Properties Of Triangles When They Undergo The Curve-Shortening Flow

Asha Ramanujam

under the direction of
Mr. Ao Sun
Department of Mathematics
Massachusetts Institute of Technology

Research Science Institute
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Abstract

In this paper, we study the properties of triangles when they undergo the curve-shortening flow. We first take the case of the isosceles triangle and prove that the legs of the triangle always decrease and the vertex angle will decrease if it is lesser than $\frac{\pi}{3}$. Our main result is that there exists a time T such that $\alpha(T) = 0$ (where $\alpha(t)$ is the vertex angle at time t) and $x(T) > 0$ (where $x(t)$ is the legs of isosceles triangle at time t) when $0 \leq \alpha(0) < \frac{\pi}{3}$, showing that the triangle becomes a straight line rather than a single point. The equilaterel triangle turns out to be a self-shrinker and hence converges to a single point. We also find expressions for the change in side lengths and angles in any general triangle and prove that the length of each side reduces during the flow.

Summary

Geometric flow refers to the movement of geometric objects in space. The curve-shortening flow is an important type of flow in which all points of the curve move inwards. It is highly applied in physics and recent research in this field addresses classical problems in topology, differential equations and geometric analysis. Our aim in this paper, is to study triangles when they undergo this flow. We find that the length of the legs of an isosceles triangle decreases. We also find that the vertex angle of an isosceles triangle decreases if it is smaller than $\frac{\pi}{3}$ and ultimately, the isosceles triangle becomes a line segment. We find that the equilaterel triangle undergoes the flow without having any change in its angles, and eventually converges to a point. We also find that in any triangle, the length of each side decreases during the flow, and we can roughly estimate what happens to the angles of the triangle .

1 Introduction

In this project we study the curve-shortening flow (CSF), a process that modifies a smooth curve in the Euclidean plane by moving its points perpendicularly to the curve at a speed proportional to the curvature at this point. This process is of fundamental interest in the calculus of variations. Recent research on curve-shortening flow addresses classical problems in topology , differential equations and geometric analysis. Furthermore it provides a setting for formulating problems in physics. It covers concepts like the area form and total curvature.

The curve-shortening flow was originally studied as a model for the annealing of metal sheets. Later, it was applied in image analysis to give a multi-scale representation of shapes. It can also model reaction diffusion systems, and the behavior of cellular automata. The curve-shortening flow can be used to find closed geodesics on Riemannian manifolds, and as a model for the behavior of higher-dimensional flows [1].

The curve-shortening flow follows the equation

$$\frac{dx_i(t)}{dt} = -k(x_i(t))\vec{N}(x_i(t))$$

where $x_i(t)$ is a point on the curve on the curve moving at time t, $\vec{N}(x_i(t))$ is the normal of the curve at the point $x_i(t)$ and $k(x_i(t))$ is the curvature of the curve at the point $x_i(t)$.

The Gage-Hamilton-Grayson theorem states how smooth curves evolve under the curve-shortening flow [2]. It states that, if a smooth simple closed curve undergoes the curve-shortening flow, it remains smoothly embedded without self-intersections. It will eventually becomes convex, and once it does so, it will remain convex. After this time, all points of the curve will move inwards, and the shape of the curve will converge to a circle as the whole curve shrinks to a single point (Figure 1).

This behavior is sometimes summarized by saying that every smooth simple closed curve shrinks to a round point. Gage proved the convergence to a circle for convex curves that

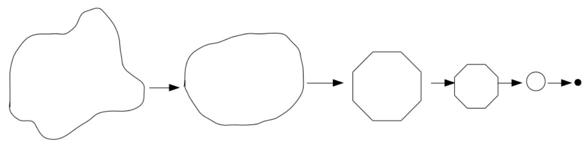

Figure 1: Evolution of triangle under CSF

contract to a point. Gage and Hamilton proved that all smooth convex curves eventually contract to a point, and Grayson proved that every non-convex curve will eventually become convex [1].

In this paper, we will study how the triangles evolve under the curve-shortening flow. In Section 2 we will present some definitions and notations involved through out the paper. We investigate the behaviour of the legs and the vertex angle of an isosceles triangle under the curve-shortening flow in Section 3, and eventually what happens to it. In Section 4 we will make some calculations for any triangle and formulate some observations based on some graphical evidence.

We find that the length of the legs of an isosceles triangle always decreases. We also find that the vertex angle decreases if it is lesser that $\frac{\pi}{3}$ and remains the same if it is $\frac{\pi}{3}$. Surprisingly, we show that unlike smooth curves, in particular for isosceles triangles, if the vertex angle is lesser that $\frac{\pi}{3}$, the triangle ultimately becomes a short line segment and not a single point. We also find that in any triangle the length of each side reduces during the curve-shortening flow. In the end we come up with some conjectures, based on some computation and graphical evidence.

2 Notations and Definitions

Let there be a triangle ABC which will undergo the curve-shortening flow. (Figure 2)

We will use the following notations in the paper.

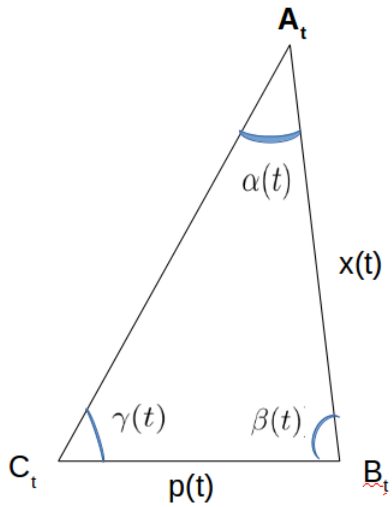

Figure 2: an arbitary triangle at time t

At any time t, A_t, B_t and C_t denote the flowed vertices of A, B and C, respectively. The lengths of the sides A_tB_t and B_tC_t are denoted by $x\,(t)$ and $p\,(t)$, respectively at any time t. For convinience, we assume C_0 to be the origin and B_0C_0 to be the x-$axis$ of the Cartesian coordinate system. The angles at the flowed vertices at any time t, are denoted by $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ respectively.

Definition 1. *Curvature at any vertex is defined as* $\pi-$*(angle at that vertex).*

Definition 2. *The normal at any vertex is defined as its angle bisector.*

Definition 3. *Self–shrinker is a figure which undergoes the curve-shortening flow without changing its angles. In other words* $\alpha\,(t) = \alpha\,(0)$, $\beta\,(t) = \beta\,(0)$, $\gamma\,(t) = \gamma\,(0)$, *for any time* t.

3 Isosceles triangle

Let us consider an isosceles triangle ABC with $AB = AC$.

According to our assumptions,

$A_0:\left(x\left(0\right)\sin\left(\frac{\alpha(0)}{2}\right),x\left(0\right)\cos\left(\frac{\alpha(0)}{2}\right)\right),$
$B_0:\left(2x\left(0\right)\sin\left(\frac{\alpha(0)}{2}\right),0\right).$

When it undergoes the curve-shortening flow in an infinitesimally small time Δt the following transformation (Figure 3) takes place:

$A_\Delta t$: $\left(x\left(0\right)\sin\left(\frac{\alpha(0)}{2}\right),x\left(0\right)\cos\left(\frac{\alpha(0)}{2}\right)-\left(\pi-\alpha\left(0\right)\right)\Delta t\right),$
$C_\Delta t$: $\left(\left(\frac{\pi}{2}+\frac{\alpha(0)}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha(0)}{4}\right)\Delta t,\left(\frac{\pi}{2}+\frac{\alpha(0)}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\alpha(0)}{4}\right)\Delta t\right),$
$B_\Delta t$: $\left(2x\left(0\right)\sin\left(\frac{\alpha(0)}{2}\right)-\left(\frac{\pi}{2}+\frac{\alpha(0)}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha(0)}{4}\right)\Delta t,\left(\frac{\pi}{2}+\frac{\alpha(0)}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\alpha(0)}{4}\right)\Delta t\right).$

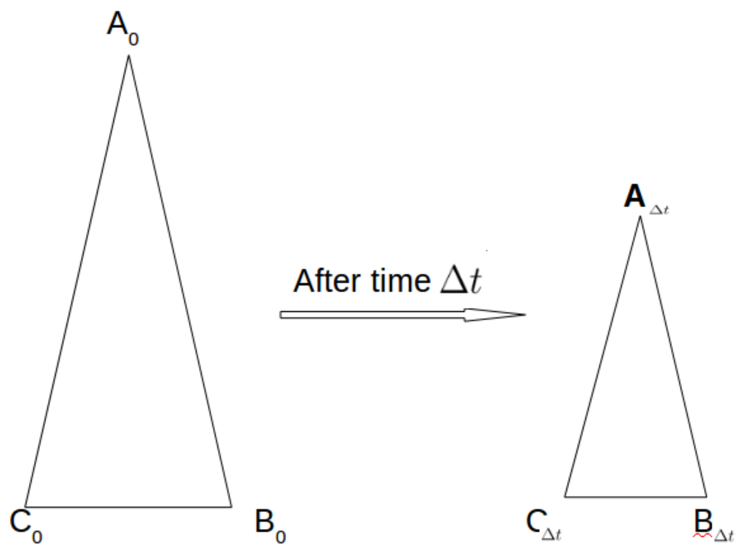

Figure 3: Isosceles triangle undergoing the curve-shortening flow

Next we prove 3 theorems about the behavior of the legs and the vertex angle during the curve-shortening flow

Theorem 4. *The length of the legs of an isosceles triangle, decreases during the curve-shortening flow.*

Proof. We find $\frac{dx(t)}{dt}$ to prove our theorem.

We find it in the following way-

$$\frac{dx\left(t\right)^{2}}{dt}|_{t=0}=2x\left(0\right)\frac{dx\left(t\right)}{dt}|_{t=0}=\lim_{\Delta t\rightarrow\ 0}\frac{x\left(\Delta t\right)^{2}-x\left(0\right)^{2}}{\Delta t}$$

$$=-2x\left(0\right)\left(\left(\frac{\pi}{2}+\frac{\alpha\left(0\right)}{2}\right)\sin\left(\frac{\pi}{4}+\frac{\alpha\left(0\right)}{4}\right)+\left(\pi-\alpha\left(0\right)\right)\cos\left(\frac{\alpha\left(0\right)}{2}\right)\right).$$

Therefore,

$$\frac{dx\left(t\right)}{dt}|_{t=0}=-\left(\left(\frac{\pi}{2}+\frac{\alpha\left(0\right)}{2}\right)\sin\left(\frac{\pi}{4}+\frac{\alpha\left(0\right)}{4}\right)+\left(\pi-\alpha\left(0\right)\right)\cos\left(\frac{\alpha\left(0\right)}{2}\right)\right).$$

By the same method,

$$\frac{dx\left(t\right)}{dt}=-\left(\left(\frac{\pi}{2}+\frac{\alpha\left(t\right)}{2}\right)\sin\left(\frac{\pi}{4}+\frac{\alpha\left(t\right)}{4}\right)+\left(\pi-\alpha\left(t\right)\right)\cos\left(\frac{\alpha\left(t\right)}{2}\right)\right).$$

Since the expression $-\left(\left(\frac{\pi}{2}+\frac{\alpha(t)}{2}\right)\sin\left(\frac{\pi}{4}+\frac{\alpha(t)}{4}\right)+\left(\pi-\alpha\left(t\right)\right)\cos\left(\frac{\alpha(t)}{2}\right)\right)$ is negative for all $\alpha\left(t\right)$, such that $0<\alpha\left(t\right)<\pi$, $\frac{dx(t)}{dt}$ is negative implying that $x\left(t\right)$ always decreases.

□

We find $\frac{d\alpha(t)}{dt}$ for further calculations.

To find $\frac{d\alpha(t)}{dt}$, we proceed as follows.

$$\frac{d\cos\left(\alpha\left(t\right)\right)}{dt}|_{t=0}=-\sin\left(\alpha\left(t\right)\right)\frac{d\alpha\left(t\right)}{dt}|_{t=0}=\lim_{\Delta t\rightarrow\ 0}\frac{\cos\left(\alpha\left(\Delta t\right)\right)-\cos\left(\alpha\left(0\right)\right)}{\Delta t}$$

.

$$= \frac{4\sin(\alpha(0))}{x(0)} \left(\left(\frac{\pi}{2} + \frac{\alpha(0)}{2} \right) \sqrt{\frac{1-\sin\left(\frac{\alpha(0)}{2}\right)}{2}} - (\pi - \alpha(0)) \sin\left(\frac{\alpha(0)}{2}\right) \right).$$

Therefore,

$$\frac{d\alpha(t)}{dt}|_{t=0} = \frac{-4}{x(0)} \left(\left(\frac{\pi}{2} + \frac{\alpha(0)}{2} \right) \sqrt{\frac{1-\sin\left(\frac{\alpha(0)}{2}\right)}{2}} - (\pi - \alpha(0)) \sin\left(\frac{\alpha(0)}{2}\right) \right).$$

By the same method

$$\frac{d\alpha(t)}{dt} = \frac{-4}{x(t)} \left(\left(\frac{\pi}{2} + \frac{\alpha(t)}{2} \right) \sqrt{\frac{1-\sin\left(\frac{\alpha(t)}{2}\right)}{2}} - (\pi - \alpha(t)) \sin\left(\frac{\alpha(t)}{2}\right) \right).$$

For convinience we set $f(s)$ to be,

$$f(s) = \left(\frac{\pi}{2} + \frac{s}{2} \right) \sqrt{\frac{1-\sin\left(\frac{s}{2}\right)}{2}} - (\pi - s) \sin\left(\frac{s}{2}\right).$$

Therefore,

$$f(s) = \frac{1}{2\sqrt{2}} \left((\pi + s) \cos\left(\frac{s}{4}\right) - (\pi + s) \sin\left(\frac{s}{4}\right) - 2\sqrt{2} (\pi - s) \sin\left(\frac{s}{2}\right) \right).$$

We use the following lemma for further calculations.

Lemma 1. *The function $f(s)$ is positive for $0 \le s < \frac{\pi}{3}$.*

Proof. We show that $\frac{df(s)}{ds} < 0$ to prove that $f(s)$ is a monotonically decreasing function in that interval.

$$\frac{df(s)}{ds} = \frac{1}{2\sqrt{2}}\left(\cos\left(\frac{s}{4}\right)\left(\frac{4-s-\pi}{4}\right) - \sin\left(\frac{s}{4}\right)\left(\frac{4+s+\pi}{4}\right) - \sqrt{2}\left(\pi - s\right)\cos\left(\frac{s}{2}\right) + 2\sqrt{2}\sin\left(\frac{s}{2}\right)\right)$$

Case 1. $0 \leq s < \frac{\pi}{4}$.

In this interval $\cos\left(\frac{s}{2}\right) > \sin\left(\frac{s}{2}\right)$. So,

$$\frac{df(s)}{ds} < \frac{1}{2\sqrt{2}}\left(\cos\left(\frac{s}{4}\right)\left(\frac{4-s-\pi}{4}\right) - \sin\left(\frac{s}{4}\right)\left(\frac{4+s+\pi}{4}\right) - \sqrt{2}\left(\pi - s - 2\right)\cos\left(\frac{s}{2}\right)\right).$$

Since $\cos\left(\frac{s}{4}\right) \leq 1$, and $\cos\left(\frac{s}{2}\right) > \cos\left(\frac{\pi}{8}\right) \approx 0.9$,

$$\frac{df(s)}{ds} < \frac{1}{2\sqrt{2}}\left(\frac{4-s-\pi}{4} - \sqrt{2}\left(\pi - s - 2\right) \times 0.9\right) < 0.$$

Case 2. $s = \frac{\pi}{4}$

$\frac{df(s)}{ds} \approx -0.836 < 0.$

Case 3. $\frac{\pi}{4} < s \leq \frac{\pi}{3}$

In this interval, $\cos\left(\frac{s}{2}\right) > \sin\left(\frac{s}{2}\right)$. So,

$$\frac{df(s)}{ds} < \frac{1}{2\sqrt{2}}\left(\cos\left(\frac{s}{4}\right)\left(\frac{4-s-\pi}{4}\right) - \sin\left(\frac{s}{4}\right)\left(\frac{4+s+\pi}{4}\right) - \sqrt{2}\left(\pi - s - 2\right)\cos\left(\frac{s}{2}\right)\right).$$

Since $\cos\left(\frac{s}{4}\right) < 1$, $\sin\left(\frac{s}{4}\right) > \sin\left(\frac{\pi}{16}\right) \approx (0.195)$ and $\cos\left(\frac{s}{2}\right) \leq \frac{\sqrt{3}}{2}$,

$$\frac{df(s)}{ds} < \frac{1}{2\sqrt{2}}\left(\frac{4-s-\pi}{4} - \frac{4+\pi+s}{4} \times 0.195 - \sqrt{2}\left(\pi - s - 2\right)\frac{\sqrt{3}}{2}\right) < 0.$$

Therefore $\frac{df(s)}{d(s)} < 0$ for $0 \leq s \leq \frac{\pi}{3}$ showing that $f(s)$ is a monotonically decreasing function in that interval.

Furthermore, $f\left(\frac{\pi}{3}\right) = 0$ and $f(0) > 0$, showing that $f(s) > 0$ for $0 \leq s < \frac{\pi}{3}$.

□

Based on the previous computations of Lemma 1, $\frac{d\alpha(t)}{dt} < 0$ for $0 \leq \alpha(t) < \frac{\pi}{3}$. Also $\frac{d\alpha(t)}{dt} = 0$ at $\alpha(t) = \frac{\pi}{3}$, leading to our Theorem 5.

Theorem 5. *The vertex angle $(\alpha(t))$ of an isosceles triangle undergoing the curve-shortening flow, decreases if $(\alpha(t)) < \frac{\pi}{3}$ and remains same if $\alpha(t) = \frac{\pi}{3}$.*

Now we will analyse what will happen to the triangle if $\alpha(0) < \frac{\pi}{3}$.

Theorem 6. *The vertex angle vanishes in a shorter time than than the legs of an isosceles triangle undergoing the curve-shortening flow, if $\alpha(0) < \frac{\pi}{3}$, i.e there exists a time T such that $\alpha(T) = 0$ and $x(T) > 0$.*

Proof. As we found in the previous part,

$$\frac{dx\,(t)}{dt} = -\left(\left(\frac{\pi}{2} + \frac{\alpha(t)}{2}\right)\sin\left(\frac{\pi}{4} + \frac{\alpha\,(t)}{4}\right) + (\pi - \alpha\,(t))\cos\left(\frac{\alpha\,(t)}{2}\right)\right)$$

and

$$\frac{d\alpha\,(t)}{dt} = -\frac{4}{x\,(t)}\left(\left(\frac{\pi}{2} + \frac{\alpha\,(t)}{2}\right)\sqrt{\frac{1-\sin\left(\frac{\alpha(t)}{2}\right)}{2}} - (\pi - \alpha\,(t))\sin\left(\frac{\alpha\,(t)}{2}\right)\right).$$

Let the function $g(\alpha(t))$ be defined as follows:

$$g\,(\alpha\,(t)) = \left(\frac{\pi}{2} + \frac{\alpha\,(t)}{2}\right)\sin\left(\frac{\pi}{4} + \frac{\alpha\,(t)}{4}\right) + (\pi - \alpha\,(t))\cos\left(\frac{\alpha\,(t)}{2}\right).$$

The function $g\,(\alpha\,(t))$ satisfies $g\,(\alpha\,(t)) < 2\pi$ for all t, since $\alpha(t) \leq \frac{\pi}{3}$.

This was found by substituting $\alpha(t) = 0$, with $g(0) = \frac{\pi}{2\sqrt{2}} + \pi < 2\pi$.

Therefore, there exists a positive integer C_2 such that, $g\,(\alpha\,(t)) < C_2$ and $C_2 \geq 2\pi$.

Also, by the proof of Lemma 1, $f\,(\alpha\,(t)) \geq f\,(\alpha\,(0)) = C_1$, where C_1 is a positive integer . For any time t in the time interval $[0, T)$, $x\,(t) > 0$.

By Lemma 1, $f(\alpha(t)) > 0$ and $g(\alpha(t)) > 0$ as it is bounded between positive real numbers.

The functions $\alpha(t)$ and $x(t)$ are decreasing in this interval.

Let $t_0 = 0$ and $t_{i+1} = t_i + \frac{x(t_i)}{2C_2}$. Then for any positve integer n, by Lagrange Mean Value Theorem, there exists time t in the interval (t_n, t_{n+1}) such that

$$x(t_{n+1}) - x(t_n) = (t_{n+1} - t_n) \frac{dx(t)}{dt}.$$

Therefore,

$$x(t_{n+1}) = x(t_n) - (t_{n+1} - t_n) g(\alpha(t)).$$

Since $g(\alpha(t)) < C_2$, we have

$$x(t_{n+1}) > x(t_n) - (t_{n+1} - t_n) C_2.$$

Hence, $x(t_{n+1}) > \frac{x(t_n)}{2}$

By induction from 0 to n, $x(t_n) > \frac{x(t_0)}{2^n}$. Thus, $x(t)$ is positive for all t in the interval $[0, t_n]$, for all positive integers n.

By applying Lagrange Mean Value Theorem again, for any positive integer n, there exists a time s in the interval (t_n, t_{n+1}) such that

$$\alpha(t_{n+1}) = \alpha(t_n) + (t_{n+1} - t_n) \frac{d\alpha(s)}{dt}.$$

Therefore,

$$\alpha(t_{n+1}) = \alpha(t_n) - (t_{n+1} - t_n) \frac{4f(\alpha(s))}{x(s)}.$$

Since $f(\alpha(s)) > C_1$, we have

$$\alpha(t_{n+1}) < \alpha(t_n) - \frac{2C_1 x(t_n)}{C_2 x(s)}.$$

Since $x(t)$ is a decreasing function, $x(t_n) > x(s)$.

Therefore, $x(t_{n+1}) < x(t_n) - \frac{2C_1}{C_2}$

By induction, $\alpha(t_n) < \alpha(t_0) - n\frac{2C_1}{C_2}$.

Suppose there is a positive integer N such that $N\frac{2C_1}{C_2} > \alpha(t_0)$. Then we know that $\alpha(t)$ vanishes in the interval $[0, t_n]$ because $\alpha(t) \geq 0$, but $x(t)$ is positive in this interval.

□

The theorem implies that the triangle will become a straight line segment and not a single point after some time(Figure 4).

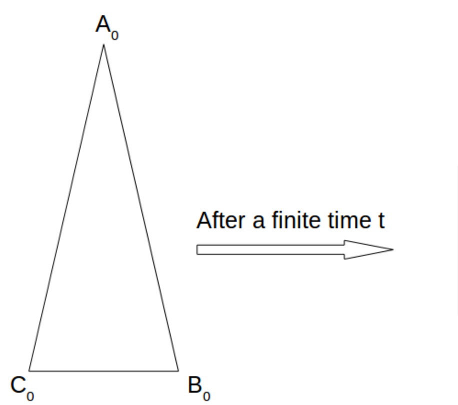

Figure 4: the end of the CSF

4 General Triangle

Let us consider the triangle ABC. (Recall that $p(t)$ denotes the side B_tC_t.)

According to our assumption,

$$A_0 : \left(\frac{p \sin (\beta (0)) \cos (\gamma (0))}{\sin (\beta (0) + \gamma (0))}, \frac{p \sin (\beta (0)) \sin (\gamma (0))}{\sin (\beta (0) + \gamma (0))} \right)$$

$$B_0 : (p (0) , 0) .$$

When the triangle undergoes the curve-shortening flow in an infinitesimally small time Δt, the following transformation (Figure 5) takes place.

The coordinates of the new vertices are -

$A_{\Delta}t = (\frac{p(0)\sin(\beta(0))\cos(\gamma(0))}{\sin(\beta(0)+\gamma(0))} - (\beta(0) + \gamma(0)) \sin(\frac{\beta(0)}{2} - \frac{\gamma(0)}{2})\Delta t, \frac{p(0)\sin(\beta(0))\sin(\gamma(0))}{\sin(\beta(0)+\gamma(0))} - (\beta(0) + \gamma(0)) \cos(\frac{\beta(0)}{2} - \frac{\gamma(0)}{2})\Delta t)$

$B_{\Delta}t = \left(p (0) - (\pi - \beta (0)) \cos \left(\frac{\beta(0)}{2}\right) \Delta t, (\pi - \beta (0)) \sin \left(\frac{\beta(0)}{2}\right) \Delta t\right)$

$C_{\Delta}t = \left((\pi - \gamma (0)) \cos \left(\frac{\gamma(0)}{2}\right) \Delta t, (\pi - \gamma (0)) \sin \left(\frac{\gamma(0)}{2}\right) \Delta t\right)$

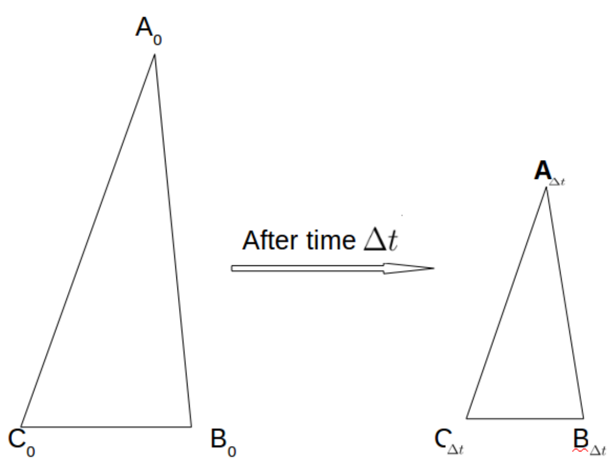

Figure 5: A general triangle undergoing the CSF

Next we prove 1 theorem and verify 2 conjectures.

Theorem 7. *In any triangle, the length of each side of the triangle, decreases during the* *curve-shortening flow.*

Proof. We find $\frac{dp(t)}{dt}$ to prove our theorem.

We find it in the following way.

$$\frac{dp\left(t\right)^2}{dt}|_{t=0} = 2p\left(0\right)\frac{dp\left(t\right)}{dt}|_{t=0} = \lim_{\Delta t \to 0} \frac{\left(p\left(\Delta t\right)^2 - p\left(0\right)^2\right)}{\Delta t}$$

$$= -2p(0)\left(\left(\pi - \beta\left(0\right)\right)\cos\left(\frac{\beta\left(0\right)}{2}\right) + \left(\pi - \gamma\left(0\right)\right)\cos\left(\frac{\gamma\left(0\right)}{2}\right)\right).$$

Therefore,

$$\frac{dp\left(t\right)}{dt}|_{t=0} = -\left(\left(\pi - \beta\left(0\right)\right)\cos\left(\frac{\beta\left(0\right)}{2}\right) + \left(\pi - \gamma\left(0\right)\right)\cos\left(\frac{\gamma\left(0\right)}{2}\right)\right).$$

By the same method,

$$\frac{dp\left(t\right)}{dt} = -\left(\left(\pi - \beta\left(t\right)\right)\cos\left(\frac{\beta\left(t\right)}{2}\right) + \left(\pi - \gamma\left(t\right)\right)\cos\left(\frac{\gamma\left(t\right)}{2}\right)\right).$$

Since the expression $\left(\pi - \beta\left(t\right)\right)\cos\left(\frac{\beta(t)}{2}\right) + \left(\pi - \gamma\left(t\right)\right)\cos\left(\frac{\gamma(t)}{2}\right)$ is positve for all $0 < \beta\left(t\right) < \pi$ and $0 < \gamma\left(t\right) < \pi$, $\frac{dp(t)}{dt}$ is negative proving that the length of side BC decreases.

The same computation can be done for all sides and a similar result will be obtained which is based on the adjacent angles, and the derivative of the length of the side with respect to time being negative.

□

4.1 Other Results and Conjectures

We find that:

$$\frac{d\beta(t)}{dt} = -\frac{\csc(\gamma(t))}{2p(t)}((\beta(t)-\pi)\cos(\frac{\beta(t)}{2}-\gamma(t))-\beta(t)\cos(3\frac{\beta(t)}{2}+\gamma(t))+\pi(\cos(\frac{\gamma(t)}{2})-\cos(\frac{3\gamma(t)}{2})+\cos(\frac{3\beta(t)}{2}+\gamma(t))) - 2\gamma(t)\sin(\frac{\gamma(t)}{2})\sin(\gamma(t)) + (\beta(t)+\gamma(t))(\sin(\frac{\beta(t)}{2}+\frac{\gamma(t)}{2}) + \sin(\frac{3(\beta(t)+\gamma(t))}{2}))).$$

$$\frac{d\gamma(t)}{dt} = -\frac{\csc(\beta(t))}{2p(t)}((\pi-\beta(t))\cos(\beta(t))+(\beta(t))-\pi)\cos(3\frac{\beta(t)}{2})+2\gamma(t)\sin(\beta(t)+\frac{\gamma(t)}{2})\sin(\gamma(t))-2\pi\sin(\beta(t)+\frac{\gamma(t)}{2})\sin(\gamma(t))+(\beta(t)+\gamma(t))(\sin(\frac{\beta(t)}{2}+\frac{\gamma(t)}{2})+\sin(\frac{3(\beta(t)+\gamma(t))}{2}))).$$

Conjecture 8. *In any triangle which is not equilateral, the smallest angle decreases during the curve-shortening flow.*

If we assume in the triangle that $\gamma(0) \geq \alpha(0) \geq \beta(0)$, then the following 3D graph (Figure 6) depicts $\frac{d\beta(t)}{dt}2p(t)$.

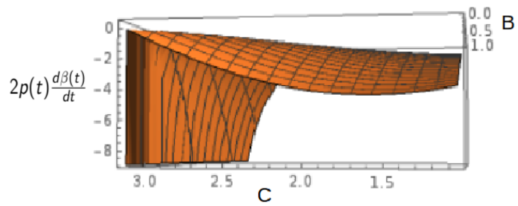

Figure 6: graph of $\frac{d\beta(t)}{dt}2p(t)$

The graph shows that $\frac{d\beta(t)}{dt}2p(t)$ is always negative, implying that $\frac{d\beta(t)}{dt}$ is negative, showing that β decreases and verifying our Conjeucture 8.

Conjecture 9. *The equilateral triangle is the only triangle which is a self–shrinker.*

If we apply $\alpha(0) = \gamma(0) = \beta(0) = \frac{\pi}{3}$, then we get $\frac{d\beta(t)}{dt} = \frac{d\gamma(t)}{dt} = \frac{d\alpha(t)}{dt} = 0$, proving that the equilateral triangle is a self–shrinker. Based on extensive computations, we can conjecture that it is the only self–shrinker.

5 Conclusion

In this paper we examined some of properties triangles undergoing the curve-shortening flow. In the case of the isosceles triangle, we found that the length of its legs decreases and its vertex angle decreases if it is smaller than $\frac{\pi}{3}$ and remains the same at $\frac{\pi}{3}$. Furthermore, the vertex angle in an isosceles triangle vanishes in a shorter time than the length, if the angle is smaller than $\frac{\pi}{3}$ and the triangle eventually becomes a line segment and not a point. In any triangle, the length of each side decreass during the flow. Our future work is to study the behaviour of isosceles triangles when the vertex angle is greater than $\frac{\pi}{3}$ and complete the observations by closely studying the behavior of any random triangle under the curve-shortening flow.

6 Acknowledgments

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