Geometric Complexity of Planar Drawings

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Abstract

We say that a planar drawing of a graph is 1-thick if the distance between the images of any two vertices, a vertex and an edge, and two non-adjacent edges is at least 1. We prove that the cylinder mesh graph $C_{M,N}$ has a 1-thick drawing inside a ball of radius $C \cdot (\sqrt{MN} + N)$ for some absolute constant C. Moreover, we prove that the value $\sqrt{MN} + N$ is sharp up to a constant factor.

Summary

We take a cylinder mesh and draw the structure on the plane. We consider different ways of representing with vertices and edges the three-dimensional structure to look at how the distances between components of the cylinder mesh change in the two-dimensional drawing. Our results have applications in electrical engineering for designing circuit boards and orienting the conductive paths so that they fit in the smallest board possible without overlapping.

1 Introduction

Kolmogorov and Barzdin's study [1] of embeddings of graphs into $\mathbb{R}^3$ was the first study that measured the geometric complexity of embeddings of simplicial complexes, and it raised subsequent important questions later studied by Gromov and Guth [2], Freedman and Krushkal [3]. Gromov and Guth generalize the results of Kolmogorov and Barzdin about embedding simplicial complexes into $\mathbb{R}^3$ to embedding graphs into $\mathbb{R}^n$. Both studies use thickness as a measure to evaluate the geometric complexity of an embedding. Kolmogorov and Barzdin discuss the possibility of applying the study of geometric complexity of embeddings to the study of neural networks. In the brain, neurons are oriented so that the cell body is aligned toward the outer brain while the axons are located in the inner brain. This specific orientation is reminiscent of how graphs can be embedded into $\mathbb{R}^3$ optimally to fit in a ball under the assumption that two axons cannot come too close to each other. The neuron's cell body corresponds to the vertices and the axons connecting the cell bodies correspond to the edges.

While the study of embeddings into $\mathbb{R}^3$ can be applied to neural networks in three-dimensional spaces such as the brain, our study of embedding into $\mathbb{R}^2$ may have applications in designing printed electric circuit boards where the conductive paths cannot come too close.

An embedding is a representation of a topological object in a given space in a way that preserves its topological structure. An embedding of a graph into $\mathbb{R}^2$ gives a planar drawing of the graph, meaning that the graph may be represented in the plane by non-intersecting curves.

We consider a specific planar graph. First take a cycle graph C_M with M vertices and M edges.

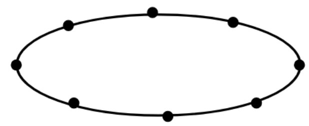

Figure 1: The cycle graph C_8

Then take N copies of this graph and connect them with edges to make a cylinder mesh graph $C_{M,N}$. In other words, $C_{M,N}$ is obtained from the $M \times N$ grid by adding N edges to make it a cylinder mesh.

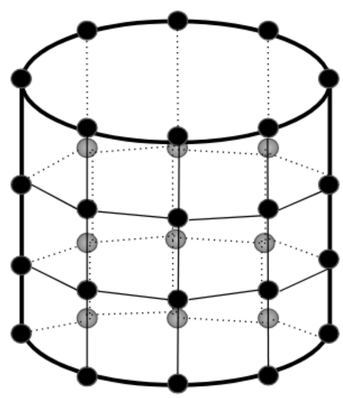

Figure 2: The cylinder mesh graph $C_{8,4}$

The cylinder mesh is one of the simplest examples of a large planar graph for which the estimations of geometric complexity of embeddings are not trivial. Can we estimate the minimal geometric complexity using the thickness of any embedding of $C_{M,N}$ into $\mathbb{R}^2$?

We say that a planar drawing of a graph has *thickness* at least 1 if the distance between the images of any two vertices, the images of any two nonadjacent edges, and the images of an edge and a vertex that are nonadjacent are all at least 1. We prove that the cylinder mesh graph $C_{M,N}$ has a 1-thick drawing inside a ball of radius $C \cdot (\sqrt{MN} + N)$ for some absolute constant C. Moreover, we prove that the value $\sqrt{MN} + N$ is sharp up to a constant factor.

In Section 2, we introduce preliminary definitions and in particular, we define Kolmogorov–

Barzdin thickness. In Section 3, we introduce our main theorem. In Sections 4 and 5, we prove the main theorem. In Section 6, possible future directions are discussed.

2 Definitions

A graph $G = (V, E)$ is given by its set of vertices $V(G)$ and its set of edges $E(G)$ (i.e., two-element subsets of $V(G)$).

Definition 2.1. A topological embedding i of G into $\mathbb{R}^d$ is given by the following data:

1) for each $v \in V(G)$, we specify a point $i(v) \in \mathbb{R}^d$;

2) for each $e = \{v, u\} \in E(G)$, we specify a continuous curve $i_e : [0, 1] \to \mathbb{R}^d$ such that $\{i_e(0), i_e(1)\} = \{i(v), i(u)\}$;

3) for all $v, u \in V(G)$, $i(v) \neq i(u)$ unless $v = u$; for all $e \in E(G), v \in V(G), t \in (0, 1)$, $i_e(t) \neq i(v)$; and for all $e, f \in E(G)$, $e \neq f$, $t, s \in [0, 1]$, $i_e(t) \neq i_f(s)$.

We will use notation $i : G \hookrightarrow \mathbb{R}^d$ for topological embedding.

Remark 2.1. Graphs admitting an embedding into $\mathbb{R}^2$ are called planar. In 1930, Kuratowski [4] gave a criterion for a graph to be planar. He proved that a graph is planar if and only if it does not contain a homeomorphic copy of either K_5 or $K_{3,3}$.

Remark 2.2. Part 3) means that the images of vertices and edges of G do not intersect unless otherwise addressed in part 2).

Remark 2.3. If the image of an embedding i belongs to the ball B_R of radius R, we use notation $i : G \hookrightarrow B_R$.

Definition 2.2. Let C_M be a cycle graph with M vertices. Take N copies of C_M and denote these cycles as $C_M^{(i)}$ where $1 \leq i \leq N$. Denote the vertices of $C_M^{(i)}$ in order as $V(C_M^{(i)}) = \{v_1^{(i)}, \ldots, v_M^{(i)}\}$. Then, we define the *cylinder mesh graph* $C_{M,N}$ as follows: Take the disjoint union of the $C_M^{(i)}$ and add edges $(v_j^{(i)}, v_j^{(i+1)})$ for all $1 \le i \le N-1$ and $1 \le j \le M$. We refer to $C_M^{(i)}$ as the *layers* of $C_{M,N}$.

Define the distance between sets A and B as

$$\mathrm{dist}(A, B) = \inf\{\|x - y\| \,|x \in A, y \in B\},$$

where $\|\cdot\|$ denotes the Euclidean norm. Given an embedding i of graph G into $\mathbb{R}^d$, Kolmogorov and Barzdin [1] define its *thickness* as follows.

Definition 2.3. Let G be a graph and let $i : G \hookrightarrow \mathbb{R}^d$ be a topological embedding. We say that i is of *Kolmogorov–Barzdin thickness*, or KB-thickness, at least T if the Euclidean distance between the images of any two vertices is at least T, the distance between the images of any nonadjacent edges is at least T, and the distance between the image of an edge and the image of a vertex not in the edge is at least T.

Definition 2.4. Given two functions $f(M, N)$ and $g(M, N)$, we write $f \lesssim g$ if there exists an absolute constant $C > 0$ such that $f(M, N) \le C \cdot g(M, N)$ for sufficiently large M, N. We write $f \sim g$ if $f \lesssim g$ and $g \lesssim f$.

3 Embedding the Cylinder Mesh Graph $C_{M,N}$ into $\mathbb{R}^2$

Our main result establishes bounds on the KB-thickness of the embedding of $C_{M,N}$ into a unit ball in the plane.

Theorem 3.1. *Any topological embedding $i : C_{M,N} \hookrightarrow B_1 \subset \mathbb{R}^2$ has KB-thickness at most $\frac{C_1}{\sqrt{MN}+N}$ for some absolute constant C_1. Moreover, there exists an embedding i_0 of thickness at least $\frac{C_2}{\sqrt{MN}+N}$ for some absolute constant C_2.*

We start from the following observation. An embedding $i : C_{M,N} \hookrightarrow B_1$ of thickness T can be inflated to get an embedding $i' : C_{M,N} \hookrightarrow B_{1/T}$ of thickness 1. Therefore, the problem of embedding $C_{M,N}$ into B_1 with largest possible thickness is equivalent to the problem of finding an embedding of $C_{M,N}$ of thickness 1 into B_R with smallest possible R, which is written as R_{opt}. We work with the second formulation.

To prove Theorem 3.1, we consider two cases: $M \leq N$ and $M > N$. The proof for the case $M \leq N$ is given in Section 4, and the proof for the case $M > N$ is given in Section 5.

4 Proof of Theorem 3.1 in the Case $M \leq N$

When $M \leq N$, we prove that $R_{\text{opt}} \sim N$. Lemma 4.1 proves the upper bound and Lemma 4.2 proves the lower bound.

Lemma 4.1. *If $M \leq N$, there exists an embedding $i_0 : C_{M,N} \hookrightarrow B_R$ for $R \lesssim N$.*

Proof. We prove this lemma by exhibiting an explicit embedding of KB-thickness 1 (see Figure 3) in the plane so that the image of $C_{M,N}$ consists of N concentric circles each 1 apart, connected by M edges.

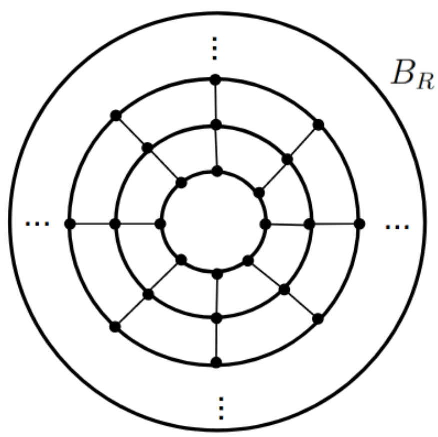

Figure 3: Explicit embedding with N concentric circles

In this embedding, every two consecutive circles are 1 unit apart. The radius R of B_R is the sum of the radius of the innermost circle R_{inner} and all the distances between the N

concentric circles. Because there are M vertices on each circle, and the vertices must be at least 1 apart according to the definition of KB-thickness, we have $R_{\text{inner}} \sim M$. Moreover, the sum of the distances between the N layers add up to $N-1$. Therefore, $R \leq M+N$, and because $M \leq N$, we have $R \lesssim N$. Thus, $R_{\text{opt}} \lesssim N$. $\square$

Lemma 4.2. *If $M \leq N$, then for any embedding $i : C_{M,N} \hookrightarrow B_R$, $R \gtrsim N$.*

Proof. We prove in several steps that given any embedding, $R \gtrsim N$. In 1932, Whitney [5] proved a result equivalent to the fact that any 3-connected planar graph has a unique drawing on $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ (see the discussion in [6]). Recall that a planar graph is 3-connected if it has at least 4 vertices and there does not exist a set of 2 vertices whose removal disconnects the graph. One way to see that $C_{M,N}$ is a 3-connected planar graph is to apply Steinitz's criterion [7]. Because $C_{M,N}$ is the edge graph of a 3-dimensional convex polytope, then by Steinitz's criterion it is a 3-connected planar graph (see Figure 4).

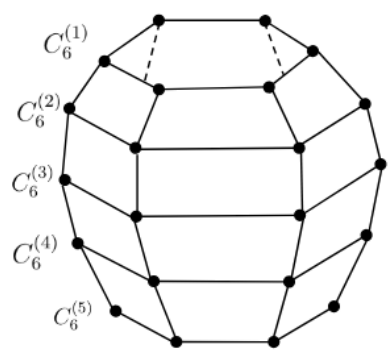

Figure 4: $C_{6,5}$ as the edge graph of a convex polytope

Therefore, any planar drawing of $C_{M,N}$ can be obtained as the stereographic projection from a point inside some face of the spherical drawing (see Figure 5).

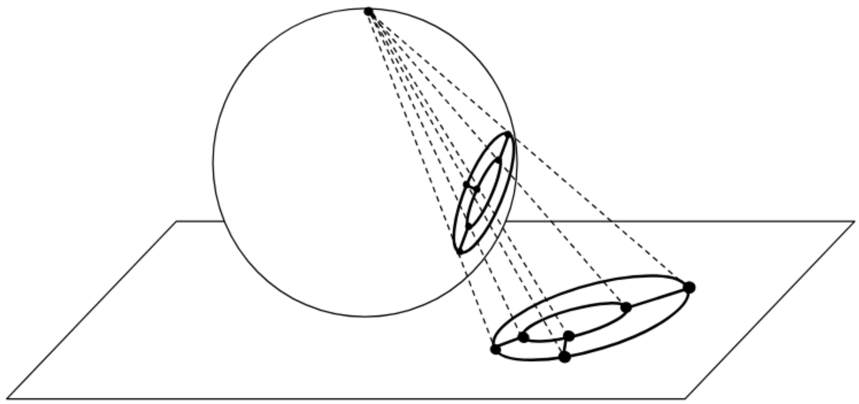

Figure 5: Stereographic projection

If the point of projection lies between $C_M^{(i)}$ and $C_M^{(i+1)}$, the planar drawing consists exactly of one family of the layers including and above $C_M^{(i)}$ and another family of the layers including and below $C_M^{(i+1)}$. Above and below are related to the indices used in Figure 4. Any planar drawing of $C_{M,N}$ consists of at most 2 families of nested closed simple curves (see Figure 6).

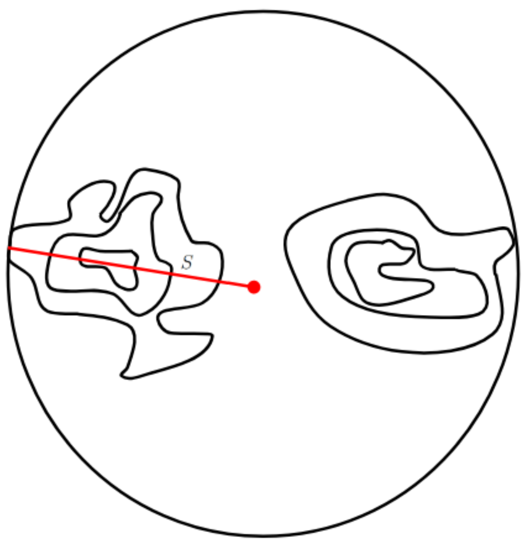

Figure 6: Example of a planar drawing of $C_{M,N}$

One family consists of at least $\frac{N}{2}$ nested closed simple curves. Call the curves $C_1, C_2, \ldots, C_m$,

$m \geq \frac{N}{2}$, in the order from the innermost curve to the outermost curve. In Figures 6 and 7, the edges connecting the layers of $C_{M,N}$ are omitted for convenience. Then, we choose a radial segment S that intersects the innermost layer which we call C_1. By the Jordan curve theorem, S intersects the entire family. Note that S does not have to cross the layers in increasing order (see Figure 7).

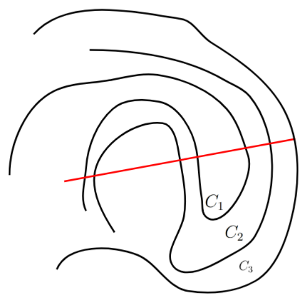

Figure 7: Radius S crossing a family of layers

Using the definition of KB-thickness, we cut out a line segment of length at least 1 between every two consecutive layers C_i and C_{i+1} that S intersects as follows. Call the endpoints of these cut-out line segments A_i, B_i, where $A_i :=$ last point of $C_i \cap S$ and $B_i :=$ first point of $C_{i+1} \cap S$ after A_i. First and last follow the order of S going outward from the innermost layer to the outermost layer.

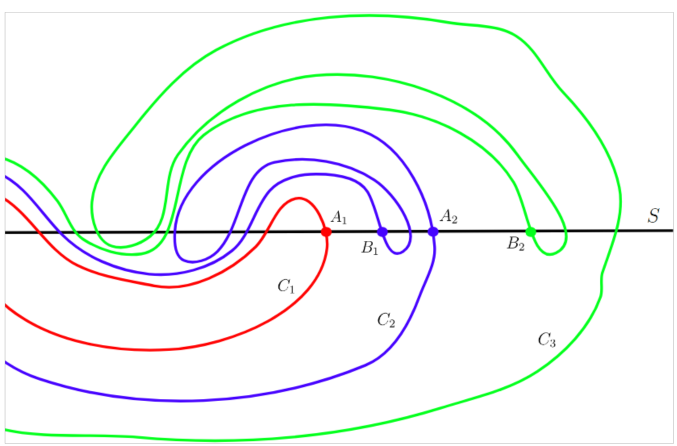

Figure 8: Spacing between consecutive layers

So far, A_i and B_i were defined informally because it is not clear whether they exist. We will define A_i, B_i formally. Parametrize the curve

$$C_1 = \{x_1(t), y_1(t) \mid t \in [0, M)\}.$$

Parametrize the line segment

$$S = \{x(r), y(r) \mid r \in [0, R]\},$$

where 0 corresponds to the center of the ball and R corresponds to the boundary of the ball. Parametrize the points of intersection of S and C_1 by introducing

$$I_1 = \{t \in [0, M) \mid (x_1(t), y_1(t)) = (x(r), y(r)) \text{ for some } r\}$$

and

$$J_1 = \{r \in [0, R] \mid (x(r), y(r)) = (x_1(t), y_1(t)) \text{ for some } t\}.$$

Notice that there is a bijection between I_1 and J_1. Let $\overline{r} = \sup J_1$. By definition of supremum, $r \leq \overline{r}$ for all $r \in J_1$. By default, it is not clear whether $\overline{r} \in J_1$. However, it can be shown using continuity that, in fact, $\overline{r} \in J_1$. The formal argument is given in Lemma 4.3. Define $A_1 = (x(\overline{r}), y(\overline{r}))$. All the remaining A_i and B_i are defined similarly. By definition of KB-thickness, any line segment A_iB_i has length at least 1.

By the definition of A_i, B_i, there are no intersections of S with either C_i or C_{i+1} between A_i and B_i. By the Jordan curve theorem, there is no other intersection of S and C_j between A_i and B_i. Thus, the line segments A_iB_i are non-overlapping. Because all non-overlapping line segments A_iB_i for all $1 \leq i \leq m-1$ have lengths at least 1, the length of S is at least $\frac{N}{2} - 1$ and $R \gtrsim N$. $\square$

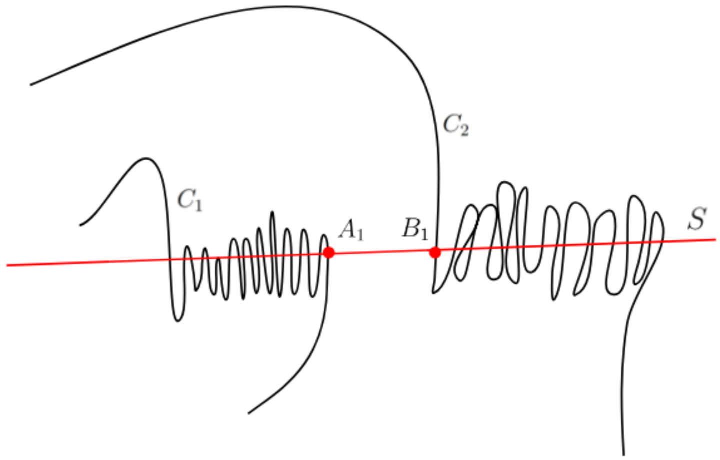

Figure 9: Points of intersection on continuous curves

Lemma 4.3. *With the notation of Lemma 4.2, the last point of intersection (see Figure 9) between C_1 and S exists, that is, $\overline{r} \in J_1$.*

Proof. By definition of sup, $J_1 \cap (\overline{r} - \epsilon, \overline{r}) \neq \emptyset$ for all $\epsilon > 0$. For all $n \in \mathbb{N}$, take $\epsilon_n = \frac{1}{n}$. Choose $r_n \in J_1 \cap (\overline{r} - \epsilon_n, \overline{r})$. As n approaches infinity, r_n approaches $\overline{r}$. Let $t_n \in I_1$ correspond to the r_n under the $J_1 - I_1$ bijection. Using the Weierstrass compactness theorem, we choose a subsequence of $(t_n)_{n=1}^{\infty}$, call it $(t_{n_k})_{k=1}^{\infty}$, such that there exists $\lim_{k\to\infty} t_{n_k} \in [0, M]$. By continuity,

$$x(\overline{r}) = x(\lim_{k\to\infty} r_{n_k}) = \lim_{k\to\infty} x(r_{n_k}) = \lim_{k\to\infty} x_1(t_{n_k}) = x_1(\lim_{k\to\infty} t_{n_k}).$$

Similarly,

$$y(\overline{r}) = y_1(\lim_{k\to\infty} t_{n_k}).$$

Therefore,

$$(x(\overline{r}), y(\overline{r})) = (x_1(\lim_{k\to\infty} t_{n_k}), y_1(\lim_{k\to\infty} t_{n_k})) \in C_1.$$

Thus, $\overline{r} \in J_1$, and the last point of intersection between S and C_1, or A_1, exists. $\square$

Combining Lemma 4.1 and Lemma 4.2, we get $R_{\text{opt}} \sim N$, thus concluding the proof of the case $M \leq N$.

5 Proof of Theorem 3.1 in the Case $M > N$

In the case $M > N$, we prove that $R_{\text{opt}} \sim \sqrt{MN}$. Lemma 5.1 proves the lower bound and Lemma 5.2 proves the upper bound.

Lemma 5.1. *If $M > N$, then for any embedding $i : C_{M,N} \hookrightarrow B_R$, $R \gtrsim \sqrt{MN}$.*

Proof. $R_{\text{opt}} \gtrsim \sqrt{MN}$ can be proven with a volumetric argument. Consider any embedding of $C_{M,N}$ into a ball with radius R. Then, replace all the vertices of $C_{M,N}$ with disks of radius $\frac{1}{2}$ centered at the vertices. Since the embedding has KB-thickness at least 1, the disks do not overlap. Then, there may exist disks that are not contained in the ball with radius R. However, if we inflate the ball with radius R to a ball with radius $R + \frac{1}{2}$, all the disks must be contained in the inflated ball. Let us denote the inflated ball as $B_{R+\frac{1}{2}}$ and the disks as $B^{(i)}_{\frac{1}{2}}$, $1 \leq i \leq M$.

Then,

$$\bigcup_{i=1}^{M} B^{(i)}_{\frac{1}{2}} \subset B_{R+\frac{1}{2}}.$$

Thus, the sum of the areas of all the non-overlapping disks is at most the area of $B_{R+\frac{1}{2}}$.

$$\frac{\pi}{4} MN \leq \pi \left(R + \frac{1}{2}\right)^2$$

$$\frac{1}{2}\sqrt{MN} \leq R + \frac{1}{2} \leq 2R.$$

Thus, we can say that $R \gtrsim \sqrt{MN}$. □

Lemma 5.2. *If $M > N$, there exists an embedding $i_0 : C_{M,N} \hookrightarrow B_R$ for $R \lesssim \sqrt{MN}$.*

Proof. We can prove $R \lesssim \sqrt{MN}$ by exhibiting an explicit embedding. Consider *tiles* of sidelength $3N \times 3N$. There are two types of tiles: *straight* and *curved*, shown in Figure 10. We use these tiles to form a grid which contains the embedding of $C_{M,N}$.

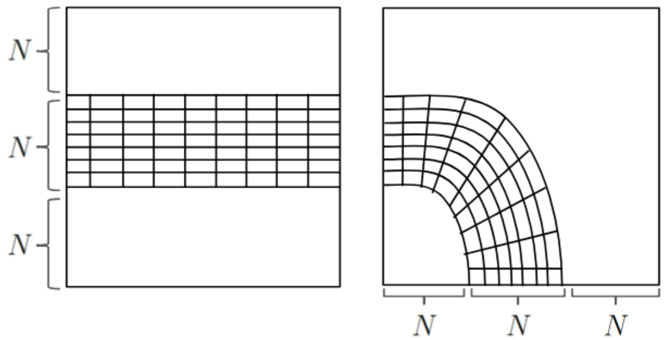

Figure 10: Straight tile and curved tile

Intersections of two edges represent vertices. The edges are placed exactly in the middle of each tile (see Figure 10). Because $M > N$, we write $M = kN + r$, where $k, r \in \mathbb{N}$ and $r < N$. Then,

$$MN = kN^2 + rN,$$

where $rN < N^2$.

Choose $L = \left[\sqrt{\frac{M}{N}}\right] + 1$. Form an $L \times L$ grid with the tiles. Note that $L^2N^2 \geq MN$. We arrange the tiles so that the edges in the tiles form a loop (see Figure 11). We consider that a tile is *occupied* if there are edges drawn on the tile. For L even, we can construct the grid so that all the tiles in the grid are occupied. We do so using exactly $2L$ curved tiles and $L^2 - 2L$ straight tiles. For L odd, we construct the grid so that all the tiles except for one are occupied. We can do so using exactly $4L - 8$ curved tiles and $L^2 - 4L + 7$ straight tiles.

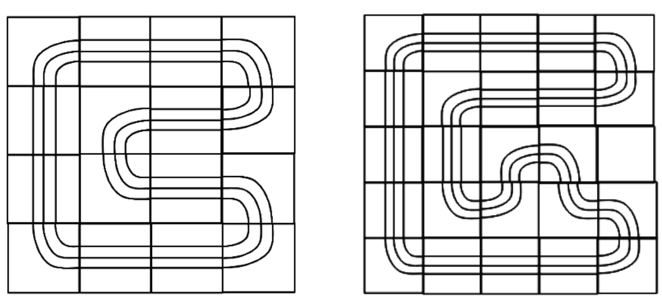

Figure 11: Constructions for $L = 4$, $L = 5$

For both cases, draw the connecting edges so that there are k tiles with exactly N^2 vertices and 1 tile with exactly rN vertices (see Figure 12). Leave the remaining $L^2 - k - 1$ tiles with no vertices in them. Then, there are a total of MN vertices drawn on the $L \times L$ grid.

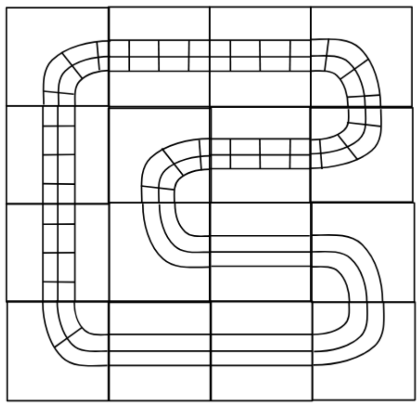

Figure 12: Embedding of $C_{28,3}$

It can be seen that this embedding has KB-thickness at least 1. Moreover, the radius of the ball containing the $L \times L$ grid is

$$R = \frac{3}{\sqrt{2}} LN = \frac{3}{\sqrt{2}} \left(\left\lceil \sqrt{\frac{M}{N}} \right\rceil + 1 \right) N \leq \frac{3}{\sqrt{2}} \left(\sqrt{\frac{M}{N}} + 1 \right) N = \frac{3}{\sqrt{2}} (\sqrt{MN} + N) \sim \sqrt{MN}.$$

□

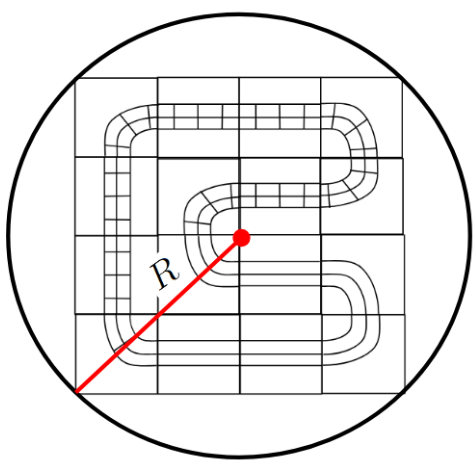

Figure 13: Radius of the ball containing the embedding of $C_{28,3}$

Combining Lemma 5.1 and 5.2, we get $R_{\text{opt}} \sim \sqrt{MN}$, thus concluding the proof of the case $M > N$. The proof of Theorem 3.1 is complete.

6 Future Work & Conclusion

Our results apply to a specific planar graph: the cylinder mesh graph $C_{M,N}$. The volumetric argument for the lower bound may be generalized for other graphs. This argument shows that for any graph $G = (V, E)$, $R_{\text{opt}} \gtrsim \sqrt{|V|}$. However, this bound is not sharp, as shown in Lemma 4.2, where a better bound is exhibited for $G = C_{M,N}$ with $M \leq N$. Moreover, the explicit embeddings that prove the upper bound are specific to $C_{M,N}$. We plan to generalize our results to other planar graphs by looking at other explicit embeddings that may work for more general planar graphs.

Moreover, we may consider other ways of measuring the geometric complexity of an embedding, such as by measuring *distortion*. The most common definition of distortion is as follows.

Definition 6.1. Let X be a graph and let $i : X \hookrightarrow \mathbb{R}^2$ be a piecewise-smooth embedding. The *distortion* is defined by

$$\delta(i) = \sup_{x,y \in i(X)} \frac{\text{dist}_{i(X)}(x, y)}{\text{dist}_{\mathbb{R}^2}(x, y)},$$

where $\text{dist}_{i(X)}$ measures distance in the intrinsic metric of $i(X) \subset \mathbb{R}^2$.

Matoušek [8] investigated the properties of finite metric spaces using a different definition of distortion.

Definition 6.2. Let X be a graph and let $i : V(X) \to \mathbb{R}^2$ be an embedding of the vertices of X. The *Matoušek distortion* of i is defined as the faithfulness of i with respect to the distances when X is treated as a finite metric space with the metric given by the edge distance.

$$\delta'(i) = \sup_{v \neq u \in V(X)} \frac{\mathrm{dist}_{\mathbb{R}^2}(i(v), i(u))}{\mathrm{dist}_X(u,v)} \cdot \sup_{v \neq u \in V(X)} \frac{\mathrm{dist}_X(u,v)}{\mathrm{dist}_{\mathbb{R}^2}(i(v), i(u))},$$

where dist_X denotes the edge distance in X.

What is the minimal possible distortion of an embedding $i : C_{M,N} \hookrightarrow \mathbb{R}^2$? Can we generalize the results for any graph G?

7 Acknowledgments

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