Combinatorial Hikita Conjecture

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Introduction

The Hikita conjecture, stated in [1] is connected with geometry and representation theory. There are some generalisations, but all of them conjecture some isomorphism between algebras. Combinatorial objects we investigate, in some sense, enumerate their bases, and our general task, loosely speaking, is to find *combinatorial* bijections. In particular, in this article, we will formulate and prove parabolic conjecture in combinatorial form. The advantage of this method over the previously known ones is the simplicity of the used objects, which means that this fact is accessible to a larger number of readers as well as it helps to understand the algebraic theorems better.

As we shall prove combinatorial facts, we only consider the case of symmetric group, but before more general treatment will be given (in sections 1 and 2). The core of our article are sections 5 and 6. Sections 3 and 4 are the connecting link between algebra and combinatorics.

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1 Coxeter groups

1.1 Basic notions

Definition 1. *A Coxeter matrix is a matrix with entries in positive integers or infinity such that the* $m_{s,s'} = 1$ *if* $s = s'$ *and* $m_{s,s'} = m_{s',s} \geq 2$ *if* $s \neq s'$.

Also let us regard W as the group defined by the generators $s (s \in S)$ and following relation:

$$(ss')^{m_{s,s'}} = 1 \tag{1}$$

Denote (W, S) as a *Coxeter group* (or a *Coxeter system*).

For all $w \in W$ let us $l(w)$ be the smallest integer $q \geq 0$ such that $w = s_1 s_2 ... s_q$, where $s_1, s_2, \ldots s_q \in S$. We say that $l(w)$ is *length* of the w and that $s_1 s_2 \ldots s_q$ is a *reduced expression*.

Also let us consider a set of all such sequences $(s_1, s_2, \ldots s_q)$ (denote it as X) that $s_1 s_2 \ldots s_q$ is a reduced element in W. We regard X as the vertices of a graph in which $(s_1, s_2, \ldots s_q), (s'_1, s'_2, \ldots, s'_{q'})$ are joined if one of them can be obtained from the other one by replacing m consecutive entries of the form $s, s', s, s', \ldots$ by the m entries $s', s, s', s, \ldots$ where $s \neq s'$ are such that $m = m_{s,s'} < \infty$. We use the notation $(s_1, s_2, \ldots s_q) \sim (s'_1, s'_2, \ldots s'_q)$ for $(s_1, s_2, \ldots s_q), (s'_1, s'_2, \ldots s'_q)$ which are in the same component in X. The main property of this graph is that is $(s_1, s_2, \ldots s_q) \sim (s'_1, s'_2, \ldots s'_q)$ if and only if $q = q'$.

1.2 Coxeter-Dynkin diagrams

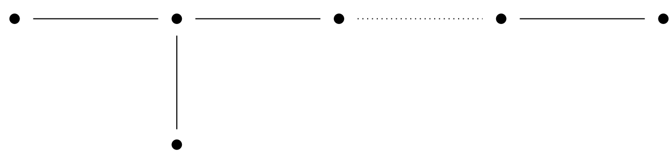
Consider a graph with the set of vertices in bijection with S where the vertices corresponding to $i \neq j$ are joined by an edge if $m_{i,j} = 3$, by a double edge if $m_{i,j} = 4$, by a triple edge if $m_{i,j} = 6$ and by a quadruple edge if $m_{i,j} = \infty$. We call this graph a *Coxeter graph*.

We will say that graph is *irreducible* if this graph is connected. Clearly, W always can be divided into a several irreducible Coxeter graphs. Also we will call all graphs *integral* if $m_{s,s'}, s \neq s'$ is equal only to 2,3,4,6 or ∞. In fact, there are only nine types of finite, irreducible and integral Coxeter graphes (they are often called Dynkin diagrams). We will later identify such of them that are pictured below using another approach to Coxeter groups.

Type A_n:

Note, that there is an obvious bijection between vertices of graph A_n and simple transpositions in symmetric group.

Type C_n:

Type D_n: 

1.3 Basis interpretation

Let E be an $\mathbb{R}$-vector space with basis $(e_s), s \in S$. For $s \in S$ define a linear map $\sigma_s : E \to E$ by $\sigma_s(e_{s'}) = e_{s'} + 2\cos\frac{\pi}{m_{s,s'}}e_s$ for all $s, s' \in S$. It can be shown that Φ defined as $\sigma_s\sigma_s'$ induces the identity map, so $\Phi : E \to E$ has order $m = m_{s,s'}$ if $s \neq s'$ so ss' has order $m_{s,s'}$ in W. In particular, Coxeter matrix is uniquely determined by Coxeter system (W, S).

In type A_n we define S as a basis $(e_1 - e_2, e_2 - e_3, \dots e_n - e_{n+1})$, so W is a set of vectors in which some coordinates are rearranged, therefore a bijection between basis elements and simple transpositions in a symmetric group S_{n+1} exists. In type D_n we define S as a basis $(e_1 - e_2, e_2 - e_3, \dots e_{n-1} - e_n, e_{n-1} + e_n)$, so W is a set of vectors in which even amount of coordinates are multiplied by (-1) and then rearranged and in type C_n we define S as a basis $(e_1 - e_2, \dots e_{n-1} - e_n, e_n)$ and then W will be the set of vectors in which some coordinates are multiplied by (-1) and then rearranged.

It can be verified that this definitions correspond to pictured diagrams, where their vertices correspond to basis elements.

2 On cells

2.1 Hecke algebra

Let (W, S) be a Coxeter system and S is indexed with a set I with a Coxeter matrix (m_{ij}). Then, the group has a presentation

$$W = \Big\langle s_i, s_j \in S \;\Big|\; (s_i s_j)^{m_{ij}} = 1 \Big\rangle.$$

As $m_{ii} = 1$ and $m_{ij} = m_{ji}$ we can rewrite the relations in the following way:

$$s_i^2 = 1, \quad s_i s_j s_i \ldots = s_j s_i s_j \ldots$$

where both sides have m_{ij} factors. Now, we can 'deform' (parametrize, i.e. something like a q-analog) the first identity to $s_i^2 = (q-1)s_i + q$ for some indeterminate q. Now, we can define Hecke algebra, but first, we extend our definition a little bit. We assign a weight to each element using a map (called weight function) $L : W \longrightarrow \mathbb{Z}$ such that $L(uv) = L(u) + L(v)$ for all elements u, v such that $\ell(uv) = \ell(u) + \ell(v)$. Then, a tuple (W, S, L) is called a weighted Coxeter group. Let R be a ring $\mathbb{Z}[q, q^{-1}]$ of Laurent polynomials (in the theory of Kazhdan-Lusztig polynomials, we extend it to $\mathbb{Z}[q^{-1/2}, q^{1/2}]$).

Definition 2 ((Iwahori-)Hecke algebra)**.** *Let $\mathcal{H}$ be an associative R-algebra given by generators $\{T_i\}_{i \in I}$ with subject of the relations:*

- $T_i T_j T_i \cdots = T_j T_i T_j \ldots$ *(where both sides have m_{ij} factors) for any $i \neq j$ and $m_{ij} < \infty$ (braid relation),*
- $(T_i + 1)(T_i - q^{L(s_i)}) = 0$ *(quadratic relation). For unweighted groups, it becomes $T_i^2 = (q-1)T_i + q$.*

2.2 Cellular algebras

2.2.1 General motivation and definition

It turns out that $\mathcal{H}$ is free as an R-module with basis $\{T_w\}_{w \in W}$ (*standard basis*). However, this fact is not trivial and an interested reader may refer to [3]. Yet, there is another basis, called Kazhdan-Lusztig, with very noticeable and useful properties, however, it is too cumbersome to introduce it here, so we again again leave a reference to [3]. One of the most essential properties of this basis is that it is *cellular*.

Let R be a commutative ring with unity, and $\mathcal{A}$ be an associative free (as a R-module) R-algebra with a basis $\{C_i\}_{i\in I}$. We introduce a relation $i \leftarrow_L j$ if there exists $a \in H$ such that $[C_i]aC_j \neq 0$ (a coefficient of C_i in aC_j). We take a transitive closure $i \leq_L j$ (if there exists a sequence $i \leftarrow_L \cdots \leftarrow_L j$). Next, we naturally introduce an equivalence $i \sim_L j$ (when $i \leq_L j$ and $j \leq_L i$). Then, (the equivalence classes) $I/\sim_L$ are called *left cells*. Similarly, one can define right and two-sided cells.

2.2.2 Cellular basis

Using definition of relation $\leq_L$, one can write

$$aC_x = \sum_{y \leq_L x} h(y,x) C_y,$$

where $h(y,x) \in R$. This identity shows that $\mathcal{H}(\leq_L x)$ (linear span of C_y such that $y \leq_L x$) is a left ideal. Also, it is easy to show that $\mathcal{H}(<_L x)$ is also a left ideal. So, we can consider $\mathcal{H}(\leq_L x)/\mathcal{H}(<_L x)$, where

$$aC_x = \sum_{y \sim_L x} h(y,x) C_y \pmod{\mathcal{H}(<_L x)}.$$

Such a module is called a *cell module*. Using that motivation, now we define *cellular basis* as follows.

Definition 3 (Cellular basis). *Let $\mathcal{A}$ be a free associative R-algebra. Also, introduce Λ, a finite poset, and associate a finite set (of indices) $M(\lambda)$ with each $\lambda \in \Lambda$. We also assume that there exists a basis $\{C^\lambda_{P,Q} \in \mathcal{A} \mid P, Q \in M(\lambda)\}$. This basis is cellular if*

- *R-linear map $(.)^*$ given by $(C^\lambda_{P,Q})^* = C^\lambda_{Q,P}$ is an anti-involution in $\mathcal{A}$,*
- *for any $a \in \mathcal{A}$ it holds that*

$$aC^\lambda_{P,Q} = \sum_{P' \in M(\lambda)} h(P,P') C^\lambda_{P',Q'} \pmod{\mathcal{H}(<_L \lambda)}.$$

for some $h(P, P') \in \mathcal{A}$ and depending only on indices P and P'.

It turns out that a Kazhdan-Lusztig basis is cellular for a Hecke algebra. In general, such cellular algebras (that is algebras which have a cellular basis) behave well in representation theory. For an original article, see [5].

In addition, Hecke algebras actually have a lot of interesting interpretations in geometry and algebra. For example, see the introduction in [3] for a quick survey on how Hecke algebras arise from reductive algebraic groups.

2.2.3 Semi-regular cells

Consider a Coxeter group (W, S) (i.e. a symmetric group), and let $\mathcal{C}$ be a set of elements $w \in W$ with the unique reduced expression. Furthermore, we split $\mathcal{C}$ into disjoint sets

$$\mathcal{C} \supset \mathcal{C}_s = \big\{ w \in W \mid ws \text{ — reducible} \big\}.$$

It can be proved that $\mathcal{C}$ is a two-sided cell, as well as $\{\mathcal{C}_s\}_{s \in S}$ are left cells (with respect to the Kazhdan-Lusztig basis), see [6].

3 RSK-correspondence and its variations

3.1 Row bumping algorithm

Robinson-Schensted correspondence gives the combinatorial bijection between permutations and pairs of standard Young tableaux of the fixed (for a particular permutation) shape, that is

$$\sum_{\lambda \in \mathbb{Y}_N} \mathrm{YT}(\lambda)^2 = N!,$$

where $\mathbb{Y}_N$ is a set of all Young diagrams of size N and $\mathrm{YT}(Shape)$ is the number of Young tableaux of a shape $Shape$.

For $\pi \in S_n$ the algorithm is

1. P_0, Q_0 are empty tableaux.
2. $P_i = P_{i-1} \leftarrow \pi_i$ (by row bumping, defined just below), add a new cell of P_i with entry i to Q_i.
3. Return (P_n, Q_n).

Insertion is usually denoted as $T \leftarrow x$, where T is a tableau and x is a value we insert. The row bumping algorithm looks the following way:

1. Keep a coordinate pair (i, j), initially set to $(1, k+1)$ where k is the first row's of T length.
2. Find the first square in i-th row with an entry larger than x (or no such an entry), for example, by running a cycle 'while $j > 1$ and $x < T_{i,j-1}$'.
3. If (i, j) is empty, add it with x. Otherwise, swap x and $T_{i,j}$, go to the next row (increase i by one) and return to second step.

In result, we obtain two tableaux P and Q, which are of the same shape and which pair is unique for any permutation π (for proof, we refer to [2]).

3.2 Construction géométrique de Viennot

One of the most notable properties of Robinson-Schensted is so called inversion theorem, which states that if $\pi \sim (P, Q)$, then $\pi^{-1} \sim (Q, P)$. The simplest proof uses construction géométrique de Viennot, see [4].

4 Conjugacy classes

4.1 Stabilizer and Q-orbit

4.1.1 Vector v_λ

From now and further we will work only with the case of the symmetric group S_n. We assign a vector v_λ to any partition λ of a specific form.

Definition 4 (Vector v_λ)**.** *Let $N = \lambda_1 + \cdots + \lambda_n$ be a partition ($\lambda_1 \geq \cdots \geq \lambda_n$) such that all the summands have the same parity. We construct an auxiliary vector*

$$v_{aux} = (Bl(\lambda_1), Bl(\lambda_2), \ldots, Bl(\lambda_k))$$

where $Bl(a) = [-(a-1)/2; (a-1)/2] \cap \mathbb{Z}$ for an odd a and $Bl(a) = [-a/2; a/2] \cap (\mathbb{Z} + 1/2)$ for an even a. Then v_λ is a sorted (in the increasing order) version of v_{aux}.

4.1.2 The longest elements

Of course, S_n acts on v_λ, permuting its coordinates. So, we can consider a stabilizer $\operatorname{Stab}(v_\lambda) \subset S_n$ (all the subgroups of this form are so-called *parabolic subgroups*) which fixes v_λ. It is clear that,

$$\operatorname{Stab}(v_\lambda) = \prod_{x=-(\lambda_1-1)/2}^{(\lambda_1-1)/2} \operatorname{Sym}(cnt_{v_\lambda}(x)) \tag{2}$$

where $cnt_v(x)$ equals the number of occurences of x in v.

Then, consider cosets $\sigma\operatorname{Stab}(v_\lambda)$ (where $\sigma \in S_n$). We take the longest element $w_0^\lambda(\sigma)$ (proved in 4.3.2) of each class and obtain a set $\operatorname{Orb}(v_\lambda)$:

$$S_n/\operatorname{Stab}(v_\lambda) \xrightarrow{w_0^\lambda} \operatorname{Orb}(v_\lambda)$$

or, in other words, $\operatorname{Orb}(v_\lambda) = \{w_0^\lambda(\sigma)\}_{\sigma \in S_n}$.

4.2 The problem statement

4.2.1 Left-cell definition

Now, let $w_0^\lambda := w_0^\lambda(\mathrm{id})$. We introduce a subset of $\mathrm{Orb}(v_\lambda)$ of elements which lie in the same left cell as w_0^λ:

$$\mathrm{QOrb}(v_\lambda) = \big\{w \mid \exists\sigma : w = w_0^\lambda(\sigma), w \sim_L w_0^\lambda\big\}.$$

The question, now, is to establish a combinatorial bijection between the $\mathrm{QOrb}(v_\lambda)$ and the set of Young tableaux of the form λ:

$$\mathrm{QOrb}(v_\lambda) \overset{?}{\longleftrightarrow} \mathrm{YTableau}(\lambda).$$

4.2.2 Purely combinatorial version

The most crucial tool here is that we can describe cells in the case of symmetric groups purely combinatorially: if $u, v \in S_n$, then

- $u \sim_L v$ if, and only if $Qu = Qv$.
- $u \sim_R v$ if, and only if $Pu = Pv$.

For proof see [2]. So, in fact,

$$\mathrm{QOrb}(v_\lambda) = \big\{w \mid \exists\sigma : w = w_0^\lambda(\sigma), Qw = Qw_0\big\},$$

which, of course, is much handier.

4.3 Centred diagrams

4.3.1 General definitions

We take a partition λ from paragraph 4.1.1 (parity of each summand is odd). Then, essentially, the centred diagram CT^λ is nothing but a Young diagram of the partition centred by the 'central' (that is, $(a-1)/2$ for a row of length a) cell. A centred tableau is a centred diagram where numbers $1, \ldots, n$ are written in the cells (with no additional requirements). Hence, the centred tableaux is just another way of representing a permutation in S_n.

There are two natural directions: vertical and horizontal. When we fill a tableau, we can also write a direction (vertical, by default) like $CT_h(\sigma)$; that is, we insert values of σ of its one-line notation one by one, filling first row, second row and so on; in $CT_v(\sigma)$ we first fill the first column, then the second, and so on. This notation allows us to consider transformations of a permutation σ of form $\mathrm{Proj}_v(CT_h(\sigma))$ (fill the diagram with σ by rows and then read it by columns). See figure 1.

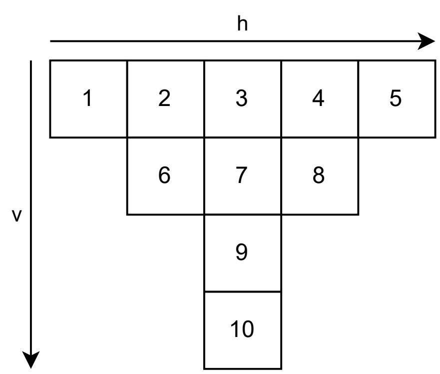


Figure 1: The picture shows $CT_h(\mathrm{id})$ for $10 = 5 + 3 + 1 + 1$. Here, for example, $\mathrm{Proj}_v(CT_h(\mathrm{id})) = (1, 2, 6, 3, 7, 9, 10, 4, 8, 5)$ (we read in vertical direction).

We can also interpret the vector v_λ with a similar construction: we index columns such that the central one is zero, left ones have negative index and right ones have positive; then, assign an index of the column to each cell and read it by rows.

4.3.2 Stabilizer and orbit

It's clear from the definitions that elements of stabilizer are just permutations inside columns (look at (2)). Moreover, each conjugacy class $\sigma\mathrm{Stab}$ can be described by first swapping some cells (action of σ) and then applying Stab action (i.e. permuting inside columns).

We can describe Orb as well, that is find how the longest elements look like. Using that the length of reduced expression is equal to the number of inversions in a permutation, we conclude that the longest element in $\sigma\mathrm{Stab}$ (recall its description using centred tableaux) has all

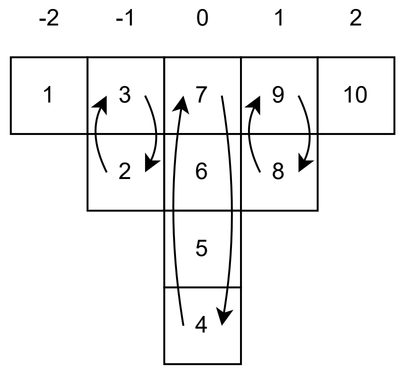


Figure 2: Here, the longest element in Stab is presented (one can obtain the permutation by reading the tableau vertically). Arrows mean that any element of Stab can be obtained by permutating there.

the columns (decreasingly) sorted, that is all the subpieces (recall formula (2)) maximise the number of inversions. For example, we can use this to prove that $Shape(Qw_0(\mathrm{id})) = \lambda$. It is very easy to understand how the row bumping algorithm works: it simply adds each column (of the centred tableau, in descending order), thus bumping all previous cells from the column (and only them) up. But the new elements are always larger than the previous ones, so nothing else is pushed and the Q-symbol is just as if we were adding columns to the table layer by layer.

Finally, we can find sizes of $\mathrm{Stab}(v_\lambda)$ and $\mathrm{Orb}(v_\lambda)$ using the column's heights $h_{-k}, \ldots, h_k$ (where $k = (\lambda_1 - 1)/2$) of the centred diagram of λ, as follows:

$$\begin{aligned} |\mathrm{Stab}(v_\lambda)| &= h_{-k}! \ldots h_k!, \\ |\mathrm{Orb}(v_\lambda)| &= n!/(h_{-k}! \ldots h_k!). \end{aligned}$$

5 The combinatorial bijection

5.1 Inversion trick

5.1.1 The reformulation of the problem

To prove the problem statement it is sufficient to prove the following fact:

Proposition 1. *All elements with Q-symbol coinciding with Qw_0 are the longest in their left cells.*

Let us formulate a main fact that is equivalent to the previous one:

Proposition 2. *All elements with P-symbol coinciding with Pw_0 are the longest in their right cells.*

In 5.2 we will prove the second proposition, which will be sufficient to prove the main fact.

5.1.2 Proof of the equivalency of propositions

Consider the bijection $M : S_n \to S_n$, which maps the elements to their opposites. Since if $w \sim (P, Q)$ under the Robinson-Schensted correspondence then $w^{-1} \sim (Q, P)$, so the left cells move to right cells, and vice versa, because $\sigma \sim_R \sigma \circ S \iff \sigma^{-1} \sim_L S^{-1} \circ \sigma^{-1}$, and S^{-1} is also a stabilizer element. So, if the element in the considered right cell was the longest and its P-symbol coincided with Pw_0, then its image Q-symbol coincides with Qw_0. This consequence is justified by the fact that on this mapping the longest element of the stabilizer maps to itself, since the image of the stabilizer is the stabilizer and the length is conserved, hence Pw_0 and Qw_0 coincide. The equivalence of the propositions is proved.

5.2 Q-symbols

Let the given number $N = \lambda_1 + \lambda_2 + \cdots + \lambda_n,\ \lambda_i \equiv\ 1\ (mod\ 2)$, and the vector, corresponding to it is $v_\lambda = (-t, -t, \ldots, -t, -(t-1), \ldots, -(t-1), -(t-2), \ldots, (t-1), t, \ldots, t)$. Let m be the last appearance of k_m in the vector. Then w_0 looks as follows:

$$w_0 = \left(k_{-t}, k_{-t} - 1, \ldots, 1, k_{-(t-1)}, \ldots, (k_{-(t-1)} - k_{-t} + 1), \ldots, (k_t - k_{t-1} + 1)\right).$$

Consider its P-symbol.

1	$k_{-(t-1)} - k_{-t} + 1$	$k_{-(t-2)} - k_{-(t-1)} + 1$	$\dots$	$k_{t-2} - k_{t-3} + 1$	$k_{t-1} - k_{t-2} + 1$	$k_t - k_{t-1} + 1$
2	$k_{-(t-1)} - k_{-t} + 2$	$k_{-(t-2)} - k_{-(t-1)} + 2$		$k_{t-2} - k_{t-3} + 2$	$k_{t-1} - k_{t-2} + 2$	$k_t - k_{t-1} + 2$
...	...	...	$\dots$	...	...	...
k_{-t}	...				...	k_t
	$k_{-(t-1)}$	...	$\dots$	...	k_{t-1}	
		$k_{-(t-2)} - 1$		$k_{t-2} - 1$		
		$k_{-(t-2)}$	$\dots$	k_{t-2}		

Figure 3: P-symbol.

Consider an arbitrary element with such P-symbol. Split the vector into blocks of identical numbers. Since the inversions between numbers in two different fixed blocks do not depend on the order of numbers in them, it is sufficient to prove that each block has the largest number of inversions, that is, the numbers there go in reverse order of the original one.

Observe that each column of P-symbols corresponds to one block of identical numbers in the vector. Since the larger number in the column is strictly below the smaller one, it occurred earlier in the vector, which means that the order of numbers in the block is reversed. This reasoning completes the proof.

6 Parabolic Hikita Conjecture

6.1 Diagrams

Fix two centred Young diagrams λ and μ. Fill them with numbers in these ways:

Filling μ: In cells of the first column of μ, we put 1s, in the second column – 2, in the third – 3, ..., and k in the last one, and also for each number which is not unique in its column we set an index of its row, for instance:

$$\mu = \begin{array}{|c|c|c|c|} \hline 1 & 2_1 & 3_1 & 4 \\ \hline \multicolumn{1}{c|}{} & 2_2 & 3_2 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}$$

Filling λ: We put numbers from μ in λ such that the smaller number is always to the left and above the larger number (not strictly), and the same numbers are always in distinct columns, and the numbers with smaller index is always to the left with the larger number. Further, if we sort numbers in the increasing order by the number itself first we can put numbers sequentially:

$$\lambda = \begin{array}{|c|c|} \hline 1 & 2_2 \\ \hline 2_1 & 3_2 \\ \hline 3_1 & 4 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}$$

The resulting tableau is considered as P-symbol of some permutation w_1. We construct its Q-symbol similarly to the previous problem, that is as in figure 3. We define the set of permutations which we can obtain by setting Q-symbol to be W_1.

6.2 Vectors

We construct vectors v_λ and v_μ by initial Young diagrams as in the previous problem. Let stabilisers of these permutations be S_λ and S_μ, respectively.

Proposition 3. *If for fixed diagrams λ and μ there exists a permutation w such that an orbit $\{g_2 \circ w \circ g_1 | g_1 \in S_\lambda, g_2 \in S_\mu\}$ is free and w is the longest element in $w \circ S_\lambda$ and the shortest element in $S_\mu \circ w$, then w is a unique element of the orbit that satisfies these properties.*

We shall prove this proposition in 6.6.

Let W_2 be the set of such permutations w with a fixed Q-symbol.

Theorem 1 (The Main Problem)**.** *The sets W_1 and W_2 are equal.*

We shall prove that $W_1 \subset W_2$ and $W_2 \subset W_1$.

6.3 Permutation w is the longest in $w \circ S_\lambda$.

From the previous problem, we know that our choice Q-symbol is equivalent to the fact that w is the longest in $w \circ S_\lambda$.

6.4 Orbit of w is free

We have to prove that orbits of all permutations from W_1 are free. First, we prove the following auxiliary proposition.

Proposition 4. *The set of consecutive equal numbers of vector v_λ is called block. Blocks of v_μ are defined in the same way. Then the orbit of w is free if and only if w sends a number from each block of v_λ to distinct blocks of v_μ.*

Proof. The condition is necessary because stabilisers can permutate only numbers inside a single block.

Now we shall prove that if w moves two numbers from a block of v_λ to two numbers of a single block of v_μ, then the orbit of w is not free. Indeed, let positions of these elements in v_λ are a_1 and a_2 and w maps then into b_1 and b_2, respectively. Then take $g_1 = (a_1, a_2) \in S_\lambda$ and $g_2 = (b_1, b_2) \in S_\mu$. Then $g_2 \circ w \circ g_1 = w$, so we reached contradiction. □

We shall also use the following proposition.

Proposition 5. *Reading P-symbol of $w \in W_1$ by columns upside-down from left to right, we obtain w.*

Proof. Consider first columns of P- and Q-symbols of $w \in W_1$. Let them contain n cells. We know numbers of the first column of P-symbol, so we surely know which number was the first in w and knowing Q-symbol, we understand that further we have decreasing sequence of $n-1$ numbers and again from the first column of P-symbol we know exactly which numbers they are the further $n-1$ numbers of w could be only upper number of the first column of P-symbol, that is first n numbers of w are written in the first column of its P-symbol reading upside-down. We can do this argument for all other columns. □

We immediately obtain the following corollary.

Corollary 1. *The numbers in a column of P-symbol of $w \in W_1$ are numbers of positions into which w sends the number of the block that corresponds to this column.*

The final ingredient is the following:

Proposition 6. *In the initial statement of the problem, v_μ can be replaced with v'_μ such that each number is equal to the number of its block and the index of this number is equal to the position of this number in its block. It saves all considered properties of v_μ.*

This proposition is obvious, albeit useful because now elements of v_μ are numbered via numbers from the filling of μ.

We recall that there is no column of the initial filling of λ such there are no two elements from one column of μ, and therefore all elements from each block v_λ were sent into different blocks of v_μ which, by proposition 4 means, that the orbit of w is free, what we wanted to prove in this subsection.

6.5 Permutation ω is the shortest in $S_\mu \circ \omega$.

Now let us consider an arbitrary block v'_μ. We know that in diagram λ numbers from this block are in the different columns, and number from column λ with smaller index is always going on to a number with smaller index. That fact and the statement 3 mean that after the permutation ω for all blocks v_μ there cannot be any new inversions between the numbers from the the same block, so ω is the shortest one in $S_\mu \circ \omega$, q.e.d.

And now we want to prove that converse statement is also true, that is the permutation $\omega \in W_2$ has $P-$symbol which can be received by algorithm which was described in section 1.1.

Consider the following operation: at first we sort numbers from μ by ascending and then replace numbers from λ for those which are corresponding to a number with an index which are on the corresponding for them places (in fact, this operation is reverse to the algorithm from section 1.1, which replace numbers from μ in λ to a consecutive natural numbers). Now suppose that after this operation there are two numbers with the same number and different index in one column. From the statement 3, at first they were in the same block, and from the statement 4 now there are also in the same block, so we get a contradiction with the statement 2. Therefore, ω has required P-symbol, q.e.d.

6.6 Proof of the proposition 3.

Suppose that permutation $\omega^* = g_2^* \circ \omega \circ g_1^*$, which lies in orbit $\omega \in W_1$, also lies in W_1. Notice, that permutation $\omega \circ g_1^*$ in vector v_λ only permutate numbers in different blocks, therefore $\omega \circ g_1^* \in S_\lambda$. So, $\omega \circ g_1^*$ match the same numbers in v_λ to a different blocks v_μ (ω has a free orbit), therefore in each block v_μ a set of numbers from v_λ which were gone in this block coincides with the set of all such numbers from ω. But ω^* is the shortest element in orbit $\{g_2 \circ \omega^* \mid g_2 \in S_2\}$, so the order of numbers in blocks v_μ also was not changed, therefore $g_2^* = id$. Then, $\omega^* = \omega \circ g_1^*$, and it can be the longest one in orbit $\{\omega^* \circ g_1 \mid g_1 \in S_1\}$ only if $g_1 = id$, because in another case ω will be longer. So, $\omega^* = \omega$, and we prove that ω^* lies in W_1 only if $\omega^* = \omega$, therefore ω is the unique permutation in it's orbit which satisfies condition, q.e.d.

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