Signatures of Multiplicity Spaces in Tensor Products of $\mathfrak{sl}_2$ and $U_q(\mathfrak{sl}_2)$ Representations, and Applications

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Abstract

We study multiplicity space signatures in tensor products of $\mathfrak{sl}_2$ and $U_q(\mathfrak{sl}_2)$ representations and their applications. We completely classify definite multiplicity spaces for generic tensor products of $\mathfrak{sl}_2$ Verma modules. This provides a classification of a family of unitary representations of a basic quantized quiver variety, one of the first such classifications for any quantized quiver variety. We use multiplicity space signatures to provide the first real critical point lower bound for generic $\mathfrak{sl}_2$ master functions. As a corollary of this bound, we obtain a simple and asymptotically correct approximation for the number of real critical points of a generic $\mathfrak{sl}_2$ master function. We obtain a formula for multiplicity space signatures in tensor products of finite dimensional simple $U_q(\mathfrak{sl}_2)$ representations. Our formula also gives multiplicity space signatures in generic tensor products of $\mathfrak{sl}_2$ Verma modules and generic tensor products of real $U_q(\mathfrak{sl}_2)$ Verma modules. Our results have relations with knot theory, statistical mechanics, quantum physics, and geometric representation theory.

Contents

1 Introduction and results

Verma modules form an important class of infinite dimensional representations of classical Lie algebras and quantum groups. Any simple Verma module with real highest weight (and $|q| = 1$ if applicable) carries a unique invariant nondegenerate Hermitian form known as the *Shapovalov form*. The signature of the Shapovalov form is closely related to several topics in representation theory and topology, including the unitary representation theory of quantized quiver varieties, the critical points of master functions, and the behavior of the Jones polynomial of knot sequences on the unit circle. In 2005, Yee used Kazhdan-Lustzig polynomials to explicitly compute the signatures of the Shapovalov forms of Verma modules [1]. This development has opened the door to studying signatures and applications arising from constructions based on Verma modules.

For any complex $\lambda \notin \mathbb{Z}_+$, the $\mathfrak{sl}_2$ Verma module M_λ is the unique simple highest weight representation of $\mathfrak{sl}_2$ of weight λ. It has a basis $\{v_i\}_{i=0}^\infty$ defined by $v_i = F^i v_0$, on which E and H act by $Ev_i = i(\lambda - i + 1)v_{i-1}$ and $Hv_i = (\lambda - 2i)v_i$. When λ is real and not in $\mathbb{Z}_+$, the Shapovalov form on M_λ is nondegenerate and satisfies the adjointness conditions $H^* = H$, $E^* = F$, and $F^* = E$. Given a set of real Verma modules $M_{\lambda_1}, M_{\lambda_2}, ..., M_{\lambda_n}$, the tensor product $\bigotimes_{i=1}^n M_{\lambda_i}$ carries a Hermitian form equal to the product of the forms on each factor.

A fundamental problem related to tensor products is the *decomposition problem*: Given a tensor product of representations, rewrite it as a direct sum. In the case of simple real $\mathfrak{sl}_2$ Verma modules satisfying the condition that the sum of highest weights is not in $\mathbb{Z}_+$, this type of decomposition exists and is unique. More precisely, we have $\bigotimes_{i=1}^n M_{\lambda_i} = \bigoplus_{m=0}^\infty M_{(\sum \lambda_i)-2m} \otimes E_m$, where each *level m multiplicity space E_m* has dimension $\binom{m+n-2}{n-2}$ and is isomorphic to the space $\mathrm{Hom}(M_{(\sum \lambda_i)-2m}, \bigotimes_i M_{\lambda_i})$. Moreover, the Shapovalov forms on the Verma modules induce a natural Hermitian form (and therefore a signature) on each multiplicity space. The direct sum over m of the product of the forms on $M_{\sum(\lambda_i)-2m}$ and E_m equals the form on the tensor product $\bigotimes_{i=1}^n M_{\lambda_i}$.

The goal of this paper is to study the multiplicity space signatures and applications arising from Verma module tensor products, in the context of the Lie algebra $\mathfrak{sl}_2$. Our main results are Theorems 1.1, 1.2, and 1.3. In particular, we completely classify the definite multiplicity spaces in any generic tensor product [1] of $\mathfrak{sl}_2$ Verma modules.

Theorem 1.1. *The classification list in Appendix A classifies all definite multiplicity spaces in any tensor product of generic $\mathfrak{sl}_2$ Verma modules.*

Remark. The definite multiplicity space classification gives the definite weight subspace classification as a corollary.

The classification of definite multiplicity spaces given by Theorem 1.1 is especially interesting because it provides a classification of a family of unitary representations of a basic quantized quiver variety, one of the first such classifications for any quantized quiver variety. The quantized quiver variety arises from the universal enveloping algebra reduction and quantum Hamiltonian reduction given by Bezrukavnikov and Losev [2].

We then use signatures of multiplicity spaces to study a problem in differential topology. There is a family of $\mathfrak{sl}_2$ *master functions* that arises in the study of Knizhnik-Zamolodchikov equations and spin chain Gaudin models [3]. Each master function $F_{z,\lambda,m} : \mathbb{C}^m \to \mathbb{C}$ associated to the Gaudin model is indexed by two sequences of n real parameters, denoted $z = (z_1, ..., z_n)$ and $\lambda = (\lambda_1, ..., \lambda_n)$, and a positive integer m. For an m-tuple $(t_1, t_2, ..., t_m) \in \mathbb{C}^m$, the master function $F_{z,\lambda,m}$ is defined as $F_{z,\lambda,m}(t_1, t_2, ..., t_m) = \mathrm{Disc}(Q) \cdot \prod_i |Q(z_i)|^{-\lambda_i}$, where $Q \in \mathbb{C}[x]$ is defined as $Q(x) = \prod_{i=1}^m (x - t_i)$ and $\mathrm{Disc}(Q) = \prod_{i<j}(t_i - t_j)^2$ is its discriminant. A critical point of $F_{z,\lambda,m}$ is said to be *real* if the corresponding Q-polynomial has all real coefficients. The computation of the number of real critical points (denoted $N_{z,\lambda,m}$) of an arbitrary master function $F_{z,\lambda,m}$ is an open and difficult problem. Recently, Mukhin and Tarasov [4] have shown that

[1] We call a set of reals generic if they are not nonnegative integers and their sum is not a nonnegative integer. We call a set of real Verma modules generic if their highest weights are generic.

signatures of Hermitian forms can be used along with the Bethe ansatz method to compute lower bounds for the numbers of real solutions to topological problems. We study real critical points of the master function using an approach based on Mukhin and Tarasov's work, a Bethe ansatz setup due to Etingof, Frenkel, and Kirillov [5], and a Bethe vector characterization due to Feigin, Frenkel, and Rybnikov [6]. Our second main result, Theorem 1.2, uses multiplicity space signatures to give the first lower bound for the number of real critical points of a generic [2] $\mathfrak{sl}_2$ master function.

Theorem 1.2. *For a generic $\mathfrak{sl}_2$ master function $F_{z,\lambda,m}$, we have $|\operatorname{sgn}(E_m)| \le N_{z,\lambda,m}$, where $\operatorname{sgn}(E_m)$ denotes the signature of the space E_m in the decomposition of $\bigotimes_{i=1}^n M_{\lambda_i}$.*

We use our Theorem 1.2 lower bound on $N_{z,\lambda,m}$ along with an upper bound on $N_{z,\lambda,m}$ given by Mukhin and Varchenko [7] to show that $\dim(E_m) = \binom{m+n-2}{n-2}$ is a good approximation for $N_{z,\lambda,m}$ for m large.

Corollary 1.1. *For fixed generic λ and z sequences, we have $\lim_{m\to\infty} \frac{N_{z,\lambda,m}}{\binom{m+n-2}{n-2}} = 1$.*

Finally, we extend our work to quantum groups and compute signatures of Hermitian forms in q-deformed multiplicity spaces. For generic [3] q on the complex unit circle, the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ is the standard q-deformation of the enveloping algebra of $\mathfrak{sl}_2$ [8]. For each nonnegative integer a, there is a unique $(a+1)$-dimensional simple representation of $U_q(\mathfrak{sl}_2)$, denoted $\widetilde{V}_a$, and this representation carries a Shapovalov form. Moreover, a tensor product of two simple finite dimensional representations $\widetilde{V}_a \otimes \widetilde{V}_b$ is a representation of $U_q(\mathfrak{sl}_2)$. The tensor product carries a unique invariant nondegenerate Hermitian form induced by the Drinfeld coboundary structure [9]. The form is induced as follows. There is a *standard universal R-matrix*, defined for our choice of coproduct as $R = q^{\frac{H\otimes H}{2}} \sum_{i\ge 0} q^{\binom{i}{2}} \cdot \frac{(q-q^{-1})^i}{[i]!} \cdot F^i \otimes E^i$, where $[i] = \frac{q^i - q^{-1}}{q-q^{-1}}$ and $[i]! = [1][2]...[i]$. The matrix $\overline{R}$ of the Drinfeld coboundary structure is defined in terms of the R-matrix by $\overline{R} = R(R^{21}R)^{-1/2}$. The form on the tensor product $\widetilde{V}_a \otimes \widetilde{V}_b$ is given by $(v_1 \otimes w_1, v_2 \otimes w_2) = \sum_i (a_i v_1, v_2) \cdot (b_i w_1, w_2)$, where $\overline{R} = \sum_i a_i \otimes b_i$ and the pairings (v_1, v_2) and (w_1, w_2) are computed using the forms on $\widetilde{V}_a$ and $\widetilde{V}_b$. This construction can be extended to a construction of a unique Hermitian form on any tensor product of finite dimensional simple $U_q(\mathfrak{sl}_2)$ representations. The form on a tensor product is contravariant due to the $\overline{R}$ factor, so the tensor product can be decomposed with multiplicity spaces, each carrying its own induced form. Our third main result, Theorem 1.3, is a formula for the multiplicity space signatures in an arbitrary tensor product of finite dimensional simple $U_q(\mathfrak{sl}_2)$ representations. A convenient feature of our formula is that it is combinatorial, in that it has only additions, and no subtractions.

Theorem 1.3. *We have the decomposition $\bigotimes_{i=1}^n \widetilde{V}_{a_i} \cong \bigoplus_{m\ge 0} \widetilde{V}_{(\sum_i a_i)-2m} \otimes \widetilde{E}_m$, where*

$$\operatorname{sgn}(\widetilde{E}_m) = \sum_{m_1+m_2+\cdots+m_{n-1}=m} \prod_{j=1}^{n-1} \operatorname{sign}\left[\binom{1+\sum_{k=1}^{j+1} a_k - \sum_{k=1}^{j} m_k}{m_j}_q \binom{\sum_{k=1}^{j} a_j - \sum_{k=1}^{j-1} m_k}{m_j}_q \binom{a_{j+1}}{m_j}_q \right].$$

Remark. Theorem 1.3 lifts to an identical result for a tensor product of $U_q(\mathfrak{sl}_2)$ Verma modules of real highest weights. When $q = 1$, the formula is still valid (even though 1 is not generic) and gives multiplicity space signatures for a tensor product of generic $\mathfrak{sl}_2$ Verma modules.

The paper is organized as follows. In Section 2, we cover definitions and background information. In Section 3, we prove the classification given by Theorem 1.1. In Section 4, we prove the critical point bound

[2] We call a master function generic if its real parameters are generic.
[3] We call a number on the complex unit circle generic if it is not a root of unity.

given by Theorem 1.2 and the approximation given by Corollary 1.1. In Section 5, we prove the signature formula given by Theorem 1.3.

2 Preliminaries

2.1 Signatures in $\mathfrak{sl}_2$ weight representations

A *weight representation* of $\mathfrak{sl}_2$ is a representation which carries a unique Hermitian form and is isomorphic to a direct sum of finite dimensional H-weight spaces, with the weights bounded from above. The Hermitian form on the weight representation must also satisfy the condition that the weight spaces are orthogonal and $H^* = H$, $E^* = F$, and $F^* = E$. For instance, the Verma modules are weight representations due to their Shapovalov forms.

To any weight representation of $\mathfrak{sl}_2$, we can associate a *signature character*, which is a (possibly infinite) sum of powers of the formal symbol e with coefficients in the ring $\mathbb{Z}[s]/(s^2-1)$. For a weight representation V, the signature character, denoted by $\mathrm{ch}_s(V)$, is defined as $\sum_\alpha (a_\alpha + b_\alpha s)e^\alpha$, where the sum is taken over all weights α. For the weight space of weight α, the number of orthogonal basis vectors which pair with themselves to give a positive real is a_α, while the number of orthogonal basis vectors which pair with themselves to give a negative real is b_α. Hence, the dimension of the α-weight space is $a_\alpha + b_\alpha$ while the signature of the α-weight space is $a_\alpha - b_\alpha$. The signature characters satisfy some nice properties for standard constructions involving weight representations. Namely, for any weight representations V_i, their tensor product and direct sum are both weight representations with $\mathrm{ch}_s(\bigotimes_i V_i) = \prod_i \mathrm{ch}_s(V_i)$ and $\mathrm{ch}_s(\bigoplus_i V_i) = \sum_i \mathrm{ch}_s(V_i)$.

Since Verma modules are weight representations, we can study their tensor product decompositions using signature characters. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be generic reals. For simplicity, from here on out, we will denote $\beta_{\lambda_i} = \mathrm{ch}_s(M_{\lambda_i})$.

Proposition 2.1. *We have* $\mathrm{ch}_s(M_{\lambda_i}) = \beta_{\lambda_i} = \begin{cases} \sum_{j=0}^{\infty} s^j e^{\lambda_i - 2j} & \text{if } \lambda_i < 0, \\ \sum_{j=0}^{\lfloor \lambda_i \rfloor} e^{\lambda_i - 2j} + \sum_{j=\lceil \lambda_i \rceil}^{\infty} s^{j+\lceil \lambda_i \rceil} e^{\lambda - 2j} & \text{if } \lambda_i > 0. \end{cases}$

There is a unique decomposition of signature characters $\prod_{i=1}^n \beta_{\lambda_i} = \sum_{m=0}^{\infty} (a_m + s b_m) \cdot \beta_{\lambda - 2m}$. This decomposition exactly encodes the tensor product decomposition $\bigotimes_{i=1}^n M_{\lambda_i} \cong \bigoplus_{m=0}^{\infty} M_{\lambda-2m} \otimes E_m$, in the sense that each multiplicity space E_m has dimension equal to $\binom{m+n-2}{n-2} = a_m + b_m$ and signature equal to $a_m - b_m$. Hence, determining when E_m is definite amounts to determining when $a_m = 0$ or $b_m = 0$. The signature character interpretation of the definite multiplicity space classification problem is useful because it translates a representation theoretic problem into a more tractable combinatorial problem.

2.2 Quantum groups case

For q on the complex unit circle, the deformed algebra $U_q(\mathfrak{sl}_2)$ over $\mathbb{C}[q, q^{-1}, \frac{1}{q-q^{-1}}]$ is generated by E, F, K, K^{-1} with defining relations $KEK^{-1} = q^2 E$, $KFK^{-1} = q^{-2} F$, and $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$. For each a, the unique $(a+1)$-dimensional simple $U_q(\mathfrak{sl}_2)$ representation $\widetilde{V}_a$ is generated by a highest weight vector v_0. In a certain basis $\{v_i\}_{i=0}^a$, the operators E, F, and K act by $Ev_i = [a-i+1]v_{i-1}$, $Fv_i = [i+1]v_{i+1}$, and $Kv_i = q^{a-2i} v_i$, where $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$. As in the classical case, the representation $\widetilde{V}_a$ carries a Shapovalov form.

Proposition 2.2. *Under the Shapovalov form* $(,)$ *and the normalization* $(v_0, v_0) = 1$*, we have* $(v_i, v_i) = \binom{a}{i}_q = \frac{\prod_{j=1}^{a}[j]}{\prod_{j=1}^{i}[j]\cdot\prod_{j=1}^{a-i}[j]}$.

From Hopf algebra theory, we know that any tensor product $\bigotimes_{i=1}^{n} \widetilde{V}_{a_i}$ can be made into a representation of $U_q(\mathfrak{sl}_2)$ under the iteration of the coproduct map defined by $\Delta(E) = E\otimes 1+K\otimes E$, $\Delta(F) = 1\otimes F+F\otimes K^{-1}$, and $\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}$. Furthermore, the Shapovalov forms and the Drinfeld coboundary induce a Hermitan form on the representation $\bigotimes_{i=1}^{n} \widetilde{V}_{a_i}$. This tensor product can be uniquely decomposed into a sum of finite dimensional simple $U_q(\mathfrak{sl}_2)$ representations tensored with multiplicity spaces, where each multiplicity space carries an induced Hermitian form. We outline the proof of the Theorem 1.3 signature formula for the induced forms of these multiplicity spaces in Section 5

3 Classification of definite multiplicity spaces

In this section we use signature characters to prove the Theorem 1.1 definite multiplicity space classification for the decomposition of an arbitrary tensor product

$$M_{\lambda_1} \otimes M_{\lambda_2} \otimes \cdots \otimes M_{\lambda_n}.$$

Throughout this section, let $\lambda = \sum_{i=1}^{n} \lambda_i$, and assume the λ_i's are reals in decreasing order, with the first p of them positive ($0 \leq p \leq n$). Refer to the tensor product $\bigotimes_{i=1}^{n} M_{\lambda_i}$ as the *original tensor*. By the formula for signature characters, we have that the signatures of the multiplicity spaces depend only on the values $\lfloor\lambda\rfloor, \lfloor\lambda_1\rfloor, ..., \lfloor\lambda_n\rfloor$. Thus, we define the *explicit type* of the original tensor to be $\langle\lfloor\lambda\rfloor, \lfloor\lambda_1\rfloor, ..., \lfloor\lambda_n\rfloor\rangle$ and the *implicit type* to be $\langle\lfloor\lambda_1\rfloor, ..., \lfloor\lambda_n\rfloor\rangle$. We first classify definite spaces in the $p = 0$, $n = 2$, and $n = 3$ cases, and then apply those results to solve the $n \geq 4$ case. Then we tie together our results to prove the classification given by Theorem 1.1

3.1 Case: $p = 0$

Take $0 > \lambda_1 > \cdots > \lambda_n$.

Theorem 3.1. *In the decomposition*

$$\bigotimes_{i=1}^{n} M_{\lambda_i} \cong \bigoplus_{m=0}^{\infty} M_{\lambda-2m} \otimes E_m,$$

each multiplicity space E_m *is definite. The even-level spaces are positive definite and the odd-level spaces are negative definite.*

Proof. The proof of this classification is a standard counting argument. □

3.2 Case: $n = 2$

Take arbitrary $\lambda_1 > \lambda_2$, and define a function g as in Appendix A.1.

Theorem 3.2. *In the decomposition*

$$M_{\lambda_1} \otimes M_{\lambda_2} \cong \bigoplus_{m=0}^{\infty} M_{\lambda-2m} \otimes E_m,$$

every multiplicity space is definite. The level m space is positive definite if $g(\lambda_1, \lambda_2, m)$ is positive and negative definite if $g(\lambda_1, \lambda_2, m)$ is negative.

Proof. See Appendix B. □

3.3 Case: $p = 1$, $n = 3$

Take $\lambda_1 > 0 > \lambda_2 \geq \lambda_3$.

Theorem 3.3. *In the decomposition*

$$M_{\lambda_1} \otimes M_{\lambda_2} \otimes M_{\lambda_3} \cong \bigoplus_{m=0}^{\infty} M_{\lambda-2m} \otimes E_m,$$

we have the following classification for definite spaces. The definite spaces are exactly those with levels 0 through $\max\{0, \lceil \frac{\lambda}{2} \rceil\}$. The even-level spaces in this range are positive definite and the odd-level spaces in this range are negative definite.

Proof. See Appendix C. □

3.4 Case: $p = 2$, $n = 3$

Take $\lambda_1 \geq \lambda_2 > 0 > \lambda_3$.

Theorem 3.4. *In the decomposition*

$$M_{\lambda_1} \otimes M_{\lambda_2} \otimes M_{\lambda_3} \cong \bigoplus_{m=0}^{\infty} M_{\lambda-2m} \otimes E_m,$$

we have the following classification for definite spaces:

Case 1. $\lambda < 0$
There are $\lceil \lambda_2 + 1 \rceil$ definite spaces. They are positive definite and have levels 0 through $\lceil \lambda_1 \rceil$.

Case 2. $\lambda > 0$ *and* $\lceil \lambda + 1 \rceil \leq \lceil \lambda_2 \rceil$
There are $\lceil \lambda_2 \rceil - \lfloor \lambda \rfloor$ definite spaces. They are positive definite and they have levels 0 and $\lceil \lambda + 1 \rceil$ through $\lceil \lambda_2 \rceil$.

Case 3. $\lambda > 0$ *and* $\lceil \lambda + 1 \rceil > \lceil \lambda_2 \rceil$
Except for one exception, only the level 0 space is definite (it is positive definite). The exception is explicit type $\langle 1, 0, 0, -1 \rangle$, in which there is one other definite space: it is the level 2 space and it is negative definite.

Proof. See Appendix D. □

3.5 Case: $p = 3$, $n = 3$

Take $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$.

Theorem 3.5. *In the decomposition*

$$M_{\lambda_1} \otimes M_{\lambda_2} \otimes M_{\lambda_3} \cong \bigoplus_{m=0}^{\infty} M_{\lambda-2m} \otimes E_m,$$

we have the following classification for definite spaces. The spaces with level 0 *through* $\lceil \lambda_3 \rceil$ *are all positive definite. Except for the following exceptions, these are the only definite spaces.*

Exceptions:

- *For* $d \geq 0$ *and explicit type* $\langle 3d, d, d, d \rangle$*, the level* $2d+1$ *and* $2d+2$ *spaces are positive definite.*
- *For* $d \geq 0$ *and explicit type* $\langle 3d+2, d, d, d \rangle$*, the level* $2d+2$ *and* $2d+3$ *spaces are negative definite.*
- *For* $d \geq 1$ *and explicit type* $\langle 3d-1, d, d, d-1 \rangle$*, the level* $2d+1$ *space is positive definite.*
- *For* $d \geq 1$ *and explicit type* $\langle 3d+1, d, d, d-1 \rangle$*, the level* $2d+2$ *space is negative definite.*
- *For* $d \geq 1$ *and explicit type* $\langle 3d-2, d, d-1, d-1 \rangle$*, the level* $2d$ *space is positive definite.*
- *For* $d \geq 1$ *and explicit type* $\langle 3d, d, d-1, d-1 \rangle$*, the level* $2d+1$ *space is negative definite.*

Proof. See Appendix E. □

3.6 Case: $p \geq 1$, $n \geq 4$

Theorem 3.6. *In the decomposition*

$$\bigotimes_{i=1}^{n} M_{\lambda_i} \cong \bigoplus_{m=0}^{\infty} M_{\lambda-2m} \otimes E_m,$$

we have the following classification for definite spaces.

Case 1. $p = 1$

There are $1+\max\{0, \lceil \frac{\lambda}{2} \rceil\}$ *definite spaces. They have levels 0 through* $\max\{0, \lceil \frac{\lambda}{2} \rceil\}$*. In this range, the even-level spaces are positive definite and the odd-level spaces are negative definite.*

Case 2. $2 \leq p \leq n-2$

There is one definite space. It is the level 0 space and it is positive definite.

Case 3. $p = n-1$

- *If* $\lambda < 0$*, there are* $\lceil \lambda_p + 1 \rceil$ *definite spaces. They are positive definite and have levels* 0 *through* $\lceil \lambda_p \rceil$*.*
- *If* $\lambda > 0$ *and* $\lceil \lambda + 1 \rceil \leq \lceil \lambda_p \rceil$*, then there are* $\lceil \lambda_p \rceil - \lfloor \lambda \rfloor$ *definite spaces. They are positive definite and they have levels* 0 *and* $\lceil \lambda + 1 \rceil$ *through* $\lceil \lambda_p \rceil$*.*
- *If* $\lambda > 0$ *and* $\lceil \lambda + 1 \rceil > \lceil \lambda_p \rceil$*, then there is one definite space. It is the level* 0 *space and it is positive definite.*

Case 4. $p = n$

Except for the exceptions listed below, there are $\lceil \lambda_p + 1 \rceil$ *definite spaces. They have levels 0 through* $\lceil \lambda_p \rceil$ *and they are positive definite.*

Exceptions:

- *For any* $p \geq 4$*, explicit type* $\langle 0, 0, 0, ..., 0 \rangle$ *has 3 definite spaces. They are the level 0,1,2 spaces and they are positive definite.*

- *For any $p \geq 4$, explicit type $\langle 1, 1, 0, ..., 0\rangle$ has 3 definite spaces. They are the level 0,1,2 spaces and they are positive definite.*
- *For $p = 4$, explicit type $\langle 3, 0, 0, 0, 0\rangle$ has 3 definite spaces. They are the level 0 and 1 spaces (which are positive definite) and the level 3 space (which is negative definite).*
- *For $p = 4$, explicit type $\langle 4, 1, 1, 1, 1\rangle$ has 4 definite spaces. They are the level 0,1,2,4 spaces and they are positive definite.*

Proof. See Appendix F. □

3.7 Proof of Theorem 1.1

Refer to the classification list in Appendix A. Theorem 3.1 gives the classification for Case 1. Theorem 3.2 gives the classification for Case 2. Theorem 3.3 gives the classification for Case 3. Theorem 3.6 gives the classification for Case 4. Theorems 3.4 and 3.6 give the classification for Case 5. Theorems 3.5 and 3.6 give the classification for Case 6.

4 Critical point bound

In this section we use the Bethe ansatz method to prove the lower bound on the number of real critical points of the master function given by Theorem 1.2. Then we use the bound to derive the asymptotic approximation for the number of real critical points of the master function given by Corollary 1.1 . Throughout this section, fix a generic λ-sequence and a generic z-sequence, each in $\mathbb{R}^n$, and consider the master function $F_{z,\lambda,m}$

$$F_{z,\lambda,m}(t_1, t_2, ..., t_m) = \prod_{i<j}(t_i - t_j)^2 \cdot \prod_{i,k}(t_i - z_k)^{-\lambda_k}.$$

4.1 Preliminaries for the proof of Theorem 1.2

We begin with some definitions and preliminary results.

Definition. For any $X \in U(\mathfrak{sl}_2)$ and any i with $1 \leq i \leq n$, define an operator X_i on $\bigotimes_{j=1}^n M_{\lambda_j}$ by $X_i = \underbrace{1 \otimes \cdots \otimes 1}_{i-1 \text{ operators}} \otimes X \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-i \text{ operators}}$.

Definition. For each i and j with $1 \leq i \neq j \leq n$, define the *Casimir tensor* Ω_{ij} by $\Omega_{ij} = E_i \otimes F_j + F_i \otimes E_j + \frac{H_i \otimes H_j}{2}$.

Definition. For each i with $1 \leq i \leq n$, define the *Gaudin model Hamiltonian* $\mathcal{H}_i$ by $\mathcal{H}_i = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j}$.

Definition. For any $(t_1, ..., t_m) \in \mathbb{C}^m$, take the polynomial Q to be $Q(x) = \prod_{i=1}^m (x - t_i)$.

Definition. For any complex t with $t \neq z_i$, define an operator $Z(t) = \sum_{i=1}^n \frac{H_i}{t - z_i}$.

Definition. For any complex t with $t \neq z_i$ for each i, define an operator $Y(t)$ by $Y(t) = \sum_{i=1}^n \frac{F_i}{t - z_i}$.

Denote the highest weight vector of each M_{λ_i} by v_i, and denote $v = \bigotimes_i v_i$.

Definition. For any complex critical point $(s_1, s_2, ..., s_m)$ of $F_{z,\lambda,m}$, define a vector b_Q by $b_Q = Y(s_1)Y(s_2)\cdots Y(s_m)v$.

We are now ready to begin with preliminary results. For the proofs of these preliminary results, we use arguments similar to those developed in [5].

Lemma 4.1. *The $\mathcal{H}_i$'s commute with each other, and they are Hermitian under the induced Hermitian form on* $\mathrm{Hom}(M_{(\sum_i \lambda_i)-2m}, \bigotimes_i M_{\lambda_i})$.

Proof. See Appendix G.1. □

Lemma 4.2. *We have $Eb_Q = 0$.*

Proof. See Appendix G.2. □

Lemma 4.3. *We have*

$$[Z(s_a), Y(s_b)] = \frac{2}{s_a - s_b}(Y(s_a) - Y(s_b)).$$

Proof. We write

$$\begin{aligned}
\frac{2}{s_a - s_b}(Y(s_a) - Y(s_b)) &= \frac{2}{s_a - s_b}\sum_{j=1}^{n}\left(\frac{F_j}{s_a - z_j} - \frac{F_j}{s_b - z_j}\right) \\
&= \frac{2}{s_a - s_b}\sum_{j=1}^{n}\left(\frac{F_j(s_b - s_a)}{(s_a - z_j)(s_b - z_j)}\right) \\
&= \sum_{j=1}^{n}\left(\frac{-2F_j}{(s_a - z_j)(s_b - z_j)}\right) \\
&= [Z(s_a), Y(s_b)],
\end{aligned}$$

where in the last line we have used the commutation relations $[H_j, F_j] = -2F_j$ and $[H_i, F_j] = 0$ ($i \neq j$). □

Lemma 4.4. *We have* $[\mathcal{H}_i, Y(s_1)Y(s_2)\cdots Y(s_m)]v = \frac{-\lambda_i Q'(z_i)}{Q(z_i)} b_Q$.

As a corollary of Lemma 4.4, we have that b_Q is an eigenvector of each $\mathcal{H}_i$ with eigenvalue $\frac{-\lambda_i Q'(z_i)}{Q(z_i)} + \left(\frac{\lambda_i}{2}\sum_{j\neq i}^{n}\frac{\lambda_j}{z_i - z_j}\right)$. (This is just the eigenvalue computed in Lemma 4.4 plus the $\mathcal{H}_i$-eigenvalue of v.)

Lemma 4.5 (known, see [6]). *The joint eigenvalues of the operators $\mathcal{H}_i$ each have multiplicity 1, and the eigenvectors of the form $Y(s_1)Y(s_2)\cdots Y(s_m)v$ are all the joint eigenvectors up to scalars.*

Lemma 4.6. *If the joint eigenvector b_Q has real joint eigenvalue, then the critical point $(s_1, ..., s_m)$ is real. That is, the corresponding Q-polynomial has real coefficients.*

Proof. See Appendix G.4. □

4.2 Proof of Theorem 1.2

We now tie together the results from Section 4.1 to prove the critical point bound. Since $Eb_Q = 0$ and $Hb_Q = (-2m + \sum_i \lambda_i)b_Q$, we have that b_Q corresponds uniquely to a vector in the space $\mathrm{Hom}(M_{(\sum_i \lambda_i)-2m}, \bigotimes_i M_{\lambda_i})$, which is isomorphic to the multiplicity space E_m in the decomposition of $\bigotimes_i M_{\lambda_i}$. Note that the Gaudin Hamiltonians $\mathcal{H}_i$ descend to commuting Hermitian operators on the space E_m. Thus, each real eigenvector of E_m gives rise to a real critical point of the master function $F_{z,\lambda,m}$. Since the absolute value of the signature of E_m is a lower bound for the number of real eigenvectors for the commuting Hermitian operators $\mathcal{H}_i$, it is also a lower bound for the number of real critical points. Hence $|\,\mathrm{sgn}(E_m)| \leq N_{z,\lambda,m}$.

4.3 Preliminaries for the proof of Corollary 1.1

Definition. Define β_n for integer n by

$$\beta_n = e^{-\epsilon}\beta_{n+\epsilon}$$

for any $0 < \epsilon < 1$. This is well-defined.

Definition. For any real μ, let β_μ^- be β_μ evaluated at $s = -1$.

Definition. For each nonnegative integer n, define a polynomial V_n by

$$V_n(x) = 1 + 2x + 2x^2 + \cdots + 2x^n.$$

We state the following results without proofs.

Lemma 4.7. *For an integer n, we have*

$$\beta_n^- = \begin{cases} \beta_{-1}^- \cdot e^{n+1} \cdot V_{n+1}(e^{-2}) & \textit{if } n \geq 0 \\ \beta_{-1}^- \cdot e^{n+1} & \textit{if } n < 0. \end{cases}$$

Lemma 4.8. *For any integer n_1 and any negative integer n_2, we have*

$$e^{n_1}\beta_{n_2} = \beta_{n_1+n_2}$$

if $n_1 + n_2 < 0$. Also, if $n_1 + n_2 \geq 0$, then $e^{n_1}\beta_{n_2}$ can be written as a finite sum of signature characters.

4.4 Proof of Corollary 1.1

Recall that $N_{z,\lambda,m}$ denotes the number of real critical points of $F_{z,\lambda,m}$. From Theorem 1.2, we know $|\operatorname{sgn}(E_m)| \leq N_m \leq \dim(E_m) = \binom{m+n-2}{n-2}$. We will show, for fixed generic λ_i's, that as m approaches infinity, the ratio

$$\frac{|\operatorname{sgn}(E_m)|}{\binom{m+n-2}{n-2}}$$

approaches 1. This will prove Corollary 1.1.

The tensor product $\pi = \bigotimes_{i=1}^n M_{\lambda_i}$ has signature character equal to $\prod_{i=1}^n \beta_{\lambda_i}$. We need to examine the signatures of the multiplicity spaces E_m in the decomposition of π, which amounts to examining the coefficients of $\beta_{\lambda-2m}^-$ in the decomposition of $\prod_{i=1}^n \beta_{\lambda_i}^-$. We have

$$\begin{aligned}
\prod_{i=1}^n \beta_{\lambda_i}^- &= e^{\sum_{i=1}^n \{\lambda_i\}} \cdot \prod_{i=1}^n \beta_{\lfloor \lambda_i \rfloor}^- \\
&= e^{\sum_{i=1}^n \{\lambda_i\}} \cdot \left(\prod_{i=1}^p \beta_{-1}^- \cdot e^{\lceil \lambda_i \rceil} \cdot V_{\lceil \lambda_i \rceil}(e^{-2}) \right) \cdot \left(\prod_{i=p+1}^n \beta_{-1}^- \cdot e^{\lceil \lambda_i \rceil} \right) \\
&= e^{\sum_{i=1}^n \lceil \lambda_i \rceil + \{\lambda_i\}} \cdot \left(\beta_{-1}^-\right)^n \cdot \left(\prod_{i=1}^p V_{\lceil \lambda_i \rceil}(e^{-2}) \right) \\
&= e^{\sum_{i=1}^n \lceil \lambda_i \rceil + \{\lambda_i\}} \cdot \left(\sum_{m=0}^\infty \beta_{-n-2m}^- \cdot (-1)^m \cdot \binom{m+n-2}{n-2} \right) \cdot \left(\prod_{i=1}^p V_{\lceil \lambda_i \rceil}(e^{-2}) \right).
\end{aligned}$$

Combining the above computation with Lemma 4.8, we obtain that for all sufficiently large m,

$$\operatorname{sgn}(E_m) \cdot (-1)^m = \sum_{i=0}^{T} \binom{m+n-2-i}{n-2} \cdot (-1)^i \cdot c_i, \tag{1}$$

where $T = \sum_{i=1}^{p} \lceil \lambda_i \rceil$ and the c_i's are defined by the polynomial identity

$$\prod_{i=1}^{p} V_{\lceil \lambda_i \rceil}(x) = \sum_{i=0}^{T} c_i x^i.$$

The RHS in (1) is a polynomial in m of degree $n-2$. Its leading term is

$$\frac{m^{n-2}}{(n-2)!} \cdot (c_0 - c_1 + \cdots + (-1)^T c_T).$$

We will show that this leading term is nonzero and compute it explicitly. We have $\sum_{i=0}^{T}(-1)^i c_i = \prod_{i=1}^{p} V_{\lceil \lambda_i \rceil}(-1)$. But $V_{\lceil \lambda_i \rceil}(-1)$ is equal to $+1$ or -1 for each i, so $\sum_{i=0}^{T}(-1)^i c_i$ also equals $+1$ or -1. Therefore, we obtain that $\pm\frac{m^{n-2}}{(n-2)!}$ is the leading term of a polynomial which equals $\operatorname{sgn}(E_m) \cdot (-1)^m$ for all sufficiently large m. We also know that $\frac{m^{n-2}}{(n-2)!}$ is the leading term of the polynomial in m defined by the binomial coefficient $\binom{m+n-2}{n-2}$. Hence

$$\lim_{m \to \infty} \frac{|\operatorname{sgn}(E_m)|}{\binom{m+n-2}{n-2}} = 1$$

as desired.

5 Signatures for the quantum group case

In this section we prove the signature formula given by Theorem 1.3. Throughout this section, fix a generic q on the complex unit circle and nonnegative integers a and b. We start by proving some preliminary results.

5.1 Preliminaries for the proof of Theorem 1.3

Fix highest weight vectors v_0 and w_0 in the simple representations $\widetilde{V}_a$ and $\widetilde{V}_b$ of $U_q(\mathfrak{sl}_2)$, respectively. We know that the tensor product $\widetilde{V}_a \otimes \widetilde{V}_b$ admits a Shapovalov form (this form is defined using an R-matrix). As a consequence of the Clebsch-Gordan formula, the tensor product decomposes as

$$\widetilde{V}_a \otimes \widetilde{V}_b \cong \bigoplus_{m=0}^{\min\{a,b\}} \widetilde{V}_{a+b-2m} \otimes \widetilde{E}_m,$$

where each multiplicity space has dimension 1 and an associated scalar. The sign of this scalar ($+1$ or -1) is the signature of the multiplicity space. For each subrepresentation $\widetilde{V}_{a+b-2m} \subset \widetilde{V}_a \otimes \widetilde{V}_b$, there is a unique highest weight vector of the form

$$v_0 \otimes w_m + \sum_{i=1}^{m} c_{m,i} \cdot v_i \otimes w_{m-i},$$

where $c_{m,i}$ are scalars. We will call the highest weight vector determined by these scalars the *unit-normalized* highest weight vector. Given the unit-normalized highest weight vector u of $\widetilde{V}_{a+b-2m} \subset \widetilde{V}_a \otimes \widetilde{V}_b$, define the *R-normalized* highest weight vector u' of the twisted subrepresentation $\widetilde{V}_{a+b-2m} \subset \widetilde{V}_b \otimes \widetilde{V}_a$ by $u' = TRu$, where T is the twist operator and R is the universal R-matrix

$$R = q^{\frac{H \otimes H}{2}} \sum_{n \geq 0} q^{\binom{n}{2}} \cdot \frac{(q - q^{-1})^n}{[n]!} F^n \otimes E^n.$$

In the next two lemmas we explicitly compute the scalars for the unit-normalized and R-normalized highest weight vectors.

Lemma 5.1. *In the subrepresentation $\widetilde{V}_{a+b-2m} \subset \widetilde{V}_a \otimes \widetilde{V}_b$, the unit-normalized highest weight vector u is*

$$\sum_{i=0}^{m} c_{m,i} \cdot v_i \otimes w_{m-i},$$

where

$$c_{m,i} = (-1)^i \cdot q^{ai - i^2 + i} \cdot \frac{\binom{b-m+i}{i}_q}{\binom{a}{i}_q}.$$

Proof. We have $c_{m,0} = 1$ by definition. Write

$$u = \sum_{i=0}^{m} c_{m,i} \cdot v_i \otimes w_{m-i}.$$

Since u is a highest weight vector, it is killed by the operator E:

$$0 = \Delta(E)u = (E \otimes 1 + K \otimes E)u = \sum_{i=1}^{m} c_{m,i} \cdot Ev_i \otimes w_{m-i} + c_{m,i-1} \cdot Kv_{i-1} \otimes Ew_{m-i+1}.$$

Next,

$$Ev_i = [a - i + 1] \cdot v_{i-1},$$

$$Kv_{i-1} = q^{a-2i+2} \cdot v_{i-1},$$

$$Ew_{m-i+1} = [b - m + i] \cdot w_{m-i}.$$

Using the construction of tensor product bases, we obtain:

$$[a - i + 1]c_{m,i} \cdot v_{i-1} \otimes w_{m-i} = -c_{m,i-1} \cdot q^{a-2i+2}[b - m + i] \cdot v_{i-1} \otimes w_{m-i}$$

so

$$c_{m,i} = (-1) \cdot c_{m,i-1} \cdot \frac{[b - m + i]}{[a - i + 1]}.$$

Solving this recursion with the initial condition $c_{m,0} = 1$, we get

$$c_{m,i} = (-1)^i \cdot q^{ai - i^2 + i} \cdot \frac{\binom{b-m+i}{i}_q}{\binom{a}{i}_q}$$

as desired. □

Lemma 5.2. *In the twisted subrepresentation $\widetilde{V}_{a+b-2m} \subset \widetilde{V}_b \otimes \widetilde{V}_b$, the R-normalized highest weight vector u' is*

$$\sum_{i=0}^{m} c'_{m,i} \cdot v_i \otimes w_m,$$

where

$$c'_{m,0} = (-1)^m \cdot q^{0.5ab-am-bm+m^2-m} \cdot \frac{\binom{b}{m}_q}{\binom{a}{m}_q}$$

and

$$c'_{m,i} = c'_{m,0} \cdot (-1)^i \cdot q^{bi-i^2+i} \cdot \frac{\binom{a-m+i}{i}_q}{\binom{b}{i}_q}.$$

Proof. Since the R-normalized highest weight vector of the twisted subrepresentation $\widetilde{V}_{a+b-2m}$ is a scalar multiple of the unit-normalized highest weight vector of the twisted subrepresentation $\widetilde{V}_{a+b-2m}$, it suffices to show that $c'_{m,0}$ is the right value. Using the operation of F, we get that

$$TRu = q^{a(b-2m)/2} \cdot w_m \otimes v_0 + \sum_{i=0}^{m-1} c'_{m,i} \cdot w_i \otimes v_{m-i}.$$

Hence, we have the equation

$$c'_{m,0} \cdot (-1)^m \cdot q^{bm-m^2+m} \cdot \frac{\binom{a}{m}_q}{\binom{b}{m}_q} = q^{0.5ab-am},$$

from which the lemma follows. $\square$

5.2 Proof of Theorem 1.3

Consider the decomposition with multiplicity spaces which preserves the Shapovalov form

$$\widetilde{V}_a \otimes \widetilde{V}_b \cong \bigoplus_{m=0}^{\min\{a,b\}} \widetilde{V}_{a+b-2m} \otimes \widetilde{E}_m.$$

To prove Theorem 1.3, it suffices to show that the signature of each $\widetilde{E}_m$ is equal to the sign ($+1$ or -1) of $\binom{a}{m}_q \binom{b}{m}_q \binom{a+b+1-m}{m}_q$. (Iterating the two-factors formula gives the expression in Theorem 1.3.) Fix an m. From the Clebsch-Gordan formula, we know that $\dim(\widetilde{E}_m) = 1$. Thus, the signature $\operatorname{sgn}(\widetilde{E}_m)$ is equal to $+1$ or -1. From R-matrix theory, we know that the $\operatorname{sgn}(\widetilde{E}_m)$ equals the sign of (u, u'), where u is the unit-normalized highest weight vector of the subrepresentation $\widetilde{V}_{a+b-2m}$ and u' is the R-normalized highest weight vector of the twisted subrepresentation $\widetilde{V}_{a+b-2m}$. By Lemmas 5.1 and 5.2, we have $u = \sum_{i=0}^{m} c_{m,i} v_i \otimes w_{m-i}$ and $u' = \sum_{i=0}^{m} c'_{m,i} w_i \otimes v_{m-i}$. From the orthogonality of the canonical bases under the Shapovalov form, we get that in (u,u'), the $c_{m,i}$ term in u interacts only with the $c'_{m,m-i}$ term in u'. We have

$$\begin{aligned}
(c_{m,i}v_i \otimes w_{m-i}, c'_{m,m-i}w_{m-i}\otimes v_i) &= c_{m,i}\overline{c}'_{m,m-i}\cdot (v_i,v_i)\cdot (w_{m-i},w_{m-i})\\
&= c_{m,i}\overline{c}'_{m,m-i}\cdot \binom{a}{i}_q\binom{b}{m-i}_q\\
&= (-1)^m\cdot \overline{c}'_{m,0}\cdot q^{-bm+m^2-m+ai+bi-2mi+2i}\cdot \frac{\binom{b-m+i}{i}_q\binom{a-i}{m-i}_q}{\binom{a}{i}_q\binom{b}{m-i}_q}\cdot \binom{a}{i}_q\binom{b}{m-i}_q\\
&= (-1)^m\cdot \overline{c}'_{m,0}\cdot q^{-bm+m^2-m}\cdot q^{(a+b+2-2m)i}\cdot \binom{b-m+i}{i}_q\binom{a-i}{m-i}_q.
\end{aligned}$$

Hence (u,u') is

$$(u,u') = (-1)^m\cdot \overline{c}'_{m,0}\cdot q^{-bm+m^2-m}\cdot \sum_{i=0}^m q^{(a+b+2-2m)i}\cdot \binom{b-m+i}{i}_q\binom{a-i}{m-i}_q.$$

Using the relation $\binom{l_1}{l_2}_q = (-1)^{l_2}\binom{l_2-l_1-1}{l_2}$ and the quantum Vandermonde identity, we get:

$$\begin{aligned}
(u,u') &= (-1)^m\cdot \overline{c}'_{m,0}\cdot q^{-bm+m^2-m}\cdot \sum_{i=0}^m q^{(a+b+2-2m)i}\cdot \binom{b-m+i}{i}_q\binom{a-i}{m-i}_q\\
&= q^{-bm+m^2-m}\cdot \overline{c}'_{m,0}\cdot \sum_{i=0}^m q^{(a+b+2-2m)i}\cdot \binom{m-b-1}{i}_q\binom{m-a-1}{m-i}_q\\
&= \overline{c}'_{m,0}\binom{2m-a-b-2}{m}_q\\
&= \overline{c}'_{m,0}\cdot (-1)^m\cdot \binom{a+b+1-m}{m}_q\\
&= q^{-0.5ab+ma+mb-m^2+m}\cdot \frac{\binom{a+b+1-m}{m}_q\binom{b}{m}_q}{\binom{a}{m}_q}.
\end{aligned}$$

Dropping the powers of q and taking the signs of the binomial coefficients gives Theorem 1.3.

6 Conclusion and further work

In this paper, we studied tensor product decompositions of $\mathfrak{sl}_2$ Verma modules and their connections to other problems. We completely classified definite multiplicity spaces in arbitrary tensor products, used multiplicity space signatures to prove a critical point bound for generic master functions, and explicitly computed multiplicity space signatures for tensor products of q-deformed finite dimensional simple representations.

There are many possible directions for future work. The existence of a Shapovalov form-respecting decomposition holds for tensor products of Verma modules over any Kac-Moody algebras. Thus, we can ask for a definite multiplicity space classification for any Kac-Moody algebra. We expect that the answers to these questions will give classifications of unitary representation families for more quantized quiver varieties.

In addition, it would be interesting to further explore our real critical point bound. Specifically, we are interested in how tight this bound is. Based on computer tests, we have found that the bound is relatively good for many λ and z sequences, in the sense that the ratio $\frac{N_m}{|\operatorname{sgn}(E_m)|}$ is usually close to 1. When m is small, however, the bound breaks down at least in some special cases. For example, we have proven the following results for $m = 2$ (the two-variable master function case).

Theorem 6.1. *For any $n > 3$, there exists some λ-sequence of length n such that for any z-sequence, we have the strict inequality*

$$|\operatorname{sgn}(E_2)| < N_2.$$

Theorem 6.2. *For n an even square, there exists some λ-sequence of length n such that for any z-sequence, we have $|\operatorname{sgn}(E_2)| = 0$ and $N_2 \geq \frac{n-\sqrt{n}}{2}$.*

It would be interesting to derive quantitative estimates, for general m, on how unusual it is for $|\operatorname{sgn}(E_m)|$ to be significantly less than N_m.

Finally, there are several interesting problems related to multiplicity space signatures for quantum groups. The problem of determining multiplicity space signatures in decompositions of tensor powers $\widetilde{V}_a^{\otimes n}$ arises in physical settings. For the $a = 1$ case, the multiplicity space signatures have been computed. However, for $a = 2$ and higher, these signatures have yet to be computed. It would also be interesting to study the relationship between multiplicity space signatures in tensor products of $U_q(\mathfrak{sl}_2)$ representations and real critical points of q-deformed master functions. We expect that these signatures will give rise to a real critical point bound for the q-deformed master function, as in the standard case.

7 Acknowledgements

I thank Mr. Gus Lonergan of the Massachusetts Institute of Technology for his invaluable guidance throughout this work. I thank Professor Pavel Etingof of MIT for his helpful conversations and the suggestion of using Mukhin and Tarasov's Bethe ansatz. I express gratitude to Dr. Tanya Khovanova and Dr. John Rickert for editing this paper. Lastly, I thank the Center for Excellence in Education, Massachusetts Institute of Technology, Research Science Institute, MIT-PRIMES program, and my sponsors for giving me the opportunity to pursue this research.

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A Classification List

Consider an arbitrary Verma module tensor product

$$\pi = M_{\lambda_1} \otimes M_{\lambda_2} \otimes \cdots \otimes M_{\lambda_n}.$$

Assume the λ_i's are in decreasing order, with the first p of them positive (here $0 \leq p \leq n$). Define $\lambda = \sum_{i=1}^{n} \lambda_i$. Define the *explicit type* of the tensor product π to be the sequence $\langle \lfloor \lambda \rfloor, \lfloor \lambda_1 \rfloor, \lfloor \lambda_2 \rfloor, ..., \lfloor \lambda_n \rfloor \rangle$. Define a function g as in Subsection A.1. Then we have the following classification for definite multiplicity spaces in the decomposition of π:

Classification List. *In the decomposition of π, the level m multiplicity space has dimension $\binom{m+n-2}{n-2}$ for each m. We have the following classification for definite multiplicity spaces in the decomposition:*

Case 1. $p = 0$

Every space is definite. The even-level spaces are positive definite and the odd-level spaces are negative definite.

Case 2. $n = 2$

Every space is definite. The level m space is positive definite if $g(\lambda_1, \lambda_2, m)$ is positive and negative definite if $g(\lambda_1, \lambda_2, m)$ is negative.

Case 3. $p = 1, n \geq 3$

There are $\max\{1, \lceil \frac{\lambda+2}{2} \rceil\}$ definite spaces. They have levels 0 through $\max\{0, \lceil \frac{\lambda}{2} \rceil\}$. In this range, the even-level spaces are positive definite and the odd-level spaces are negative definite.

Case 4. $2 \leq p \leq n-2, n \geq 4$

There is one definite space. It is the level 0 space and it is positive definite.

Case 5. $p = n-1, n \geq 3$

a. *If $\lambda < 0$, there are $\lceil \lambda_p + 1 \rceil$ definite spaces. They are positive definite and have levels 0 through $\lceil \lambda_p \rceil$.*

b. *If $\lambda > 0$ and $\lceil \lambda + 1 \rceil \leq \lceil \lambda_p \rceil$, then there are $\lceil \lambda_p \rceil - \lfloor \lambda \rfloor$ other definite spaces. They are positive definite and they have levels 0 and $\lceil \lambda + 1 \rceil$ through $\lceil \lambda_p \rceil$.*

b. *If $\lambda > 0$ and $\lceil \lambda + 1 \rceil > \lceil \lambda_p \rceil$, then except for one exception only the level 0 space is definite (it is positive definite). The exception is explicit type $\langle 1, 0, 0, -1 \rangle$, in which there is one other definite space: it is the level 2 space and it is negative definite.*

Case 6. $p = n, n \geq 3$

The first $\lceil \lambda_p + 1 \rceil$ spaces (levels 0 through $\lceil \lambda_p \rceil$) are positive definite. Except for the following exceptions, there are no other definite spaces.

- *For explicit type $\langle 3d, d, d, d \rangle$ where $d \geq 0$, the level $2d+1$ and $2d+2$ spaces are positive definite.*
- *For explicit type $\langle 3d+2, d, d, d \rangle$ where $d \geq 0$, the level $2d+2$ and $2d+3$ spaces are negative definite.*
- *For explicit type $\langle 3d-1, d, d, d-1 \rangle$ where $d \geq 1$, the level $2d+1$ space is positive definite.*
- *For explicit type $\langle 3d+1, d, d, d-1 \rangle$ where $d \geq 1$, the level $2d+2$ space is negative definite.*
- *For explicit type $\langle 3d-2, d, d-1, d-1 \rangle$ where $d \geq 1$, the level $2d$ space is positive definite.*
- *For explicit type $\langle 3d, d, d-1, d-1 \rangle$ where $d \geq 1$, the level $2d+1$ space is negative definite.*

– *For explicit type* $\langle 3,0,0,0,0\rangle$*, the level* 3 *space is negative definite.*
– *For explicit type* $\langle 4,1,1,1,1\rangle$*, the level* 4 *space is positive definite.*
– *For explicit type* $\langle 0,0,0,...,0\rangle$ *where* $p \geq 4$*, the level* 2 *space is positive definite.*
– *For explicit type* $\langle 1,1,0,...,0\rangle$ *where* $p \geq 4$*, the level* 2 *space is positive definite.*

A.1 Definitions

Define a function sign by $\operatorname{sign}(x) = x/|x|$. Define a function g on a triple of two nonintegral reals $x_1 > x_2$ and a nonnegative integer k as follows:

If $0 > x_1 > x_2$ then

$$g(x_1, x_2, k) = (-1)^k$$

If $x_1 > 0$, $x_2 < 0$, $x_1 + x_2 < 0$, then

$$g(x_1, x_2, k) = \operatorname{sign}\left[\binom{x_1}{k}\right]$$

If $x_1 > 0$, $x_2 < 0$, $x_1 + x_2 > 0$ then

$$g(x_1,x_2,k) \quad = \quad \left\{\begin{array}{rr} (-1)^k & : 0 \leq k \leq \lfloor \frac{x_1+x_2}{2} \rfloor \\ (-1)^{\lceil \frac{x_1+x_2}{2} \rceil} & : \lceil \frac{x_1+x_2}{2} \rceil \leq k \leq \lfloor \frac{x_1+x_2+1}{2} \rfloor \\ (-1)^{\lceil \frac{x_1+x_2}{2} \rceil + \lceil \frac{x_1+x_2+1}{2} \rceil + k} & : \lceil \frac{x_1+x_2+1}{2} \rceil \leq k \leq \lceil x_1 + x_2 \rceil \\ 1 & : \lceil x_1 + x_2 \rceil + 1 \leq k \leq \lceil x_1 \rceil \\ (-1)^{\lceil x_1 \rceil + k} & : \lceil x_1 \rceil + 1 \leq k \end{array}\right\}$$

If $x_1 > x_2 > 0$, then

$$g(x_1,x_2,k) \quad = \quad \left\{\begin{array}{rr} 1 & : 0 \leq k \leq \lfloor x_2 \rfloor \\ g(x_1, x_2 - 2\lceil x_2 \rceil, k - \lceil x_2 \rceil) & : \lceil x_2 \rceil \leq k \leq \max(\lceil x_2 \rceil, \lfloor x_1 \rfloor) \\ g(x_1, x_2 - 2\lceil x_2 \rceil, k - \lceil x_2 \rceil) & \\ +2\lfloor x_1 \rfloor + 2\lfloor x_2 \rfloor - 2\lfloor x_1 + x_2 \rfloor & : \max(\lceil x_2 \rceil, \lfloor x_1 \rfloor) + 1 \leq k \leq \lceil x_1 \rceil + \lceil x_2 \rceil \\ (-1)^{k - \lceil x_1 \rceil - \lceil x_2 \rceil} & : \lceil x_1 \rceil + \lceil x_2 \rceil + 1 \leq k \end{array}\right\}$$

B Proof of Theorem 3.2

Since every multiplicity space has dimension 1, every multiplicity space is definite. We just need to determine which spaces are positive definite and which are negative definite, which we do using the following result (Lemma B.1):

Definition. Let β_μ^- be β_μ evaluated at $s = -1$.

Definition. Let sign be a function on the nonzero reals defined by $\operatorname{sign}(x) = x/|x|$.

Lemma B.1. *We have*

$$e^\mu = \beta_\mu^- - \operatorname{sign}(\mu)\beta_{\mu-2}^- + (\operatorname{sign}(1-\mu) - 1)\beta_{\mu - 2\lceil \mu \rceil}^-.$$

Proof. Write out expression on the right hand side for each of the intervals $\mu > 1$, $1 > \mu > 0$, and $0 > \mu$. □

We divide into cases based on the number of positive highest weights.

B.1 Proof of Theorem 3.2 for $p = 0$

This case is straightforward by the $p = 0$ classification already computed in Theorem 3.1.

B.2 Proof of Theorem 3.2 for $p = 1$

We only treat the case when $\lfloor \lambda_1 \rfloor$ is even, as the $\lfloor \lambda_1 \rfloor$ odd case is almost identical. The decomposition for the $\lambda_1 + \lambda_2 < 0$ follows easily from the relation $e^t \beta_\mu = \beta_{t+\mu}$ for $\mu < 0$, $t + \mu < 0$. The other case is that $\lambda_1 + \lambda_2 > 0$. In this case, we have

$$\begin{aligned}\beta^-_{\lambda_1}\beta^-_{\lambda_2} &= (e^{\lambda_1} + e^{\lambda_1 - 2} + \cdots + e^{\lambda_1 - 2\lfloor \lambda_1 \rfloor} + \beta^-_{\lambda_1 - 2\lceil \lambda_1 \rceil})\beta^-_{\lambda_2} \\ &= (e^{\lambda_1} + e^{\lambda_1 - 2} + \cdots + e^{\lambda_1 - 2\lfloor \lambda_1 \rfloor})\beta^-_{\lambda_2} + \sum_{m \geq \lceil \lambda_1 \rceil}^{\infty} \beta^-_{\lambda - 2m} \cdot (-1)^{m + \lceil \lambda_1 \rceil},\end{aligned}$$

where in the last line we have used the $p = 0$ classification. Expanding the product $(e^{\lambda_1} + e^{\lambda_1 - 2} + \cdots + e^{\lambda_1 - 2\lfloor \lambda_1 \rfloor})\beta^-_{\lambda_2}$, we obtain

$$\beta^-_{\lambda_1}\beta^-_{\lambda_2} = e^{\lambda} + e^{\lambda - 2} + \cdots + e^{\lambda - 2\lfloor \lambda_1 \rfloor} + \sum_{m = \lceil \lambda_1 + 1 \rceil}^{\infty} \beta^-_{\lambda - 2m} \cdot (-1)^{m+1}.$$

Carefully rewriting the powers of e as signature characters using Lemma B.1 gives the decomposition in Theorem 3.2.

B.3 Proof of Theorem 3.2 for $p = 2$

For any positive real t, write $L_t = e^t + e^{t-2} + \cdots + e^{t - 2\lfloor t \rfloor}$. We have

$$\begin{aligned}\beta^-_{\lambda_1}\beta^-_{\lambda_2} &= (L_{\lambda_1} + \beta^-_{\lambda_1 - 2\lceil \lambda_1 \rceil})(L_{\lambda_2} + \beta^-_{\lambda_2 - 2\lceil \lambda_2 \rceil}) \\ &= \beta^-_{\lambda_1}\beta^-_{\lambda_2 - 2\lceil \lambda_2 \rceil} + \beta^-_{\lambda_2}\beta^-_{\lambda_1 - 2\lceil \lambda_1 \rceil} - \beta^-_{\lambda_1 - 2\lceil \lambda_1 \rceil}\beta^-_{\lambda_2 - 2\lceil \lambda_2 \rceil} + L_{\lambda_1} L_{\lambda_2},\end{aligned}$$

and

$$\begin{aligned}L_{\lambda_1} L_{\lambda_2} = & \sum_{m=0}^{\lfloor \lambda_2 \rfloor} (m+1) e^{\lambda - 2m} \\ & + \sum_{m = \lceil \lambda_2 \rceil}^{\lfloor \lambda_1 \rfloor} \lceil \lambda_2 \rceil e^{\lambda - 2m} \\ & + \sum_{m = \lceil \lambda_1 \rceil}^{\lceil \lambda_1 \rceil + \lceil \lambda_2 \rceil} (\lfloor \lambda_1 \rfloor + \lfloor \lambda_2 \rfloor - m + 1) e^{\lambda - 2m}.\end{aligned}$$

The classifications for $p \leq 1$ and $n = 2$ give a closed form for each of $\beta^-_{\lambda_1}\beta^-_{\lambda_2 - 2\lceil \lambda_2 \rceil}$, $\beta^-_{\lambda_2}\beta^-_{\lambda_1 - 2\lceil \lambda_1 \rceil}$, and $\beta^-_{\lambda_1 - 2\lceil \lambda_1 \rceil}\beta^-_{\lambda_2 - 2\lceil \lambda_2 \rceil}$, as a sum of signature characters. Carefully rewriting the powers of e in $L_{\lambda_1} L_{\lambda_2}$ as signature characters using Lemma B.1 gives the decomposition in Theorem 3.2.

C Proof of Theorem 3.3

Our strategy is to identify a finite range of possible definite spaces and examine the spaces in the range using characters. We start with a lemma.

Lemma C.1. *No space with level more than $\lceil \frac{\lambda}{2} \rceil$ is definite.*

Proof (Lemma C.1). We address two cases, based on the sign of λ.

Case 1. $\lambda < 0$

We need to show that there are no definite spaces other than the level 0 positive definite space. Note that $\beta_{\lambda_2} \cdot \beta_{\lambda_3}$ contains $\beta_{\lambda_2+\lambda_3} + s\beta_{\lambda_2+\lambda_3-2}$. Then from the $n = 2$ classification, it is clear from the product $\beta_{\lambda_1}(\beta_{\lambda_2+\lambda_3} + s\beta_{\lambda_2+\lambda_3-2})$, which is contained in $\beta_{\lambda_1}\beta_{\lambda_2}\beta_{\lambda_3}$, that no space other than the level 0 space is definite.

Case 2. $\lambda > 0$

From the $n = 2$ classification, the product $\beta_{\lambda_2}\beta_{\lambda_3}$ contains $\beta_{\lambda_2-2\lceil \frac{\lambda}{2} \rceil}\beta_{\lambda_3}$. Hence, $\beta_{\lambda_1}\beta_{\lambda_2}\beta_{\lambda_3}$ contains $\beta_{\lambda_1}\beta_{\lambda_2-2\lceil \frac{\lambda}{2} \rceil}\beta_{\lambda_3}$. Since

$$\lambda_1 + \lambda_2 - 2\lceil \frac{\lambda}{2} \rceil + \lambda_3 < 0,$$

the same argument as the previous subcase shows that no space with level more than $\lceil \frac{\lambda}{2} \rceil$ is definite, as desired.

□

From Lemma C.1, the only possible definite spaces are those with level between 0 and $\lceil \frac{\lambda}{2} \rceil$ (inclusive). Upon developing the product

$$\beta_{\lambda_1}\beta_{\lambda_2}\beta_{\lambda_3} = \beta_{\lambda_1}(\sum_{m=0}^{\infty} s^m \beta_{\lambda_2+\lambda_3-2m})$$

with repeated use of the $n = 2$ classification, we obtain that the even-level spaces in our range are positive definite and the odd-level spaces in our range are negative definite.

D Proof of Theorem 3.4

Our strategy is to identify a finite range of possible definite spaces and examine spaces in the range using characters. We divide into cases based on the sign of λ.

D.1 Proof of Theorem 3.4 for $\lambda < 0$

Lemma D.1. *No space with level greater than $\lceil \lambda_2 \rceil$ is definite.*

Proof. Since $\lambda_2 + \lambda_3 < 0$, the closed form for two tensors gives that the product of β_{λ_2} and β_{λ_3} contains the product of $\beta_{\lambda_2-2\lceil \lambda_2 \rceil}$ and β_{λ_3}. Hence, $\beta_{\lambda_1} \cdot \beta_{\lambda_2} \cdot \beta_{\lambda_3}$ contains $\beta_{\lambda_1} \cdot \beta_{\lambda_2-2\lceil \lambda_2 \rceil} \cdot \beta_{\lambda_3}$. By the partial closed form for three tensors we get the claim. □

Lemma D.2. *Every space with level at most $\lceil \lambda_2 \rceil$ is positive definite.*

Proof. From the closed form for two tensors, we obtain that in the decomposition of $M_{\lambda_1} \otimes M_{\lambda_2}$, the spaces with level at most $\lceil \lambda_2 \rceil$ are all positive definite. Applying the closed form for two tensors again, we get that if the following condition is satisfied for $0 \le k \le \lceil \lambda_2 - 1 \rceil$:

$$\lambda_1 + \lambda_2 - 2k - 2\lceil \lambda_1 + \lambda_2 - 2k \rceil \le \lambda_1 + \lambda_2 - 2\lceil \lambda_2 \rceil$$

then the claim holds. This is equivalent to:

$$\lceil \lambda_1 + \lambda_2 \rceil \ge 2\lceil \lambda_2 \rceil - 1$$

which is true. □

Proof of $\lambda < 0$ case of Theorem 3.4. Lemma D.1 shows that there are no definite spaces with level more than $\lceil \lambda_2 \rceil$. Lemma D.2 shows that all spaces with level at most $\lceil \lambda_2 \rceil$ are positive definite. Thus, the $\lambda < 0$ case of Theorem 3.4 holds. □

D.2 Proof of Theorem 3.4 for $\lambda > 0$

From the closed form for two tensors, the product $\beta_{\lambda_1} \cdot \beta_{\lambda_2} \cdot \beta_{\lambda_3}$ contains the product of $\beta_{\lambda_1 - 2\lceil \lambda_1 \rceil} \cdot \beta_{\lambda_2} \cdot \beta_{\lambda_3}$ if $\lceil \lambda_1 + \lambda_3 \rceil < \lceil \lambda_1 \rceil$, and it contains $\beta_{\lambda_1 - 2\lceil \lambda_1 + 1 \rceil} \cdot \beta_{\lambda_2} \cdot \beta_{\lambda_3}$ if $\lceil \lambda_1 + \lambda_3 \rceil = \lceil \lambda_1 \rceil$. From this we get that all definite spaces have levels at most $\lceil \lambda_1 + 1 \rceil$, so we only need to focus on this finite set of spaces. Our strategy here is to write $\beta_{\lambda_1} \cdot \beta_{\lambda_2} \cdot \beta_{\lambda_3}$ as a sum of powers of e and use Lemma B.1 (see Appendix B.2) to determine the definite spaces. We begin by writing out the e-decomposition for the first $\lceil \lambda_1 + 1 \rceil$ powers of e in $(\beta_{\lambda_1}\beta_{\lambda_2}\beta_{\lambda_3})^-$:

Lemma D.3. *If $k \le \lceil \lambda_2 \rceil$, then the coefficient of $e^{\lambda - 2k}$ in the $(\beta_{\lambda_1}\beta_{\lambda_2}\beta_{\lambda_3})^-$ is $\lceil \frac{k+1}{2} \rceil$. If $\lceil \lambda_1 \rceil \ge k > \lceil \lambda_2 \rceil$, then the coefficient of $e^{\lambda - 2k}$ is $\lceil \frac{k+1}{2} \rceil$ if $2|(k - \lceil \lambda_2 \rceil)$ and $\lceil \frac{k+1}{2} \rceil - (k - \lfloor \lambda_2 \rfloor)$ otherwise.*

Proof. Use the fact that each term of $e^{\lambda - 2k}$ in the final product arises from a sum of products of $e^{\lambda_1 - 2i_1}, e^{\lambda_2 - 2i_2}, e^{\lambda_3 - 2i_3}$ for $i_1 + i_2 + i_3 = k$. There are $\binom{k+2}{2}$ such products, and we can write them all out. If $k \le \lceil \lambda_2 \rceil$, these terms are easy to analyze. If $k > \lceil \lambda_2 \rceil$, we must write out four cases based on the parities of k and $\lceil \lambda_2 \rceil$. After writing down the answer in each case, it is easy to see that the description provided covers all cases. □

Corollary D.1. *Define a function r by $r(i) = 0$ if i is even and $r(i) = 1$ if i is odd. Define a function t by $t(i) = 1$ if i positive and $t(i) = 0$ if i nonpositive. Define a function f by*

$$f(\lambda_1, \lambda_2, k) = \lceil \frac{k+1}{2} \rceil - (k - \lfloor \lambda_2 \rfloor) \cdot r(k - \lceil \lambda_2 \rceil) \cdot t(k - \lceil \lambda_2 \rceil)$$

if $0 \le k \le \lceil \lambda_1 + 1 \rceil$ and $f(\lambda_1, \lambda_2, k) = 0$ if k is not in that range. Then an equivalent statement of Lemma D.3 is that the coefficient of $e^{\lambda - 2k}$ in the $(\beta_{\lambda_1}\beta_{\lambda_2}\beta_{\lambda_3})^-$ is $f(\lambda_1, \lambda_2, k)$ for $0 \le k \le \lceil \lambda_1 \rceil$.

Corollary D.2. *Using the same logic as in the proof of Lemma D.3, we get that the coefficient of $e^{\lambda - 2\lceil \lambda_1 + 1 \rceil}$ in the $(\beta_{\lambda_1}\beta_{\lambda_2}\beta_{\lambda_3})^-$ is $f(\lambda_1, \lambda_2, \lceil \lambda_1 + 1 \rceil) - 2$.*

We now have an explicit description of the original signature character product written as a sum of powers of e. We will now find the definite spaces (which have levels k between 0 and $\lceil \lambda_1 + 1 \rceil$), tackling the $k \le \lceil \lambda_1 \rceil$ and $k = \lceil \lambda_1 + 1 \rceil$ cases separately. We begin by solving the $k \le \lceil \lambda_1 \rceil$ case. Using Lemma B.1, the fact that the function $x \to x - 2\lceil x \rceil$ is cyclic, and a division of the interval $[0, \lceil \lambda_1 + 1 \rceil]$ into several intervals based on the signs of $\lambda - 2k$ and $2k + 1 - \lambda$, we get the following result (Lemma D.4):

Lemma D.4. *We have the following results on the regular signatures of the first $\lceil \lambda_1 + 1 \rceil$ spaces. (Checking positive/negative definiteness of each level k space is the same as checking when the regular signature is $k+1$ or $-k-1$.)*

- *If $0 \leq k \leq \lceil \frac{\lambda}{2} \rceil$, then the signature of the level k space is*

$$f(\lambda_1, \lambda_2, k) - f(\lambda_1, \lambda_2, k-1).$$

- *If $\lceil \frac{\lambda+2}{2} \rceil \leq k \leq min(\lceil \lambda_1 \rceil, \lceil \lambda \rceil)$, then the signature of the level k space is*

$$f(\lambda_1, \lambda_2, k) + f(\lambda_1, \lambda_2, k-1) - 2f(\lambda_1, \lambda_2, \lceil \lambda \rceil - k).$$

- *If $\lceil \lambda + 1 \rceil \leq k \leq \lceil \lambda_1 \rceil$, then the signature of the level k space is*

$$f(\lambda_1, \lambda_2, k) + f(\lambda_1, \lambda_2, k-1)$$

Lemma D.5. *If $1 \leq k \leq \lceil \lambda_1 \rceil$, then the level k space is positive definite iff $\lceil \lambda + 1 \rceil \leq k \leq \lceil \lambda_2 \rceil$. In particular if $\lceil \lambda_1 + 1 \rceil > \lceil \lambda_2 \rceil$, then the only positive definite space in the first $\lceil \lambda_1 + 1 \rceil$ spaces is the level 0 space.*

Proof. If $\lceil \lambda + 1 \rceil \leq k \leq \lceil \lambda_2 \rceil$, then Corollary D.1 and Lemma D.4 give that the level k space is definite. Now we show that no other $\lceil \lambda_1 \rceil \geq k > 0$ has level k space positive definite. If $1 \leq k \leq \lceil \frac{\lambda}{2} \rceil$, then using Lemma D.4, the maximum possible value for the signature of the level k space is

$$1 + (k-1) - \lfloor \lambda_2 \rfloor < k + 1.$$

If $\lceil \frac{\lambda+2}{2} \rceil \leq k \leq \min(\lambda_1, \lambda)$, then we must address several cases. If $k > \lceil \lambda_2 \rceil$, $\lceil \lambda \rceil - k \leq \lceil \lambda_2 \rceil$, or $2 | \lceil \lambda \rceil - k - \lceil \lambda_2 \rceil$, then some bounding shows that the signature of the level k space is less than $k+1$. The other possibility is that $k > \lceil \lambda_2 \rceil$, $\lceil \lambda \rceil - k > \lceil \lambda_2 \rceil$, and $2 | \lceil \lambda \rceil - k - \lceil \lambda_2 \rceil - 1$. In this case the signature of the level k space is equal to

$$k + 1 - 2\lceil \frac{\lambda - k + 1}{2} \rceil + 2\lceil \lambda - k \rceil - 2\lfloor \lambda_2 \rfloor.$$

This level k space is positive definite when

$$2\lceil \lambda - k \rceil - 2\lceil \frac{\lambda - k + 1}{2} \rceil = 2\lfloor \lambda_2 \rfloor.$$

Viewing the LHS as a function of k, we see that as k increases by 2, the LHS decreases by 2. Evaluating the LHS at $k = 0$ and $k = 1$ gives that the above equality holds only if k is equal to one of

$$\frac{\lceil \lambda - 1 \rceil}{2} - \lfloor \lambda_2 \rfloor$$

or

$$\frac{\lceil \lambda - 2 \rceil}{2} - \lfloor \lambda_2 \rfloor.$$

But k can't equal either of these as $k \geq \frac{\lceil \lambda + 2 \rceil}{2}$. So there are no positive definite spaces of level k when $\lceil \frac{\lambda+2}{2} \rceil \leq k \leq \min(\lambda_1, \lambda)$.

If $\lceil \lambda_2 + 1 \rceil \le k \le \lceil \lambda_1 \rceil$, then the maximum possible value of the signature of the level k space is

$$k + 1 - (k - \lfloor \lambda_2 \rfloor) < k + 1.$$

So, the only positive definite spaces are the ones in the given range. □

Lemma D.6. *There are no negative definite spaces with level at most $\lceil \lambda_1 \rceil$.*

Proof. Consider a level k space with $0 \le k \le \lceil \lambda_1 \rceil$. If $k \le \lceil \frac{\lambda}{2} \rceil$, then the minimum possible signature of the level k space is:

$$0 - (k - \lfloor \lambda_2 \rfloor) > -k - 1$$

If $\lceil \frac{\lambda+2}{2} \rceil \le k \le \min(\lceil \lambda_1 \rceil, \lceil \lambda \rceil)$, then the minimum possible value of the signature of the level k space is

$$\begin{aligned}(k+1) - 2\lceil \frac{\lambda - k + 1}{2} \rceil - k + \lfloor \lambda_2 \rfloor &\ge 1 - \lceil \lambda - k + 2 \rceil \\ &\ge k - 1 - \lceil \lambda \rceil \\ &\ge -k.\end{aligned}$$

If $\lceil \lambda + 1 \rceil \le k \le \lceil \lambda_1 \rceil$, then the minimum possible value of the signature of the level k space is

$$k + 1 - (k - \lfloor \lambda_2 \rfloor) > -k - 1.$$

So there are no negative definite spaces. □

Lemmas D.5 and D.6 classify definite spaces with level at most $\lceil \lambda_1 \rceil$. All that remains is to determine when the level $\lceil \lambda_1 + 1 \rceil$ space is definite.

Lemma D.7. *The only tensors with the level $\lceil \lambda_1 + 1 \rceil$ space definite are tensors with explicit type $\langle 1, 0, 0, -1 \rangle$. In this explicit type, the level 2 space is negative definite.*

Proof. First we check when the $\lceil \lambda_1 + 1 \rceil$ space is positive definite. As $\lceil \lambda \rceil - \lceil \lambda_1 \rceil \le \lceil \lambda_2 \rceil$ the maximum possible value of the signature of the level $\lceil \lambda_1 \rceil$ space is:

$$f(\lambda_1, \lambda_2, \lceil \lambda_1 + 1 \rceil) - 2 + f(\lambda_1, \lambda_2, \lceil \lambda \rceil) \le \lceil \lambda_1 \rceil < \lceil \lambda_1 + 2 \rceil.$$

So this space can never be positive definite. For negative definiteness, we have that the minimum possible value of the signature is:

$$f(\lambda_1, \lambda_2, \lceil \lambda_1 + 1 \rceil) - 2 + f(\lambda_1, \lambda_2, \lceil \lambda \rceil) - 2f(\lambda_1, \lambda_2, \lceil \lambda \rceil - \lceil \lambda_1 \rceil),$$

which is at least $\lceil \lambda_1 \rceil - \lceil \lambda \rceil - 2 + \lfloor \lambda_2 \rfloor$. For this space to be negative definite, then, we must have

$$-\lceil \lambda_1 \rceil \ge \lceil \lambda_1 \rceil - \lceil \lambda \rceil + \lfloor \lambda_2 \rfloor$$

Rearranging and using $\lceil \lambda \rceil \le \lceil \lambda_1 \rceil + \lceil \lambda_2 \rceil + 1$, we get that for this space to be negative definite, we must have $2 \ge \lceil \lambda_1 \rceil$. Checking the (finitely many) explicit types with this condition yields the given exception. □

Proof of $\lambda > 0$ subcase of Theorem 3.4. From initial observations, only spaces with level at most $\lceil \lambda_1 + 1 \rceil$ can be definite. Lemmas D.5 and D.6 classify definite spaces with level at most $\lceil \lambda_1 \rceil$, while Lemma D.7 shows when the level $\lceil \lambda_1 + 1 \rceil$ is definite. Pulling this claims together gives the $\lambda > 0$ subcase of Theorem 3.4. □

Subsections D.1 and D.2 together prove the classification in Theorem 3.4.

E Proof of Theorem 3.5

An easy corollary of the $n = 2$ classification is that the first $\lceil \lambda_3 + 1 \rceil$ multiplicity spaces are indeed positive definite. Thus, we need only classify multiplicity spaces with level greater than $\lceil \lambda_3 \rceil$, which we will call the exceptional multiplicity spaces. To do this, we first determine all possible exceptional spaces in Subsections E.1 and E.2. Then in Subsection E.3 we use character computations to determine which of those instances actually produce exceptional spaces.

We begin with some notation. Write $A = \lfloor \lambda_1 \rfloor$, $a = \lambda_1 - \lfloor \lambda_1 \rfloor$, and similarly define B, b and C, c for λ_2 and λ_3 respectively. Define $\epsilon = a + b + c$. We divide into subcases based on the floor of ϵ, which is between 0 and 2 (inclusive).

E.1 Possible exceptional multiplicity spaces when $\epsilon < 2$

Lemma E.1. *If the original tensor does not have explicit type $\langle 3d-1, d, d, d-1 \rangle$, $\langle 3d-2, d, d-1, d-1 \rangle$, $\langle 3d, d, d, d \rangle$, or $\langle 3d+1, d, d, d \rangle$, then it has no definite spaces with level more than $\lceil \lambda_3 \rceil$. If the original tensor has one of those types, then the possible exceptional definite spaces have levels $2d+1$ (first type), $2d$ (second type), $2d+1$ and $2d+2$ (third type), and $2d+2$ (fourth type).*

Proof. By "fudging" the fractional parts of λ_1, λ_2, λ_3, we can ensure that $b + c < 1$ while maintaining the original tensor's explicit type. If this condition is satisfied, than by the closed form for two tensors, the signature character product $\beta_{\lambda_1} \cdot \beta_{\lambda_2} \cdot \beta_{\lambda_3}$ contains $\beta_{\lambda_1} \cdot \beta_{\lambda_2} \cdot \beta_{\lambda_3 - 2\lceil \lambda_3 \rceil}$. From the closed form of two positive and one negative Vermas, we obtain that the only spaces with level more than $\lceil \lambda_3 \rceil$ that can be definite in the original tensors are spaces with level k, where k satisfies

$$\lceil \lambda_1 + \lambda_2 + \lambda_3 - 2\lceil \lambda_3 \rceil + 1 \rceil + \lceil \lambda_3 \rceil \leq k \leq \lceil \lambda_2 \rceil + \lceil \lambda_3 \rceil.$$

This inequality is equivalent to

$$\lceil A + B - C + \epsilon - 1 \rceil + \lceil \lambda_3 \rceil \leq k \leq B + 1 + \lceil \lambda_3 \rceil.$$

Using some case analysis and bounding, we obtain that the only possible exceptional definite spaces occur in types:

- Explicit type $\langle 3d-1, d, d, d-1 \rangle$, in which the level $2d+1$ space could be definite.
- Explicit type $\langle 3d-2, d, d-1, d-1 \rangle$, in which the level $2d$ space could be definite.
- Explicit type $\langle 3d, d, d, d \rangle$, in which the level $2d+1$ and $2d+2$ spaces could be definite.
- Explicit type $\langle 3d+1, d, d, d \rangle$, in which the level $2d+2$ space could be definite.

□

E.2 Possible exceptional multiplicity spaces when $2 < \epsilon < 3$

Lemma E.2. *If the original tensor does not have explicit type $\langle 3d, d, d-1, d-1\rangle$, $\langle 3d+2, d, d, d\rangle$, or $\langle 2d+d'+2, d, d, d'\rangle$ for $d > d'$, then it has no definite spaces with level more than $\lceil \lambda_3 \rceil$. If the original tensor has one of those types, then the possible exceptional definite spaces have levels $2d+1$ (first type), $2d+2$ and $2d+3$ (third type), and $d+d'+3$ (fourth type).*

Proof. We divide into cases based on whether $\lfloor \lambda_3 \rfloor = \lfloor \lambda_2 \rfloor$ or not.
Suppose $\lfloor \lambda_2 \rfloor = \lfloor \lambda_3 \rfloor$. Then from the closed form of two tensors, the signature character product $\beta_{\lambda_1} \cdot \beta_{\lambda_2} \cdot \beta_{\lambda_3}$ contains $\beta_{\lambda_1} \cdot \beta_{\lambda_2 - 2\lceil \lambda_3 \rceil - 2} \cdot \beta_{\lambda_3}$. By the definite space classification for negative/positive/positive products, the only possible exceptional definite spaces are those with level k where k satisfies

$$\lceil \lambda_1 + \lambda_2 + \lambda_3 - 2\lceil \lambda_2 \rceil - 2 + 1 \rceil + \lceil \lambda_3 + 1 \rceil \le k \le \lceil \lambda_3 \rceil + \lceil \lambda_3 + 1 \rceil.$$

This inequality is equivalent to

$$A + \lceil \lambda_3 + 1 \rceil \le k \le C + 1 + \lceil \lambda_3 + 1 \rceil.$$

Using some case analysis and bounding, we get that the only possible exceptional definite spaces occur in

- $\langle 3d, d, d-1, d-1\rangle$, in which the level $2d+1$ space could be definite.
- $\langle 3d+2, d, d, d\rangle$, in which the level $2d+2$ and $2d+3$ spaces could be definite.

The other possibility is that $\lfloor \lambda_3 \rfloor < \lfloor \lambda_2 \rfloor$. Using the closed form for two tensors, the signature character product $\beta_{\lambda_1} \cdot \beta_{\lambda_2} \cdot \beta_{\lambda_3}$ contains $\beta_{\lambda_1} \cdot \beta_{\lambda_2 - 2\lceil \lambda_2 \rceil - 1} \cdot \beta_{\lambda_3 + 1}$. Using the definite space classification for negative/positive/positive products, we get that the only possible exceptional definite spaces are spaces with level k where k satisfies

$$\lceil \lambda_1 + \lambda_2 + \lambda_3 - 2\lceil \lambda_2 \rceil + 1 \rceil + \lceil \lambda_2 \rceil \le k \le \lceil \lambda_3 + 1 \rceil + \lceil \lambda_2 \rceil$$

Using some case analysis and bounding, we get the only possible exceptional definite spaces occur in $\langle 2d + d' + 2, d, d, d'\rangle$, in which the $d + d' + 3$ space could be definite. Here $d > d'$. ☐

E.3 Classification of exceptions

Here we will address the exceptions found in the previous two subsections and show that six of the seven potential families of exceptions are always exceptional, and the seventh family is never exceptional. We begin with the seventh family.

Lemma E.3. *In explicit type $\langle 3d+1, d, d, d\rangle$, the level $2d+2$ space is nondefinite.*

Proof. Fudge around the fractional parts of λ_1, λ_2, and λ_3, so that $a + c > 1$ and $b + c < 1$. (We can do this and maintain the explicit type of the original tensor.) By multiplying $\beta_{\lambda_1} \cdot \beta \lambda_3$ and using the closed form for two tensors, we get that the level $2d+2$ space is not positive definite. By multiplying the $\beta_{\lambda_2} \cdot \beta_{\lambda_3}$ and using the closed form for two tensors, we get that the level $2d+2$ space is not negative definite. We are done. ☐

Now we will show that the other six exceptional families are always exceptional, and we will determine the positivity/negativity of the exceptional definite spaces. Our strategy here is to write the signature character product as a sum of powers of e and use the following results along with Lemma B.1 to compute the regular signatures of the multiplicity spaces:

Definition. Define a function f_2 on an integer j and a real x by

$$f_2(x, j) = 0$$

if $j \leq \lceil x \rceil$ and

$$f_2(x, j) = (j - \lceil x \rceil - \lfloor \frac{j - \lceil x+1 \rceil}{2} \rfloor)(1 + \lfloor \frac{j - \lceil x+1 \rceil}{2} \rfloor)$$

if $j \geq \lceil x+1 \rceil$.

Lemma E.4. *For $k \leq \lceil \lambda_2 \rceil + \lceil \lambda_3 \rceil + 1$, the coefficient of $e^{\lambda - 2k}$ in the original signature character product (evaluated at $s = -1$) is*

$$\binom{k+2}{2} - 2f_2(\lambda_1, k) - 2f_2(\lambda_2, k) - 2f_2(\lambda_3, k)$$

Proof. Each term of $e^{\lambda-2k}$ arises from a product of $e^{\lambda_1 - 2i_1}$, $e^{\lambda_2 - 2i_2}$, and $e^{\lambda_3 - 2i_3}$, where $i_1 + i_2 + i_3 = k$. If all of these terms were 1, then by a standard counting argument, the coefficient would be $\binom{k+2}{2}$. However, some of the terms are -1. So if we can count the number of terms that are -1 and subtract twice that from $\binom{k+2}{2}$, we will be done.

A -1 arises as a product of $-1 \cdot 1 \cdot 1$ or $-1 \cdot -1 \cdot -1$. But since k is bounded by $\lceil \lambda_3 \rceil + \lceil \lambda_2 \rceil + 1$, the three -1's cannot happen. So we need only count the number of $-1, 1, 1$ triples.

We claim that the number of valid triples with a -1 coming from β_{λ_1} is $f_2(\lambda_1, k)$. To see this, count the number of solutions to:

$$2i_1' + i_2 + i_3 = k - \lceil \lambda_1 + 1 \rceil$$

using standard methods. By symmetry, the total number of valid triples is $f_2(\lambda_1, k) + f_2(\lambda_2, k) + f_2(\lambda_3, k)$. Subtracting twice this from our original count gives the result. □

Lemma E.5. *We have the following exceptional definite spaces:*

- *For $d \geq 0$ and explicit type $\langle 3d, d, d, d \rangle$, the level $2d+1$ and $2d+2$ spaces are positive definite.*
- *For $d \geq 0$ and explicit type $\langle 3d+2, d, d, d \rangle$, the level $2d+2$ and $2d+3$ spaces are negative definite.*
- *For $d \geq 1$ and explicit type $\langle 3d-1, d, d, d-1 \rangle$, the level $2d+1$ space is positive definite.*
- *For $d \geq 1$ and explicit type $\langle 3d+1, d, d, d-1 \rangle$, the level $2d+2$ space is negative definite.*
- *For $d \geq 1$ and explicit type $\langle 3d-2, d, d-1, d-1 \rangle$, the level $2d$ space is positive definite.*
- *For $d \geq 1$ and explicit type $\langle 3d, d, d-1, d-1 \rangle$, the level $2d+1$ space is negative definite.*

Proof. We will outline the proof for $\langle 3d, d, d, d \rangle$. The other cases are exactly the same.

For $\langle 3d, d, d, d \rangle$, we know the only possible exceptional definite spaces are levels $2d+1$ and $2d+2$. We claim that these are both positive definite. Based on whether d is odd or even, the claim that in $\langle 3d, d, d, d \rangle$, levels $2d+1$ and $2d+2$ are positive definite is by Lemmas B.1,E.4 and the cyclicity of the map $x \to x - 2\lceil x \rceil$, equivalent to two quadratic polynomials being identically 0. That is, if $d = 2a$ (respectively $d = 2a - 1$), then the statement that the level $2d+1$ and $2d+2$ spaces are positive definite is equivalent to a quadratic in a being identically 0. To check that these quadratics are 0 everywhere, we only need to check that they are 0 at three different points. Testing $d = 2, 4, 6$ and $d = 1, 3, 5$ verifies the claim. □

Lemma E.6. *In explicit type $\langle 2d+d'+2, d, d, d'\rangle$ where $d > d'$, the only potential exceptionally definite space (level $d+d'+2$) is definite exactly when $d' = d-1$. If it is definite then it is negative definite.*

Proof. Set $a = \lceil n/2 \rceil$ and $b = \lceil m/2 \rceil$. Using Lemmas B.1 and E.4 we get that the signature of the level $d+d'+2$ space is negative, so we only need to check when it is equal to $-d-d'-3$. If d and d' have oppositive parities, than using Lemmas B.1 and E.4 gives that this happens exactly when $d' = d-1$. If d and d' have the same parity, this happens either when $d = d'$ (impossible) or never. □

From the classification of exceptional multiplicity spaces proved in Subsection E.3, we obtain Theorem 3.5.

F Proof of Theorem 3.6

We start with some new definitions. Define a function Λ by $\Lambda(k) = \sum_{i=1}^{k} \lambda_i$. Let $\lambda_+ = \Lambda(p)$ and $\lambda_- = \lambda - \lambda_+$.

F.1 Proof of Theorem 3.6 for $p = 1$

The definite spaces in the decomposition of $\bigotimes_i M_{\lambda_i}$ are exactly the same as the definite spaces in the decomposition of $M_{\lambda_1} \otimes M_{\lambda_-/2} \otimes M_{\lambda_-/2}$. Applying the $p = 1$ and $n = 3$ classification proves the $p = 1$ case of Theorem 3.6.

F.2 Proof of Theorem 3.6 for $p \geq 2$, $n - p \geq 2$

Lemma F.1. *If $p \geq 2$, $n-p \geq 2$, and $|\lambda_+| > |\lambda_{p+1}|$, then there is exactly one definite space in the original tensor. It is the level 0 space and it is positive definite.*

Proof. When we multiply $\beta_{\lambda_1}, \beta_{\lambda_2}, ..., \beta_{\lambda_{p+1}}$, the product is:

$$e^{\lambda_{p+1}+\lambda_+} + (p+s)e^{\lambda_{p+1}+\lambda_+-2} = \beta_{\lambda_{p+1}+\lambda_+} + (p-1+s)\beta_{\lambda_{p+1}+\lambda_+-2}$$

Since the level 1 space is nondefinite, when we multiply by $\beta_{\lambda_{p+2}}$, we get that all spaces with level at least 1 are nondefinite. Thus, in the original tensor product, all spaces with level at least 1 are nondefinite. In the original tensor product, the level 0 space is positive definite. Our claim follows. □

Lemma F.2. *If $p \geq 1$, $n-p \geq 2$, and $|\lambda_-| > |\lambda_p|$, then there is exactly one definite space in the original tensor. It is the level 0 space and it is positive definite.*

Proof. The product of the $n-p-1$ signature characters $\beta_{\lambda_{p+2}}, ..., \beta_{\lambda_n}$ contains $\beta_{-\lambda_{p+1}+\lambda_-}$. So the product of $\beta_{\lambda_p}, \beta_{\lambda_{p+1}}, ...\beta_{\lambda_n}$ contains the product of $\beta_{\lambda_p}, \beta_{\lambda_{p+1}}, \beta_{-\lambda_{p+1}+\lambda_-}$. From the classification for $p = 1$ and $n = 3$ case, we deduce that there is exactly one definite space. That is the level 0 space which is positive definite. □

Lemma F.3. *If $p \geq 2$ and $n-p \geq 2$, then there is exactly one definite space in the original tensor. It is the level 0 space and it is positive definite.*

Proof. Assume the claim is false. Then by Lemma F.1, $\lambda_{p+1} + \lambda_+ < 0$. By Lemma F.2, $0 < \lambda_p + \lambda_-$. Adding these yields

$$\lambda_+ + \lambda_{p+1} < \lambda_p + \lambda_-$$

or

$$\sum_{i<p} \lambda_i < \sum_{i>p+1} \lambda_i$$

which is a contradiction as the LHS is positive and the RHS is negative. So our assumption was false and the claim holds. □

This proves the classification in Theorem 3.6 for the case $p \geq 2$ and $n - p \geq 2$.

F.3 Proof of Theorem 3.6 for $p = n - 1$

Lemma F.4. *If $|\Lambda(p-1)| > |\lambda_n|$, then there is exactly one definite space in the original tensor. It is the level 0 space and it is positive definite.*

Proof. We know $p \geq 3$ since $n \geq 4$. By the same logic as in the proof of Lemma F.1, in the tensor product of $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_{p-1}} \otimes M_{\lambda_n}$, the level 1 space is nondefinite. So when we tensor this with M_p, all the spaces with level at least 1 are nondefinite. The level 0 space is positive definite. The lemma follows. □

Lemma F.5. *If $|\Lambda(p-1)| < |\lambda_n|$, then no space after the level $\lceil \lambda_p \rceil$ is definite in the original tensor.*

Proof. The product of β_{λ_p} and β_{λ_n} contains the product of $\beta_{\lambda_p - 2\lceil \lambda_p \rceil}$ and β_{λ_n}. Thus, the product of all our n signature characters contains the product of $\beta_{\lambda_1}, ..., \beta_{\lambda_{p-1}}, \beta_{\lambda_p - 2\lceil \lambda_p \rceil}, \beta_{\lambda_n}$. Applying Lemma F.2 gives that in the original tensor product, no space after the level $\lceil \lambda_p \rceil$ space is definite. □

Lemma F.6. *If $|\Lambda(p-1)| < |\lambda_n| < |\Lambda(p)|$, then there are exactly $\lceil \lambda_p \rceil - \lfloor \lambda \rfloor$ definite spaces in the original tensor. They are all positive definite; one of them is level 0 and the others are levels $\lceil \lambda + 1 \rceil$ through $\lceil \lambda_p \rceil$.*

Proof. When we multiply $\beta_{\lambda_1}, ..., \beta_{\lambda_p}$, its easy to see that the first $\lceil \lambda_p \rceil$ signature characters in the sum decomposition are positive definite. (Use the fact that ss dont appear in the series for each β_x until $\lceil x + 1 \rceil$ when x is positive.) Using the fact that the product of $\beta_{\lambda_1}, ..., \beta_{\lambda_p}$ contains β_{λ_+} and $p\beta_{-2+\lambda_+}$, along with the classification for $n = 2$ and Lemma F.5, gives that the definite spaces in the original tensor product are positive and have levels 0 and $\lceil \lambda + 1 \rceil$ through $\lceil \lambda_p \rceil$. □

Lemma F.7. *If $|\lambda_n| > \Lambda(p)$, then there are exactly $\lceil \lambda_p + 1 \rceil$ definite spaces in the original tensor. They are all positive definite and they have levels 0 through $\lceil \lambda_p \rceil$.*

Proof. From Lemma F.5, its enough to show that spaces with level 0 to $\lceil \lambda_p \rceil$ are positive definite. When we multiply $\beta_{\lambda_1}, ..., \beta_{\lambda_p}$, its easy to see that the first $\lceil \lambda_p \rceil$ signature characters in the sum decomposition are positive definite. (Use the fact that ss dont appear in the series for each β_x until $\lceil x + 1 \rceil$ when x is positive.) Using this, along with the closed form for two tensors, gives that the first $\lceil \lambda_p + 1 \rceil$ spaces are definite in the original tensor product. □

Combining the results of this subsection gives the $p = n - 1$ case of Theorem 3.6. The form of the $p = n - 1$ classification stated in Theorem 3.6 is slightly different than the form of the classification we have proved here, but it can be shown to be equivalent.

F.4 Proof of Theorem 3.6 for $p = n$

Lemma F.8. *Suppose the original tensor does not satisfy either the following special conditions:*

- *$p > 4$, and the original tensor has implicit type either $\langle 1, 0, ..., 0\rangle$ or $\langle 0, 0, ..., 0\rangle$.*
- *$p = 4$, and the original tensor has $\lfloor \lambda_1 \rfloor \in \{0, 1\}$.*

Then the original tensor has exactly $\lceil \lambda_p + 1\rceil$ definite spaces. They are all positive definite and they have levels 0 through $\lceil \lambda_p \rceil$.

Proof. Since the original tensor contains $M_{\lambda_1} \otimes M_{\lambda_2} \otimes M_{\lambda_n}$ as a sub-tensor product, the $n = 3$ classification gives that the only possible implicit types with exceptional definite spaces are $\langle 0, 0, ..., 0\rangle$, $\langle 1, 1, ..., 1\rangle$, and $\langle 1, 0, ..., 0\rangle$. None of the (finitely many) explicit types for $n = 5$ that have implicit type $\langle 1, 1, ..., 1\rangle$ has exceptional definite spaces, so the claim is proven. □

Lemma F.9. *Suppose $p \geq 4$ and the original tensor has explicit type $\langle 0, 0, ..., 0\rangle$. Then it has exactly three definite spaces.*

Proof. Check that in a tensor of three Vermas of the original set of Vermas, the level 3 space has multiplicity signature $1 + 3s$. Thus no space after level 2 is definite in the original tensor. Check that levels 0,1,2 are definite. □

Lemma F.10. *Suppose $p > 4$ and the original tensor has implicit type $\langle 0, 0, ..., 0\rangle$. Then if $\lambda_+ < 1$, the original tensor has exactly 3 definite spaces: They are levels 0,1,2 and they are positive definite. If $\lambda_+ > 1$, then the original tensor has exactly 2 definite spaces: They are levels 0,1 and they are positive definite.*

Proof. If $\lambda_+ < 1$, then the claim follows from Lemma F.9. Otherwise suppose $\lambda_+ > 1$. Let i be minimal such that $\Lambda(i) > 1$. From assumptions, $2 \leq i \leq p$. If $2 < i < p$, then in the tensor of $\Delta_{\lambda_1}, ...\Delta_{\lambda_i}$, we can check that the level 2 space is nondefinite, so our claim holds for the original tensor. Otherwise we have $i = p$ or $i = 2$. If $i = p$, tensor the first $p - 1$ Vermas and apply Lemma F.9 and the classification for $n = 2$ to get the claim. If $i = 2$, check that all the (finitely many) explicit types of tensors for $n = 5$ which have $\lambda_+ > 1$ and implicit type $\langle 0, 0, 0, 0, 0\rangle$ have two definite spaces. So in all cases the claim holds. □

Lemma F.11. *Suppose $p > 4$ and the original tensor has implicit type $\langle 1, 0, ..., 0\rangle$. Then if $\lambda_+ < 2$, the original tensor has exactly 3 definite spaces: They are levels 0,1,2 and they are positive definite. If $\lambda_+ > 2$, then the original tensor has exactly 2 definite spaces: They are levels 0,1 and they are positive definite.*

Proof. Check that the claim holds for all (finitely many) explicit types of tensors which have $p = 5$ and implicit type $\langle 1, 0, .., 0\rangle$. For $p > 5$, tensoring the 5 Vermas with largest highest weights and using the result in the last sentence gives that no space after level 3 can be definite. Check that if $\lambda_+ < 2$ there are three definite spaces and if $\lambda_+ > 2$ there are two definite spaces as claimed. □

Lemma F.12. *Suppose $p = 4$. Then there are $\lceil \lambda_p + 1\rceil$ definite spaces in the original tensor as described before except for the following exceptions:*

- *Explicit type $\langle 0, 0, 0, 0, 0\rangle$, in which level 0,1,2 spaces are positive definite.*
- *Explicit type $\langle 3, 0, 0, 0, 0\rangle$, in which level 0,1 spaces are positive definite and level 3 space is negative definite.*
- *Explicit type $\langle 1, 1, 0, 0, 0\rangle$, in which level 0,1,2 spaces are positive definite.*

- *Explicit type* $\langle 4,1,1,1,1\rangle$*, in which the level 0,1,2,4 spaces are positive definite.*

Proof. From Lemma F.8, we only need to check the (finitely many) explicit types for which $\lambda_1 < 2$. $\square$

Combining the results in this subsection gives the $p = n$ case of Theorem 3.6.

G Proofs of preliminary results in Section 4

G.1 Proof of Lemma 4.1

We need to show that $[h_i, h_j] = 0$, where the bracket denotes the commutator map. We have

$$[h_i, h_j] = \sum_{k_1 \neq i, k_2 \neq j} \frac{\Omega_{ik_1}\Omega_{jk_2} - \Omega_{jk_2}\Omega_{ik_1}}{(z_i - z_{k_1})(z_j - z_{k_2})}$$

Next, $\Omega_{ik_1}\Omega_{jk_2} - \Omega_{jk_2}\Omega_{ik_1}$ equals 0 unless $k_1 = i$, $k_2 = j$, or $k_1 = k_2$. It is easy to check that these three are mutually exclusive. Thus, we have

$$[h_i, h_j] = \sum_{k \neq i,j} \frac{(z_i - z_k)([\Omega_{ik}, \Omega_{jk}]) + (z_j - z_k)([\Omega_{ik}\Omega_{ij}]) + (z_i - z_j)([\Omega_{ik}, \Omega_{jk}])}{(z_i - z_j)(z_i - z_k)(z_j - z_k)}$$

Using the relation $[\Omega_{xz}, \Omega_{yz}] = \sum_\sigma \operatorname{sgn}(\sigma) E_{\sigma(1)} \otimes F_{\sigma(2)} \otimes H_{\sigma(3)}$ (where σ runs over all six permutations of (x, y, z)), we obtain $[h_i, h_j] = 0$.

G.2 Proof of Lemma 4.2

By a combinatorial argument, we have

$$b_Q = \sum_{a_1+a_2+\ldots+a_n=m} \bigotimes_i F^{a_i} v_i \cdot \left(\sum_\sigma \frac{1}{\prod_{j=1}^m (s_j - z_{\sigma(j)})} \right),$$

where σ runs over all permuations of the list consisting of a_k k's. Then applying $E = E_1 + E_2 + \ldots E_n$ to b_Q and using the identity $EF^k = HF^{k-1} + FHF^{k-2} + \ldots F^{k-1}H$ gives

$$Eb_Q = \sum_{a_1+a_2+\ldots+a_n=m-1} \bigotimes_i F^{a_i} v_i \cdot \sum_{i=1}^n \sum_\sigma \frac{(a_i+1)(\lambda_i - a_i)}{\prod_{j=1}^m (s_j - z_{\sigma(j)})}, \tag{2}$$

where for each i, the sum over σ runs over all distinguishable permutations of the list consisting of a_k k's for $k \neq i$, and $a_i + 1$ i's. Fix an arbitrary n-tuple of nonnegative integers $(a_1, a_2, \ldots, a_n)$ with sum $m - 1$. Using the construction of a tensor product basis, we see that showing that the expression in equation (2) equals 0 is equivalent to showing the inner sum

$$\sum_{i=1}^n \sum_\sigma \frac{(a_i+1)(\lambda_i - a_i)}{\prod_{j=1}^m (s_j - z_{\sigma(j)})} \tag{3}$$

in (2) equals 0. Multiplying by $a_1!a_2!...a_n!$ gives that (3) being equal to 0 is equivalent to

$$\sum_{i=1}^{n}\sum_{\sigma\in S_m}\frac{\lambda_i}{\prod_j(s_j-z_{\sigma(j)})}=\sum_{i=1}^{n}\sum_{\sigma\in S_m}\frac{a_i}{\prod_j(s_j-z_{\sigma(j)})}, \tag{4}$$

where for each i, the permutation σ acts on the list $\{i,1,1,..,1,2,2,..,2,...,n,n,...n\}$. We will separately develop the LHS and RHS of (4) and show that they are equal, and then the theorem will be proven.
Start with the LHS. It is equal to

$$\sum_{i=1}^{n}\sum_{\sigma\in S_m}\frac{\lambda_i}{\prod_j(s_j-z_{\sigma(j)})}=\sum_{\sigma\in S_m}\frac{1}{\prod_{j\neq\sigma^{-1}(1)}(s_j-z_{\sigma(j)})}\cdot\sum_{i=1}^{n}\frac{\lambda_i}{s_{\sigma^{-1}(1)-z_i}}.$$

Since $(s_1,s_2,...,s_m)$ is a critical point, we know that

$$\sum_{i=1}^{n}\frac{\lambda_i}{s_{\sigma^{-1}(1)-z_i}}=\sum_{i\neq\sigma^{-1}(1)}\frac{2}{s_{\sigma^{-1}(1)}-s_i},$$

whence

$$\sum_{i=1}^{n}\sum_{\sigma\in S_m}\frac{\lambda_i}{\prod_j(s_j-z_{\sigma(j)})}=\sum_{\sigma\in S_m}\frac{1}{\prod_{j\neq\sigma^{-1}(1)}(s_j-z_{\sigma(j)})}\cdot\sum_{i\neq\sigma^{-1}(1)}\frac{2}{s_{\sigma^{-1}(1)}-s_i}. \tag{5}$$

Rewriting the RHS of (5) based on the value of $k:=\sigma^{-1}(1)$, we get

$$\sum_{i=1}^{n}\sum_{\sigma\in S_m}\frac{\lambda_i}{\prod_j(s_j-z_{\sigma(j)})}=\sum_{k=1}^{m}\sum_{\sigma\in S_{m-1}}\frac{1}{\prod_j(s_j-z_{\sigma(j)})}\cdot\left(\sum_{i\neq k}\frac{2}{s_k-s_i}\right), \tag{6}$$

where σ permutes all the z's except one. Note that the term $\left(\sum_{i\neq k}\frac{2}{s_k-s_i}\right)$ in the sum in the RHS of (6) is independent of everything else, so we can sum over expressions of the form $\frac{2}{s_x-s_y}$. Thus, (6) becomes

$$\sum_{i=1}^{n}\sum_{\sigma\in S_m}\frac{\lambda_i}{\prod_j(s_j-z_{\sigma(j)})} \tag{7}$$

$$=\sum_{k_1<k_2}\frac{2}{s_{k_1}-s_{k_2}}\cdot\left[\left(\sum_{\sigma\in S_{m-1}}\frac{1}{\prod_{j\neq k_1}(s_j-z_{\sigma(j)})}\right)-\left(\sum_{\sigma\in S_{m-1}}\frac{1}{\prod_{j\neq k_2}(s_j-z_{\sigma(j)})}\right)\right]. \tag{8}$$

The permutations in the two sums $\left(\sum_{\sigma \in S_{m-1}} \frac{1}{\prod_{j \neq k_1}(s_j - z_{\sigma(j)})}\right)$ and $\left(\sum_{\sigma \in S_{m-1}} \frac{1}{\prod_{j \neq k_2}(s_j - z_{\sigma(j)})}\right)$ can be summed in a way that allows for factorization and cancellation:

$$\begin{aligned}
&\sum_{k_1<k_2} \frac{2}{s_{k_1}-s_{k_2}} \cdot \left[\left(\sum_{\sigma \in S_{m-1}} \frac{1}{\prod_{j \neq k_1}(s_j - z_{\sigma(j)})}\right) - \left(\sum_{\sigma \in S_{m-1}} \frac{1}{\prod_{j \neq k_2}(s_j - z_{\sigma(j)})}\right)\right] \\
&= \sum_{k_1<k_2} \frac{2}{s_{k_1}-s_{k_2}} \cdot \sum_{j=1}^{n} \left(\frac{1}{s_{k_2}-z_j} - \frac{1}{s_{k_1}-z_j}\right) \cdot \sum_{\sigma \in S_{m-2}} \frac{1}{\prod_{j \neq k_1,k_2}(s_j - z_{\sigma(j)})} \\
&= \sum_{k_1<k_2} \frac{2}{s_{k_1}-s_{k_2}} \cdot \sum_{j=1}^{n} \left(\frac{s_{k_1}-s_{k_2}}{(s_{k_2}-z_j)(s_{k_1}-z_j)}\right) \cdot \sum_{\sigma \in S_{m-2}} \frac{1}{\prod_{j \neq k_1,k_2}(s_j - z_{\sigma(j)})} \\
&= \sum_{k_1<k_2} 2 \cdot \sum_{j=1}^{n} \left(\frac{1}{(s_{k_2}-z_j)(s_{k_1}-z_j)}\right) \cdot \sum_{\sigma \in S_{m-2}} \frac{1}{\prod_{j \neq k_1,k_2}(s_j - z_{\sigma(j)})} \\
&= \sum_{k_1<k_2} 2 \cdot \sum_{j=1}^{n} \left(\frac{1}{(s_{k_1}-z_j)}\right) \cdot \sum_{\sigma \in S_{m-1}} \frac{1}{\prod_{j \neq k_1}(s_j - z_{\sigma(j)})} \\
&= \sum_{k_1,k_2} \sum_{j=1}^{n} \left(\frac{1}{(s_{k_1}-z_j)}\right) \cdot \sum_{\sigma \in S_{m-1},\sigma(k_2)=j} \frac{1}{\prod_{j \neq k_1}(s_j - z_{\sigma(j)})} \\
&= \sum_{k_1} \sum_{j=1}^{n} \left(\frac{1}{(s_{k_1}-z_j)}\right) \sum_{k_2} \sum_{\sigma \in S_{m-1},\sigma(k_2)=j} \frac{1}{\prod_{j \neq k_1}(s_j - z_{\sigma(j)})}.
\end{aligned}$$

Since for every permutation $\sigma \in S_{m-1}$, there are exactly a_j values of k_2 for which $\sigma(k_2) = j$, we get that the LHS of (4) equals

$$\begin{aligned}
&\sum_{k_1} \sum_{j=1}^{n} \left(\frac{1}{(s_{k_1}-z_j)}\right) \sum_{k_2} \sum_{\sigma \in S_{m-1},\sigma(k_2)=j} \frac{1}{\prod_{j \neq k_1}(s_j - z_{\sigma(j)})} \\
&= \sum_{k_1} \sum_{j=1}^{n} \left(\frac{a_j}{(s_{k_1}-z_j)}\right) \sum_{\sigma \in S_{m-1}} \frac{1}{\prod_{j \neq k_1}(s_j - z_{\sigma(j)})}.
\end{aligned}$$

Next, we tackle the RHS of (4). Since the σ in the RHS of (4) acts on $\{i, 1, 1, ...1, 2, 2, ..., 2, ..., n, n, ..., n\}$, we can write

$$\sum_{i=1}^{n} \sum_{\sigma \in S_m} \frac{a_i}{\prod_j (s_j - z_{\sigma(j)})} = \sum_{\sigma \in S_m} \frac{1}{\prod_{j \neq \sigma^{-1}(1)}(s_j - z_{\sigma(j)})} \cdot \left(\sum_{i=1}^{n} \frac{a_i}{s_{\sigma^{-1}(1)} - z_i}\right) \tag{9}$$

Breaking the RHS of (9) into cases based on $k := \sigma^{-1}(1)$, we get

$$\sum_{i=1}^{n} \sum_{\sigma \in S_m} \frac{a_i}{\prod_j (s_j - z_{\sigma(j)})} = \sum_{k=1}^{m} \left(\sum_{\sigma \in S_{m-1}} \frac{1}{\prod_{j \neq k}(s_j - z_{\sigma(j)})}\right) \cdot \left(\sum_{i=1}^{n} \frac{a_i}{s_k - z_i}\right).$$

This is the same as the LHS of (4), so we are done.

G.3 Proof of Lemma 4.4

Fix $i = 1$ (we can do this since the Y's commute with each other), and write

$$\begin{aligned}
[h_1, Y_1Y_2...Y_m] &= (h_1Y_1Y_2...Y_m) - (Y_1Y_2...Y_mh_1) \\
&= (h_1Y_1Y_2...Y_m) - (Y_1h_1Y_2...Y_m) + (Y_1h_1Y_2...Y_m) - (Y_1Y_2...Y_mh_1) \\
&= [h_1, Y_1](Y_2...Y_m) + Y_1[h_1, Y_2Y_3...Y_m] \\
&= Y_1Y_2...Y_mH_1 - \frac{F_1}{s_1 - z_1}\Big(2\sum_{j=2}^{m}(\frac{1}{s_j - z_1}Y_1...\hat{Y_j}...Y_m) \\
&+ \hat{Y_1}Y_2...Y_mZ_1 + [Z_1, Y_2...Y_m]\Big) + Y_1[h_1, Y_2...Y_m].
\end{aligned}$$

Next, we manipulate the expression $[Z_1, Y_2...Y_m]$

$$\begin{aligned}
[Z_1, Y_2...Y_m] = &\frac{2}{s_1 - s_2}(Y_1 - Y_2)(Y_3...Y_m) \\
&+ Y_2(\frac{2}{s_1 - s_3})(Y_1 - Y_3)(Y_4...Y_m) \\
&... \\
&+ (Y_2...Y_{m-1})(\frac{2}{s_1 - s_m})(Y_1 - Y_m) \\
&= (\sum_{j=2}^{m}\frac{-2}{s_1 - s_j})(Y_2...Y_m) \\
&+ \sum_{j=2}^{m}\frac{2}{s_1 - s_j}Y_1...\hat{Y_j}...Y_m \\
&= 2\sum_{j=2}^{m}\frac{1}{s_1 - s_j}\big(Y_1...\hat{Y_j}...Y_m - \hat{Y_1}Y_2...Y_m\big).
\end{aligned}$$

We plug this back in to the relation for $[h_1, Y_1...Y_m]$ and obtain

$$\begin{aligned}
&[h_1, Y_1Y_2...Y_m] \\
&= Y_1Y_2...Y_mH_1 - \frac{F_1}{s_1 - z_1}\Big(2\sum_{j=2}^{m}(\frac{1}{s_j - z_1}Y_1...\hat{Y_j}...Y_m) + \hat{Y_1}Y_2...Y_mZ_1 \\
&+ 2\sum_{j=2}^{m}\frac{1}{s_1 - s_j}\big(Y_1...\hat{Y_j}...Y_m - \hat{Y_1}Y_2...Y_m)\Big) \\
&+ Y_1[h_1, Y_2...Y_m]
\end{aligned}$$

Iterating this type of expression with $Y_1[h_1, Y_2...Y_m]$ until there are no commutators, we obtain in the end:

$$
\begin{aligned}
[h_1, Y_1...Y_m] = \sum_{k=1}^{m} \Big(& \frac{1}{s_k - z_1} Y_1...Y_m H_1 \\
& - \frac{F_1}{s_k - z_1} \Big[2 \sum_{j=k+1}^{m} \big(\frac{1}{s_j - z_1} Y_1...\hat{Y}_j...Y_m \big) \\
& + (Y_1...\hat{Y}_k...Y_m) \sum_{i=1}^{n} \big(\frac{H_i}{s_k - z_1} \big) + 2 \sum_{j=k+1}^{m} \frac{1}{s_k - s_j} \big(Y_1...\hat{Y}_j...Y_m - Y_1...\hat{Y}_k...Y_m \big) \Big] \Big) \\
= \sum_{k=1}^{m} \Big(& \frac{1}{s_k - z_1} Y_1...Y_m H_1 \\
& - \frac{F_1}{s_k - z_1} \Big[2 \sum_{j=k+1}^{m} \big(\frac{1}{s_j - z_1} Y_1...\hat{Y}_j...Y_m \big) \\
& + (Y_1...\hat{Y}_k...Y_m Z_k) + 2 \sum_{j=k+1}^{m} \frac{1}{s_k - s_j} \big(Y_1...\hat{Y}_j...Y_m - Y_1...\hat{Y}_k...Y_m \big) \Big] \Big).
\end{aligned}
$$

The coefficient of each operator $Y_1...\hat{Y}_k...Y_m$ in the above sum is

$$
\sum_{j>k} \big(\frac{2}{(s_j - s_k)(s_k - z_1)} \big) + \sum_{j<k} \big(\frac{2}{(s_j - s_k)(s_j - z_1)} + \frac{2}{(s_j - z_1)(s_k - z_1)} \big) = \frac{2}{s_k - z_1} \sum_{j \neq k} \big(\frac{1}{s_j - s_k} \big).
$$

Thus

$$
\begin{aligned}
[h_1, Y_1...Y_m] = & \sum_{k=1}^{m} \frac{1}{s_k - z_1} Y_1...Y_m \\
& - F_1 \sum_{k=1}^{m} \frac{2}{s_k - z_1} \left(\sum_{j \neq k} \frac{1}{s_j - s_k} \right) Y_1...\hat{Y}_k...Y_m \\
& - F_1 \sum_{k=1}^{m} \frac{1}{s_j - z_1} Y_1...\hat{Y}_k...Y_m Z_k.
\end{aligned}
$$

We are now ready to compute $[h_1, Y_1...Y_m]v$. We have

$$
\begin{aligned}
[h_1, Y_1...Y_m]v = & \left(\sum_{k=1}^{m} \frac{1}{s_k - z_1} Y_1...Y_m \right) v \\
& - \left(F_1 \sum_{k=1}^{m} \frac{2}{s_k - z_1} \left(\sum_{j \neq k} \frac{1}{s_j - s_k} \right) Y_1...\hat{Y}_k...Y_m \right) v \\
& - \left(F_1 \sum_{k=1}^{m} \frac{1}{s_j - z_1} Y_1...\hat{Y}_k...Y_m Z_k \right) v.
\end{aligned}
$$

Finally, note that

$$\begin{aligned}\left(\sum_{k=1}^{m}\frac{1}{s_k - z_1}Y_1...Y_m\right)v &= \left(\sum_{k=1}^{m}\frac{\lambda_1}{s_k - z_1}\right)v \\ &= -\lambda_1 \cdot \frac{Q'(z_1)}{Q(z_1)}b_Q,\end{aligned}$$

while

$$\begin{aligned}&\left(\sum_{k=1}^{m}\frac{2}{s_k - z_1}\left(\sum_{j\neq k}\frac{1}{s_j - s_k}\right)Y_1...\hat{Y_k}...Y_m\right)v \\ &+\left(\sum_{k=1}^{m}\frac{1}{s_j - z_1}Y_1...\hat{Y_k}...Y_m Z_k\right)v \\ &=\left(\sum_{j\neq k}\left(\frac{2}{s_j - s_k}\right)+\left(\sum_{j=1}^{n}\frac{\lambda_j}{s_k - z_j}\right)\right)v \\ &= 0\end{aligned}$$

where in the last step we have used the fact that $(s_1, ...s_m)$ is a critical point. We conclude that

$$[h_i, Y_1Y_2...Y_m]v = \frac{-\lambda_i Q'(z_i)}{Q(z_i)}b_Q,$$

as desired.

G.4 Proof of Lemma 4.6

First, by combinatorial argument similar to the one used in the beginning of Lemma 4.2, we have

$$b_Q = \sum_{a_1+a_2...+a_n=m}\left(\bigotimes_i F^{a_i}v_i\right)\cdot\prod_{\sigma}\frac{1}{\prod_{k=1}^{m}(t_k - z_{\sigma(j)})}.$$

where σ runs over all distinguishable permuations consisting of a_i i's. By assumption, the coefficient of each $\bigotimes_i F^{a_i}v_i$ is real. We show by induction on k that the k^{th} derivative $Q^k(z_i)$ (here $0 \leq k \leq m$) is real for each i, whence we will be done.

Step 1. Base cases

First we show $Q(z_i)$ and $Q'(z_i)$ are real. Consider the ordered partition $p_0 = (m, 0, ..., 0)$ of m. From the coefficient of the term in b_Q indexed by p_0, we obtain that $Q(z_1)$ is real, so by symmetry all $Q(z_i)$'s are real. Next, consider the ordered partition $p_1 = (m-1, 1, ..., 0)$ of m. From coefficient of the term in b_Q indexed by p_1, we obtain that

$$\frac{1}{Q(z_1)}\cdot\sum_{k=1}^{m}\frac{t_k - z_1}{t_k - z_2}$$

is real, whence $\sum_{k=1}^{m} \frac{t_k - z_1}{t_k - z_2}$ is real. Subtracting 1 from each fraction gives that

$$\sum_{k=1}^{m} \frac{z_2 - z_1}{t_k - z_2}$$

is real, whence $Q'(z_2)$ (and by symmetry, all $Q'(z_i)$'s) is real.

Step 2. Induction
Assume that $Q(z_i), Q'(z_i), ..., Q^k(z_i)$ are all real for each i. We show that $Q^k(z_i)$ is real for each i. Consider the ordered partition $p_k = (m-k, k, 0, ..., 0)$ of m. From the coefficient of the term of b_Q indexed by p_k, we obtain that

$$\frac{1}{Q(z_1)} \cdot \sum_{k\text{-sets } S \text{ of } \{1,2,\ldots,m\}} \frac{\prod_{i\in S}(t_i - z_1)}{\prod_{i\in S}(t_i - z_2)}$$

Define a polynomial associated to each k-set S by $R_S(z) = \prod_{i\in S}(t_i - z)$. We know that

$$\sum_{k\text{-sets } S} \frac{R_S(z_1)}{R_S(z_2)}$$

is real. Using the relation

$$\frac{R_S(z_1) - R_S(z_2)}{z_1 - z_2} = R'_S(z_2) + (\frac{z_1 - z_2}{2!}) R''_S(z_2) + ...,$$

we get that

$$\sum_{k\text{-sets } S} \frac{R'_S(z_2) + \left(\frac{z_1 - z_2}{2!}\right) R''_S(z_2) + ...}{R_S(z_2)} = \sum_{k\text{-sets } S} \sum_{i\in S} \frac{1}{t_i - z_2} + (\frac{z_1 - z_2}{2!}) \sum_{i\neq j\in S} \frac{1}{(t_i - z_2)(t_j - z_2)} + ...$$

is real. When we sum over all k-sets S, by symmetry we get that for some positive integers $a_0, a_2, ..., a_k$, the quantity

$$\sum_{i=1}^{k} a_i \cdot \sum_{i\text{-sets } S \text{ of } \{1,2,\ldots,m\}} \frac{1}{\prod_{i\in S}(t_i - z_2)}$$

is an integer. The first $k-1$ summands in the sum are integer multiples of $Q(z_2), Q'(z_2), ..., Q^{k-1}(z_2)$, while the last summand is an integer multiple of $Q^k(z_2)$. By the inductive hypothesis, we get that $Q^k(z_2)$ (and by symmetry, all $Q^k(z_i)$'s) is real. The inductive step is complete.

Since the first m derivatives of Q evaluated at a real point (say, z_1) are all real, we get that Q has real coefficients as desired.