu])$ is the set of linear maps that commute with the S_n-action, namely

$$h(g \cdot x) = g \cdot h(x).$$

Remark

Homomorphisms, with respect to a fixed basis, are matrices.

Definition

The Schur Algebra of degree n is $\mathrm{End}_{S_n}(\bigoplus_\lambda \mathbb{F}[S_n/S_\lambda])$.

Thick String Diagrams and Homomorphisms

Remark

A general result tells us there is an isomorphism of vector spaces,

$$\mathbb{F}[K\backslash G/H] \cong \mathrm{Hom}_G(\mathbb{F}[G/H], \mathbb{F}[G/K]).$$

Proposition

We have a bijection:

$$\text{Diagrams from } \lambda \text{ to } \mu \longleftrightarrow \text{Basis of } \mathrm{Hom}_{S_n}(\mathbb{F}[S_n/S_\lambda], \mathbb{F}[S_n/S_\mu]).$$

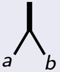
Goal

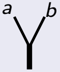
Make string diagrams work like homomorphisms.

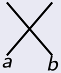
Polynomial Web Algebra

Definition

The Polynomial Web Algebra is generated by thick string diagrams,



:(a,b) $\to (a + b)$, 

$: (a + b) \to (a, b)$, 

$: (a, b) \to (b, a)$.

Question

What relations do the Polynomial Web Algebra need to be like the Schur Algebra?

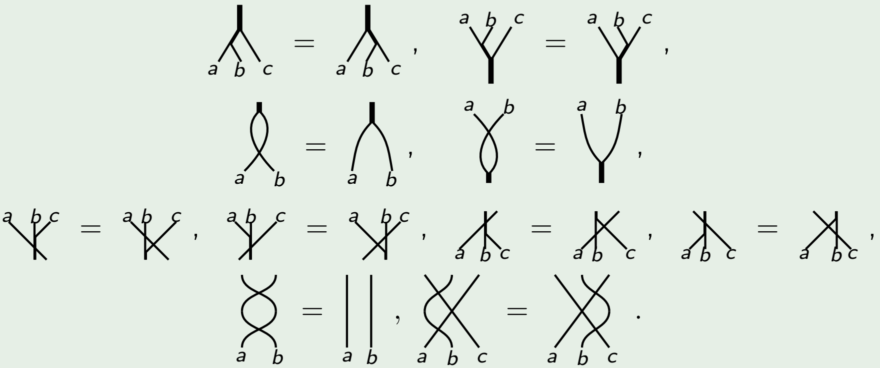
How do we use structure to deduce visual observations?

Required Relations of the Polynomial Web Algebra

Topological Relations

Double cosets only care about how inputs go to outputs. We expect that diagrams with inputs going to the same outputs will be equal.

Example



Required Relations of the Polynomial Web Algebra

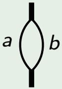
Evaluative Relations

Basic homomorphism compositions (matrix multiplications) should be true.

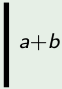
Remark

$$|S_{a+b}/S_{(a,b)}| = \frac{(a+b)!}{a! \cdot b!} = \binom{a+b}{a}, \qquad |S_{a+b}/S_{a+b}| = 1.$$

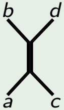
Example



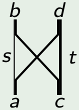
$= \binom{a+b}{a}$



$\longleftrightarrow \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix},$



$= \sum_{s,t}$

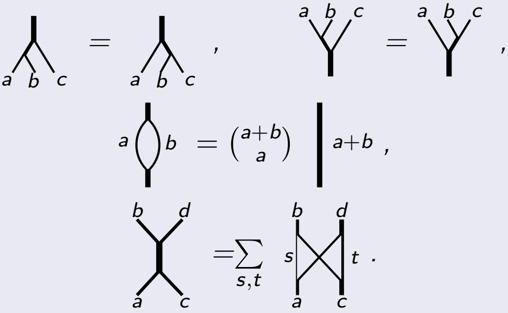


$\longleftrightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$

Polynomial Web Presentation for Schur Algebra

Theorem (Brundan–(Entova–Aizenbud)–Etingof–Ostrik)

The Schur Algebra is isomorphic to the Polynomial Web Algebra with below relations,



Key Idea

All topological relations follow from the above three.

Our Work

Research Question

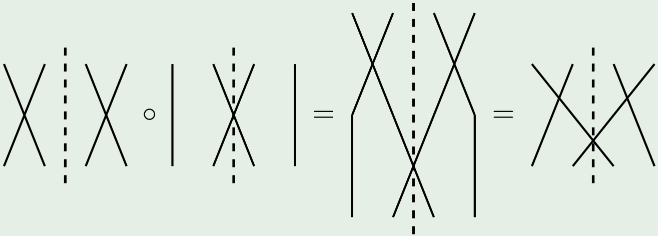
Can we create a Schur Algebra and a Polynomial Web Algebra for another group?

Hyperoctahedral Group and Hyperoctahedral Schur Algebra

Definition

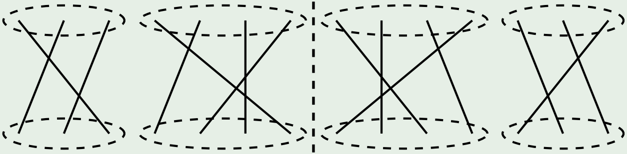
The hyperoctahedral group H_n is the subgroup of S_{2n} with a vertical line of symmetry.

Example (in H_2)

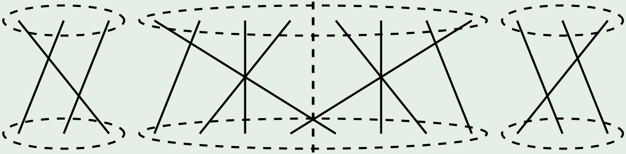


Subgroups of H_n

Examples (in H_7)



$\in H_{(3,4,4,3)}$.

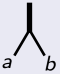


$\in H_{(3,8,3)}$.

Hyperoctahedral Web Algebra

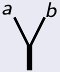
Definition

The Hyperoctahedral Web Algebra is generated by thick string diagrams,



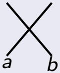
: (a,b,b,a) $\to (a+b, a+b)$,


: $(a, 2b, a) \to (2a+2b)$,



:(a+b, a+b)$\to$ (a, b, b, a),


: $(2a+2b) \to (a, 2b, a)$,

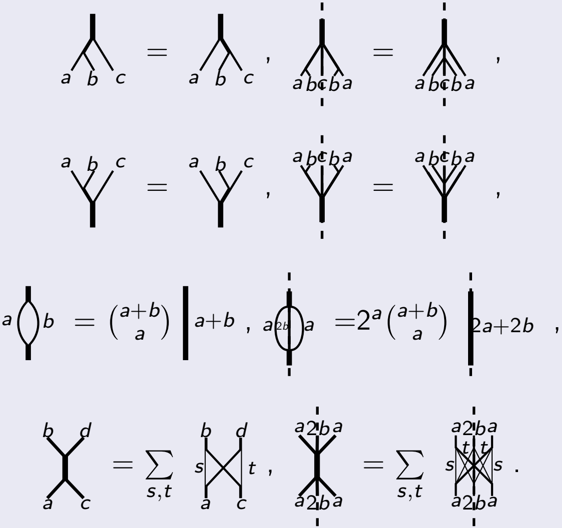


:(a,b, b, a) $\to (b, a, a, b)$,


: $(a, 2b, a) \to (a, 2b, a)$.

It also has relations....

Definition (Cont.)



Hyperoctahedral Schur Algebra and Hyperoctahedral Web Algebra

Definition

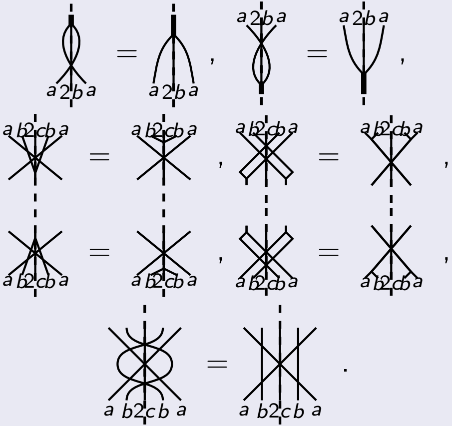
The Hyperoctahedral Schur Algebra of degree n is $\mathrm{End}_{H_n}(\bigoplus_\lambda \mathbb{F}[H_n/H_\lambda])$.

Conjecture

The Hyperoctahedral Schur Algebra and the Hyperoctahedral Web Algebra are isomorphic.

Conjectured Topological Relations

Conjectured Topological Relations



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Bibliography

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- J. Brundan, I. Entova-Aizenbud, P. Etingof, V. Ostrik: *Semisimplification of the category of tilting modules for* GL_n, Advances Math **375**, 2020.