A Classification of Formal Group Laws over Torsion-free Rings
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Abstract

In this paper we obtain a classification of formal group laws up to strict isomorphism over any ring which is torsion-free as a $\mathbb{Z}$-module. We obtain our results mainly through the development of the theory of formal group law chunks. We also consider the case of rings with torsion briefly and find a reduction of the full classification problem which may be more tractable.

1 Introduction

Formal group laws are a particular piece of algebraic data which are sometimes found attached to other objects as an invariant.

We start by introducing the basic notions of formal group law theory. In section 2 we collect various previous results that we will need to prove our main theorem. In section 3 we introduce formal group law chunks and study their properties. The bulk of the section is spent on a representability theorem for the groupoid of p-typical chunks. In section 4 we prove several lemmas relating to a certain fibration of groupoids and then use these results in combination with those from section 4 to classify formal group law chunks over a torsion-free ring in the p-typical case. In section 5 we prove our main theorems (5.3 and 5.4).

Our main result generalizes several previously known classification theorems. However, the proofs of these two do not bear any close resemblance. We finish section 6 with a discussion of how to possibly tackle the case of rings with torsion.

Throughout this paper all formal group laws will be commutative and 1-dimensional.

Definition 1.1. *A formal group law over a ring, R is an $F \in R[[x,y]]$ such that,*

a. $F(0,x) = F(x,0) = x$.

b. $F(x,F(y,z)) = F(F(x,y),z)$.

c. $F(x,y) = F(y,x)$.

In order to motivate this definition we provide the following example. Consider a real 1-dimensional analytic lie group, G. Let, F be the power series expansion of the multiplication on G at the identity. It is easy to veriy that this does in fact give a formal group law.

A much better motivation for the study of formal group laws is the fact that to each multiplicative, even, complex oriented cohomology theory there is an associated formal group law.

Examples:

- $F_a(x,y) = x + y$. Over any ring R.
  This is called the additive formal group law and is usually denoted F_a.

- $F_m(x,y) = x + y - xy$. Over any ring R.
  This is called the multiplicative formal group law and is usually denoted F_m.

- $F(x,y) = \frac{x\sqrt{1-y^4}+y\sqrt{1-x^4}}{1+x^2y^2}$. Over the ring $\mathbb{Z}[1/2]$.
  This is the elliptic formal group law originially discovered by Euler.

In fact for each of the examples given above there is a multiplicative complex oriented cohomology theory associated to it. The additive formal group law is associated to regualr cohomology, $H^*(-)$. The multiplicative formal group law is associated to complex K-theory, $K^*(-)$. The elliptic formal group law above is associated to elliptic cohomology $Ell^*(-)$.

We now recount some of the basic properties of formal group laws which will be relevant to us. It is easy to see from the properties of formal power series that there exists an $i \in R[[x]]$ such tha t such that $F(x, i(x)) = 0$. Using the fact that $F(x, 0) = F(0, x) = x$, we obtain that

$$F(x, y) \equiv x + y \mod (x, y)^2$$

this result is a first result which will be extended much further in the section on comparison lemmas.

In analogy with groups we now introduce morphisms between formal group laws over the same base ring R.

Definition 1.2. *Let F and G be formal group laws over a ring R and let $f \in R[[x]]$, then we say f is homomorphism from F to G if,*

$$f(F(x, y)) = G(f(x), f(y)).$$

We must also specify that $f = 0 \mod (x)$ otherwise the composition of power series is not necessarily defined. As usual an isomorphism is an invertible morphism. It is easy to show that an morphism is invertible iff its degree 1 componenet is invertible in R. If an isomorphism has $f(x) = x \mod (x)^2$, then we call it a strict isomorphism. For the bulk of this paper we will concern ourselves with strict isomorphism. If $F \xrightarrow{f} G$ is a isomorphism of formal group laws, then there exists an $f^{-1}(x)$ such that $f^{-1}(f(x)) = x$. Then, each of (F, f) and (f, G) determine the entire triple becuase we have that $G(x, y) = f(F(f^{-1}(x), f^{-1}(y)))$ and $F(x, y) = f^{-1}(G(f(x), f(y)))$.

Examples:

- The morphism $f(x) = x$ is the identity morphism for every formal group law.
- $f = log(1 + x)$ is a strict isomorphism from F_m to F_a over $\mathbb{Q}$.

$$\begin{aligned} log(1 + F_m(x, y)) = log(1 + x + y + xy) &= log(1 + x) + log(1 + y) \\ &= F_a(log(1 + x), log(1 + y)) \end{aligned}$$

At this point we introduce some notation. A formal group law F can be used as a formal addition, so we define,

$$x +_F y := F(x, y) \text{ and } \sum_{i \in I}^{F} a_i := F(a_{i_1}, F(a_{i_2}, \dots)).$$

Now we introduce a way to change the base ring of a formal group law. Suppose that F is a formal group law over R and $f : R \to S$ is a ring homomorphism. Then, there is a unique $\tilde{f}$ such that

$$\begin{array}{ccc} R[[x,y]] & \xrightarrow{\tilde{f}} & S[[x,y]] \\ \downarrow & & \downarrow \\ R & \xrightarrow{f} & S \end{array}$$

is commutative, $\tilde{f}(x) = x$ and $\tilde{f}(y) = y$. Using this we have a direct image of a formal group law which we denote f_*F. Similarly, we also have a direct image of formal group law morphisms. With this, we now know enough to turn formal group laws over a given ring into a category.

Definition 1.3.

a. *FG : **Ring** $\to$ **Cat** takes a ring R to the category of formal groups laws over R and morphisms between them.*

b. *FG_s : **Ring** $\to$ **Grpd** is the functor which takes a ring R to the groupoid of formal group laws over R and strict isomorphisms between them.*

Theorem 1.4.

a. *$FG(R)$ is enriched over Ab.*

b. *The full single object subcategories of $FG(R)$ give us an endomorphism ring for each formal group law F over R which we will denote $End_R(F)$.*

c. *There is a natural transformation $(i \circ FG_s) \to FG$ for which the induced functors $i(FG_s(R)) \to FG(R)$ are bijective on objects and faithfull. i is the inclusion of **Grpd** in **Cat**.*

Proof. In order to prove a.) we first turn the hom sets into abelian groups. The additive structure we give them is that of the formal sum at the target.

For $f, g \in hom_R(F,G)$ we must show that $f +_G g \in hom_R(F,G)$ as well.

$$\begin{aligned}(f +_G g)(x +_F y) = f(x +_F y) +_G g(x +_F y) &= f(x) +_G f(y) +_G g(x) +_G g(y) \\ &= (f +_G g)(x) +_G (f +_G g)(y)\end{aligned}$$

Commutativity follows from the fact that $f +_G g = g +_G f$. Identity follows from $f +_G 0 = 0 +_G f = f$. Finally, we verify that $i \circ f$ is an inverse for f where $i(x)$ is the inverse for x mentioned above. Letting $\alpha(x) = x$ be denoted 1,

$$(f +_G (i \circ f))(x) = ((1 +_G i) \circ f)(x) = (0 \circ f)(x) = 0.$$

Now, in order to finish the proof of a.) we must show that composition is bilinear. Suppose we have $f, g \in hom_R(F,G)$ and $a, b \in hom_R(G,H)$, then

$$a \circ (f +_G g) = a(f +_G g) = (a \circ f) +_H (a \circ g)$$

and for the other direction,

$$(a +_H b) \circ f = (a \circ f) +_H (b \circ f).$$

b.) is a direct consequence of the definition of preadditive.

The proof of part c.) is rather straightforward, it just uses the inclusions that one gets from $FG_s(R)$ being a subcategory of $FG(R)$ with the same objects. $\square$

Example: For a ring which is torsion-free as a $\mathbb{Z}$-module we have $End_R(F_a) \cong R$. Any endomorphism of F_a will have $f(x+y) = f(x) + f(y)$ from which we can determine that $f(x) = rx$ for some $r \in R$. Each $f(x) = rx$ is also an endormophism, so we have the desired result.

With the setup complete we can now introduce the three main questions asked about formal group laws over a given ring.

a. Which formal group laws are (strictly) isomorphic to a given one?

b. What are the (strict) isomorphism classes of formal group laws?

c. What is the endomorphism ring of a given formal group law ?

Hazewinkel's book [5] collects many of the known answers to these questions in section 18. It provides several good answers to the first question and only partial results on the second and third. Recall that π_0 is the functor from **Grpd** to **Set** which sends a groupoid to the set of its connected components. The second question can be restated as understanding $\pi_0 FG$ and $\pi_0 FG_s$.

2 Background Material

In this secion we introduce necessary background material. We will discuss: logarithms, p-typicality, representability, local-global results and comparison lemmas. Most of the results recounted here can be found in [5].

2.1 Logarithms

In analogy with the example of $log(1+x)$ as a morphism from F_m to F_a we make the following definition.

Definition 2.1. *A logarithm for a formal group law, F, is a strict isomorphism between F and F_a.*

Logarithms are useful because F_a is simplest formal group law. In particular, $End_R(F_a)$ is the easiest endomorphism ring to compute. In [6], Lazard proved the result on existence of logarithms below. The uniqueness part comes from our previous calculation of $End_R(F_a)$ in the torsion-free case.

Lemma 2.2 (logarithm existence and uniqueness). *If R is a $\mathbb{Q}$-algebra then every formal group law over R has a unique strict isomorphism to F_a.*

This result allows us to answer all three of the questions posed at the end of the previous section in the case of a $\mathbb{Q}$-algebra. For torsion-free rings there is an inclusion,

$$A \hookrightarrow A \otimes \mathbb{Q}$$

so logaritms are unique and existence is reduced to an integrality question. In fact, there is an explicit formula for the logarithm over a torsion-free ring,

$$log_F(x) = \int_0^x \frac{1}{(\partial_y F)(t,0)} dt$$

This formula was found in Appendix A.2 of Ravenel's book [7].

Corollary 2.3. *If F is a formal group law over a torsion-free ring A and $log_F(x) = \sum m_i x^{i+1}$, then $(i+1)m_i \in A$.*

Proof. This is clear from the formula above. □

Corollary 2.4. *If A is a torsion-free ring, then there is a strict isomorphism $F \xrightarrow{f} G$ iff $log_G^{-1}(log_F(x)) \in R[[x]]$.*

Proof. Over $A \otimes \mathbb{Q}$ there is only one strict isomorphism class and only one strict automorphism of each formal group law. The corollary follows easily from this. □

2.2 p-typicality

p-typicallity is an analog of the localization at a prime one sees in topology and has several definitons which are of varying generality. We provide the most general of these in order for our work to be carried out over all $\mathbb{Z}_{(p)}$-algebras. The introduction we give to p-typicallity follows section 15 and 16 of [5] and the reader is encouraged to consult that or a similar source.

In order to introduce p-typicallity we must introduce curves. A curve is a $\gamma \in R[[x]]$ such that $\gamma(0) = 0$. Given a group law F over R the curves over R forma a group under formal addition. We now define the frobenius operator $\mathbf{f}_q$ which operates on the group of curves.

$$b(Z_1, \ldots, Z_q; t^{1/q}) := \gamma(Z_1 t^{1/q}) +_F \gamma(Z_2 t^{1/q}) +_F \cdots +_F \gamma(Z_q t^{1/q})$$

b is a power series in $t^{1/q}$ with coeffecients in $R[Z_1, \ldots, Z_q]$. The coefficients of b are homogenious and symmetric, so if $\{\sigma_i\}$ are the elementary symmetric functions on $Z_1, \ldots, Z_q$, then there is b' such that,

$$b'(\sigma_1, \ldots, \sigma_q; t^{1/q}) = b(Z_1, \ldots, Z_q; t^{1/q}).$$

Then, we make the definition

$$\mathbf{f}_q \gamma(t) := b'(0, \ldots, 0, (-1)^{q-1}; t^{1/q})$$

The reader should note that $\mathbf{f}_q \gamma(t)$ is in fact a formal power series in t, not $t^{1/q}$.

Definition 2.5. *A formal group law F is called p-typical if $\mathbf{f}_q t = 0$ for all prime numbers $q \neq p$.*

This definition can be stated much more simply in the case of a torsion-free $\mathbb{Z}_{(p)}$-algebra.

Lemma 2.6. *Let F be a formal group law over a torsion-free $\mathbb{Z}_{(p)}$-algebra A. F is p-typical iff its logarithm can be written in the form $\sum_{i\geq 0} l_i x^{p^i}$ with $l_i \in A \otimes \mathbb{Q}$.*

Theorem 2.7. *If R is a $\mathbb{Z}_{(p)}$-algebra, then each formal group law over R is strictly isomorphic to some p-typical formal group law.*

Isomorphisms between p-typical formal group laws also take on a simpler form similar to the logarithm. The following lemma is the same as lemma A2.1.26 in [7].

Theorem 2.8. *If $F \xrightarrow{f} G$ is an isomorphism between formal group laws over a $\mathbb{Z}_{(p)}$-algebra R and F is p-typical, then G is p-typical iff*

$$f^{-1}(x) = \sum_{i\geq 0}^{F} t_i x^{p^i}$$

for some $t_i \in R$ and t_0 is a unit in R.

Before we go further it is convenient to introduce two functors which capture the behavoir of p-typical formal group laws.

Definition 2.9.

*a. $FG_p : \mathbb{Z}_{(p)}$**-alg** $\to$ **Cat** sends a $\mathbb{Z}_{(p)}$-algebra R to the full sub-category of $FG(R)$ whose objects are the p-typical formal group laws.*

*b. $FG_{p,s} : \mathbb{Z}_{(p)}$**-alg** $\to$ **Grpd** sends a $\mathbb{Z}_{(p)}$-algebra R to the full sub-groupoid of $FG_s(R)$ whose objects are the p-typical formal group laws.*

To end this subsection we introduce the formal p series which will become important in the next section. The formal p series is a formal multiplication by p power series.

$$[p]_F(x) := \sum_{0<i\leq p}^{F} x$$

2.3 Representability

As a rule of thumb most of the functors used in this paper are representable. This is made precise in the theorems that follow. The first such representability result is due to Lazard [6] and is as follows.

Theorem 2.10. *$L \cong \mathbb{Z}[x_1, x_2, \ldots]$ with $|x_i| = 2i$ represents formal group laws. That is, there is a formal group law F over L such that for any formal group law G over R there is a unique map $f : L \to R$ such that $f_* F = G$.*

This result can be restated as $[L,-] \cong Ob(FG(-))$. Later, in [3, 4] Hazewinkel explicitly determines the structure of several universal formal group laws and gives the representing rings.

Theorem 2.11.

a. *$LB \cong L \otimes \mathbb{Z}[b_1, \dots]$ with $|b_i| = 2i$ has $[LB, -] \cong FG_s$.*

b. *$V \cong \mathbb{Z}_{(p)}[v_1, \dots]$ with $|v_i| = 2(p^i - 1)$ has $[V, -] \cong Ob(FG_p(-))$.*

c. *$VT \cong V \otimes \mathbb{Z}_{(p)}[t_1, \dots]$ with $|t_i| = 2(p^i - 1)$ has $[VT, -] \cong FG_{p,s}$.*

Let F_V be the universal p-typical formal group law. Then, we can choose the generators v_i of V such that $[p]_{F_V}(x) = \sum_{i \geq 0}^{F} v_i t^{p^i}$, with the convention that $v_0 = p$. This choice of generators first appeared in [1].

2.4 Local-Global Results

This section recounts a result first appearing in section 20.5 of [5]. It allows us to reconstruct information about strict isomorphism classes of formal group laws from information about strict isomorphism classes of p-typical formal group laws.

Theorem 2.12. *Let A be a torsion-free ring.*

a. *If F and G are formal group laws over A, then they are strictly isomorphic iff they are strictly isomorphic over $A \otimes \mathbb{Z}_{(p)}$ for all p.*

b. *If we have a formal group law $F_{(p)}$ over $A \otimes \mathbb{Z}_{(p)}$ for every prime p, then there exists an F over A such that F is strictly isomorphic to $F_{(p)}$ over $A \otimes \mathbb{Z}_{(p)}$ for all p.*

As remarked in [5] the second of these results holds even when A is not torsion-free. Rewriting this result in terms of π_0 and FG_S it appears as,

Corollary 2.13. *If A is a torsion-free ring, then the map,*

$$\pi_0 FG_s(A) \to \prod_{p \ prime} \pi_0 FG_s(A \otimes \mathbb{Z}_{(p)}).$$

is a bijection.

The previous remark becomes a statement that this map is surjectivity even when A is not torsion-free. Due to this theorem it is sufficient for us to consider only the case of p-typical formal group laws for the remainder of the paper.

2.5 Comparison Lemmas

This section relates several results about formal group laws that are nearly congruent modulo some degree. In order to introduce our results we first make some combinatorial definitions.

Definition 2.14.

a. $B_n(x,y) := x^n + y^n - (x+y)^n$

b. $C_n(x,y) := \begin{cases} p^{-1}B_n(x,y) & n = p^k \text{ for some prime } p \\ B_n(x,y) & \text{otherwise} \end{cases}$

C_n is the unique multiple of B_n that is still integral, but has coffecients with gcd 1. A comparison lemma first appeared in [6] and our extended version can be proved from the results in [3].

Lemma 2.15 (Comparison lemma)**.** *Let R be a ring and A a $\mathbb{Z}_{(p)}$-algebra.*

a. If F and G are formal group laws over R and $F(x,y) \equiv G(x,y) \mod (x,y)^n$, then there is $a \in R$ such that,

$$F(x,y) \equiv G(x,y) + aC_n(x,y) \mod (x,y)^{n+1}.$$

b. If F and G are p-typical formal group laws over A and $F(x,y) \equiv G(x,y) \mod (x,y)^n$ with $p^{k-1} < n \leq p^k$, then there is $a \in A$ such that,

$$F(x,y) \equiv G(x,y) + aC_{p^k}(x,y) \mod (x,y)^{p^k+1}.$$

c. If F and G are p-typical formal group laws over A represented by $f, g \in [V, A]$ respectively, then $f(v_i) = g(v_i)$ for all $i \leq n$ iff

$$F(x,y) \equiv G(x,y) \mod (x,y)^{p^n+1}.$$

d. If F and G are p-typical formal group laws over A represented by $f, g \in [V, A]$ respectively and $f(v_i) = g(v_i)$ for all $i < n$, then

$$F(x,y) \equiv G(x,y) + (f(v_n) - g(v_n))C_{p^n}(x,y) \mod (x,y)^{p^n+1}.$$

3 Introduction to Formal Group Law Chunks

A formal group law chunk is a truncated form of a normal formal group law. We make this precise as follows.

Definition 3.1. *A formal group law chunk over a ring R of size n is a $F \in R[x,y]/(x,y)^{n+1}$, such that*

a. $F(0,x) = F(x,0) = x$.

b. $F(x, F(y,z)) = F(F(x,y),z)$.

c. $F(x,y) = F(y,x)$.

A morphism of formal group law chunks of size n, F and G, is a $f \in R[x]/(x)^{n+1}$ *such that,*

$$f(F(x,y)) = G(f(x), f(y)).$$

These conditions are analogous to those for a formal group law, but chunks turn out to be easier objects to study. As usual the invertible morphisms are the ones with degree 1 component a unit. An isomorphism is called strict if $f(x) = x \mod (x)^2$. In [6] Lazard proved the following result:

Theorem 3.2. *Every formal group law chunk is the truncated form of some formal group law.*

Using 3.2 we can make a definition of p-typicallity for chunks which is rather more succint than it would otherwise be.

Definition 3.3.

a. Ch^n : $\boldsymbol{Ring} \to \boldsymbol{Cat}$ *sends a ring, R, to the category of formal group law chunks of size n over R and morphisms between them.*

b. Ch^n_s : $\boldsymbol{Ring} \to \boldsymbol{Grpd}$ *sends a ring, R, to the groupoid of formal group law chunks of size n over R and strict isomorphisms between them.*

c. A formal group law chunk is called p-typical if it is the truncated form of a p-typical formal group law.

d. $Ch^n_p : \mathbb{Z}_{(p)}$**-alg** $\to$ $\boldsymbol{Cat}$ *sends a* $\mathbb{Z}_{(p)}$*-algebra R to the full sub-category of* $Ch^{p^n}(R)$ *whose objects are the p-typical chunks.*

e. $Ch^n_{p,s} : \mathbb{Z}_{(p)}$**-alg** $\to$ $\boldsymbol{Grpd}$ *sends a* $\mathbb{Z}_{(p)}$*-algebra R to the full sub-groupoid of* $Ch^{p^n}_s(R)$ *whose objects are the p-typical chunks.*

The intuitive idea of truncating a formal group law or chunk can be captured by a set of related natural transformation. Each of the projection homomorphisms $R[[x,y]] \to R[x,y]/(x,y)^{n+1}$ and $R[x,y]/(x,y)^{m+1} \to R[x,y]/(x,y)^{n+1}$ where $m > n$ induce a truncation natural transformations which we denote by ρ_n and $\rho_{m,n}$ respectively. When the source and target are clear from context we will simply write ρ for brevity. We will also use ρ_* to denote $\rho(R)$ when the ring we are working over is clear.

It is straightforward to see that,

$$\rho_*(FG_p(R)) \subseteq \mathrm{Ch}^n_p(R).$$

Lemma 3.4.

a. If R is a $\mathbb{Z}_{(p)}$*-algebra and* $F \in Ch^n(R)$*, then F is strictly isomorphic to some p-typical chunk.*

b. *If R is a $\mathbb{Q}$-algebra, then logarithms exist in $Ch_s^n(R)$.*

c. *The only endomorphism of the additive formal group law chunk over a torsion-free ring which is also a strict isomorphism is the identity.*

d. *If A is a torsion-free ring, then for $F, G \in Ch_s^n(A)$ there is at most one strict isomorpishm $F \xrightarrow{f} G$.*

e. *If we restrict $\pi_0 Ch_s^{p^n}$ and $\pi_0 Ch_{p,s}^n$ to $\mathbb{Z}_{(p)}$-alg, they are naturally isomorphic.*

Proof. a.) By Theorem 3.2 there is some $F' \in Ob(FG(R))$ such that $\rho_*(F') = F$. By Theorem 2.7 there is a strict isomorphism $F' \xrightarrow{f} F''$ where F'' is p-typical. Then, $\rho_*(f)$ is a strict isomorphism from F to some p-typical formal group law chunk.

b.) Let $F \in Ob(\mathrm{Ch}_s^n(A))$, then by theorem 3.2 there is some $F' \in Ob(FG_s(A))$ such that $\rho_*(F') = F$. By lemma 2.2 there is a strict isomorphism $F' \xrightarrow{f} F_a$. Then, $\rho_*(f)$ is a strict isomorphism from F to the additive formal group law chunk.

c.) Suppose f is an endomorphism as hypothesized, then $f(x+y) \equiv f(x) + f(y) \mod (x,y)^{n+1}$ which shows that $f(x) = x \in R[x]/(x)^{n+1}$.

d.) Suppose that f, g are two distinct strict isomorphisms between F and G. Then, over $A \otimes \mathbb{Q}$ we have $log_F \circ f^{-1} \circ g \circ \log_F^{-1} \neq Id$, which contradicts part c.).

e.) This is just a restatement of part a.). □

In light of the various representability results we will from here on silently identify a formal group law (or chunk) with the map representing it. For example, if F is a p-typical formal group law then we will say $F \in Ob(FG_p(R))$ and $F \in [V, R]$. This allows us to speak of both $F(x,y)$ and $F(v_i)$ without needlessly introducing more variables. It can be determined which one we are treating F as at a given moment based on how many arguments it has.

Theorem 3.5 (Representability for p-typical Chunks)**.**
$V_n = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$ where $|v_i| = 2(p^i - 1)$ represents objects in Ch_p^n.

Proof. Let $r : V \to V_n$ be the map for which,

$$r(v_i) = \begin{cases} v_i & i \le n \\ 0 & i > n \end{cases}.$$

This induces a natural transformation $r : [V_n, -] \to [V, -]$. It now suffices to prove that $\rho \circ r$ is a natural isomorphism.

First we show surjectivity. Let $F \in Ob(\mathrm{Ch}_p^n(R))$. By the definition of p-typicality for chunks there is a p-typical formal group law G such that $\rho_*(G) =$

F. Let $H \in [V_n, R]$ be the map such that $H(v_i) = G(v_i)$ for $i \leq n$. Then, by the comparison lemma,

$$G(x, y) \equiv r(H)(x, y) \mod (x, y)^{p^n+1}.$$

Thus, $F = \rho_*(G) = (\rho \circ r)(H)$.

Next we show injectivity. Let $f, g \in [V_n, R]$ and suppose that $(\rho \circ r)(f) = (\rho \circ r)(g)$. This is equivalent to $r(f)(x, y) \equiv r(g)(x, y) \mod (x, y)^{p^n+1}$. Which, by the comparison lemma means that $r(f)(x_i) = r(g)(x_i)$ for all $i \leq n$. But, for $i \leq n$, $r(f)(x_i) = f(x_i)$ and $r(g)(x_i) = g(x_i)$. So, we get $f(x_i) = g(x_i)$ for all $i \leq n$ which implies $f = g$. □

This result explains why we only look at the p-typical chunks of certain sizes. The ones we consider are exactly when a new generator pops up. For a formal group law chunk F we get a formal sum just like with formal group laws.

Lemma 3.6. *If $F \xrightarrow{f} G$ is a strict isomorphism of formal group law chunks of size n over a $\mathbb{Z}_{(p)}$-algebra R and F is p-typical, then G is p-typical iff*

$$f^{-1}(x) = \sum_{0 \leq i \leq k}^{F} t_i x^{p^i}$$

with $p^k \leq n < p^{k+1}$, $t_i \in A$ and $t_0 = 1$.

Proof. First we prove the if part. Pick a $F' \in Ob(FG_p(R))$ such that $\rho_*(F') = F$. Note that the triple (F, f, G) is determined by (F, f). Let $\tilde{f} \in R[[x]]$ be such that,

$$\tilde{f}^{-1}(x) = \sum_{0 \leq i \leq k}^{F'} t_i x^{p^i}$$

then, $\rho_*(\tilde{f}) = f$. By Theorem 2.8 $(F', \tilde{f})$ is a strict isomorphism between p-typical formal group laws. Applying ρ_* we get (F, f) back again so it must be a morphism between p-typical chunks. p Next we prove the only if part. We proceed by contradiction. Let n be the smallest size where a counterexample (F, f, G) occurs. By the hypothesis $\rho_*(f)$ does not provide a counterexample so we have,

$$f^{-1} \equiv \sum_{i \geq 0}^{F} t_i x^{p^i} \mod (x^n)$$

Thus, there exists some $r \in R$ such that $f^{-1} = rx^n + \sum_{i \geq 0}^{F} t_i x^{p^i}$.

Case 1: $n = p^k$. we have that,

$$f^{-1} \equiv rx^{p^k} + \sum_{0 \leq i \leq k-1}^{F} t_i x^{p^i} \equiv rx^{p^k} +_F \sum_{0 \leq i \leq k-1}^{F} t_i x^{p^i} \mod (x^{p^k+1})$$

This expresses f^{-1} in exactly the form it wasn't allowed to be in so we are done.

Case 2: $n = p^k + a$. This time we seek to show that $r = 0$. Let $g^{-1}(x) = f^{-1}(x) - rx^n$, by the first half of this theorem (F, g) has a target G' which is p-typical. Then,

$$G \equiv G' \mod (x^n)$$

which by the comparison lemma means that,

$$G \equiv G' \mod (x^{n+1})$$

$$\begin{aligned}(g(x) - rx^n) \circ f^{-1}(x) &\equiv g(f^{-1}(x)) - r(f^{-1}(x))^n \\ &\equiv g(g^{-1}(x) + rx^n) - rx^n \\ &\equiv g(g^{-1}(x)) + rx^n - rx^n \\ &\equiv x \mod (x^{n+1})\end{aligned}$$

This gives us a way to write f in terms of g. To finish the proof we perform a calculation which implies that $r = 0$.

$$\begin{aligned}G &\equiv f(F(f^{-1}(x), f^{-1}(y))) \\ &\equiv f(F(g^{-1}(x) + rx^n, g^{-1}(y) + ry^n)) \\ &\equiv f(F(g^{-1}(x), g^{-1}(y)) + r(x^n + y^n)) \\ &\equiv f(F(g^{-1}(x), g^{-1}(y))) + r(x^n + y^n) \\ &\equiv g(F(g^{-1}(x), g^{-1}(y))) - r(F(g^{-1}(x), g^{-1}(y)))^n + r(x^n + y^n) \\ &\equiv G' - r(g^{-1}(x) + g^{-1}(y))^n + r(x^n + y^n) \\ &\equiv G' - r(x + y)^n + r(x^n + y^n) \\ &\equiv G' - r((x + y)^n - x^n - y^n) \mod (x, y)^{n+1}\end{aligned}$$

□

Equipped with this lemma we can now attack the problem of representability of $\mathrm{Ch}^n_{p,s}$.

Theorem 3.7. *$VT_n = V_n \otimes \mathbb{Z}_{(p)}[t_1, \ldots, t_n]$ where $|t_i| = 2(p^i - 1)$ represents $Ch^n_{p,s}$.*

Proof. A strict isomorphism (F, f, G) can be specified by giving (F, f). By Lemma 3.6 f can be chosen independently of F. Lemma 3.6 also says that specifying f is the same thing as specifying each of the t_i. The expansion of f^{-1} as a formal sum of t_i is unique. Thus, specifying a (F, f) is that same as giving a map $F \in [V_n, R]$ and a map $f \in [\mathbb{Z}_{(p)}[t_1, \ldots, t_n], R]$. By the universal property for colimits these two maps are exactly a map $VT_n \to R$. □

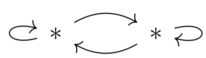
Corollary 3.8. *In the p-typical case ρ_* is induced by $i : VT_n \to VT_m$ and is surjective on objects and arrows.*

Proof. We note that $\rho_*(F)(v_i) = F(v_i)$ for $i \leq n$ and that this determines the behavior of ρ_* i obviously has the same property so they are the same. The second part is an obvious corollary of the first. $\square$

4 Classification of Chunks

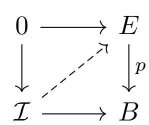
In this section we give a classification of the strict isomorphism classes of p-typical formal group law chunks over a torsion-free $\mathbb{Z}_{(p)}$-algebra. For a formal group law or chunk, F, we let $[F]$ denote its strict isomorphism class.

Before proceeding further we introduce fibrations of groupoids as seen in [2]. We let $\mathcal{I}$ denote the groupoid with two objects pictured below.



This grouoid is an analog of the unit interval in topology.

Definition 4.1. *Let E and B be groupoids and $p : E \to B$ a morphism between them. p is a fibration if for every commutative square as shown we have the lift indicated.*



Theorem 4.2. $\rho_* : Ch^n_{p,s}(R) \to Ch^{n-1}_{p,s}(R)$ *is a fibration.*

Proof. A map $\mathcal{I} \to \mathrm{Ch}^{n-1}_{p,s}(R)$ is the same thing as a strict isomorphism $(F, f) \in \mathrm{Ch}^{n-1}_{p,s}(R)$. A map $0 \to \mathrm{Ch}^n_{p,s}(R)$ is the same thing as a $F' \in [V_n, R]$. Commutativity of the square is the same as saying $\rho_*(F') = F$. What we need to find is a $(F', f') \in [VT_n, R]$ such that $\rho_*(f') = f$.

ρ_* is surjective so we can choose some pair $(G, g) \in \mathrm{Ch}^n_{p,s}(R)$ such that $\rho_*(G, g) = (F, f)$. Now, consider the pair $(F', g) \in \mathrm{Ch}^n_{p,s}(R)$, it obviously has the desired properties. $\square$

As a corollary of this result we have that $(\pi_0\rho_*)^{-1}([F]) \cong \pi_0(\rho_*^{-1}(F))$. Our goal will be to determine the right hand side of this bijection.

Lemma 4.3. *There is an action of R (as an abelian group) on $Ch^n_{p,s}(R)$ such that $Ch^n_{p,s}(R)/R \cong Ch^{n-1}_{p,s}(R)$.*

Proof. First we define the action of R on objects. Let $F \in Ob(\mathrm{Ch}^n_{p,s}(R))$ and $a \in R$, then

$$(a.F)(v_i) := F(v_i) + \begin{cases} 0 & i \neq n \\ a & i = n \end{cases}.$$

By the comparison lemma this tells us that, $(a.F)(x,y) = F(x,y) + aC_{p^n}(x,y)$.

Now, define $a.(F,f) = (a.F,f)$. The only non-trivial thing to verify in order for this to be an action on a groupoid is that a is a morphism of groupoids. Thus, we must show that for a strict isomorphism (F,f,G) that $(a.F,f,a.G)$ is also a strict isomorphism.

$$\begin{aligned} f((a.F)(x,y)) &= f(F(x,y) + aC_{p^n}(x,y)) \\ &= f(F(x,y)) + aC_{p^n}(x,y) \\ &= G(f(x),f(y)) + aC_{p^n}(f(x),f(y)) \\ &= (a.G)(f(x),f(y)) \end{aligned}$$

Because this action is exactly modifying the value of v_n the second part of the lemma is clear. □

For the remainder of this section A will denote a torsion-free $\mathbb{Z}_{(p)}$-algebra.

Theorem 4.4. *If $F \in Ch^{n-1}_{p,s}(A)$, $G, H \in Ch^n_{p,s}(A)$ and $\rho_* G = \rho_* H = F$, then G and H are strictly isomorphic iff $G(v_n) \equiv H(v_n) \mod (p)$.*

Proof. Any strict isomorphism $G \xrightarrow{f} H$ will have $\rho_*(f) = Id(x) = x$ by Lemma 4.4.d. Thus, we can write f in the form $f(x) = x + bx^n$. By the comparison lemma,

$$G(x,y) \equiv H(x,y) + (G(v_n) - H(v_n))C_n(x,y) \mod (x,y)^{n+1}$$

We put all of this together and get,

$$\begin{aligned} f(G(x,y)) &\equiv H(f(x),f(y)) \mod (x,y)^{n+1} \\ G(x,y) + b(G(x,y))^n &\equiv G(x+bx^n, y+by^n) + (H(v_n) - G(v_n))C_n(f(x),f(y)) \mod (x,y)^{n+1} \\ G(x,y) + b(x+y)^n &\equiv G(x,y) + b(x^n + y^n) + (H(v_n) - G(v_n))C_n(x,y) \mod (x,y)^{n+1} \\ bB_n(x,y) &\equiv (H(v_n) - G(v_n))C_n(x,y) \mod (x,y)^{n+1} \\ bp &= (H(v_n) - G(v_n)) \end{aligned}$$

This last equation is solvable exactly when $G(v_n) \equiv H(v_n) \mod (p)$, which completes the theorem. □

Theorem 4.4 is the crux of all of our results. It allows us to compute the the fibers of ρ_* and thereby determine strict isomorphism classes.

Lemma 4.5. *There is a free action of A/pA on $\pi_0 Ch^n_{p,s}(A)$ such that*

$$\pi_0 Ch^n_{p,s}(A)/(A/pA) \cong \pi_0 Ch^{n-1}_{p,s}(A).$$

Proof. Applying π_0 to the action in Lemma 4.3 we get an induced action of A on $\pi_0\mathrm{Ch}^n_{p,s}(A)$ which is defined by $a.[F] = [a.F]$. This actions factors to be an action of A/pA iff $[F] = [(pa).F]$ for all $a \in A$. This is exactly the statement of Theorem 4.4 so we have the desired action. This action is free because for $a \in A$ such that $a \notin pA$ Theorem 4.4 gaurantees that $F \notin [a.F] = a.[F]$.

In order to prove the last part of the theorem, let F, G and H be such that $[\rho_*(G)] = [\rho_*(H)] = [F]$, then by Theorem 5.2 we can pick $F' \in [G]$ and $F'' \in [H]$ such that $\rho_*(F') = \rho_*(F'') = F$. Thus, there is a so that $a.F' = F''$ which completes the proof. □

Corollary 4.6.

$$\pi_0\, Ch^n_{p,s}(A) \cong (A/pA) \times \pi_0\, Ch^{n-1}_{p,s}(A)$$

Proof. The free action in Lemma 4.5 gives us this. □

Corollary 4.7.

$$\pi_0\, Ch^n_{p,s}(A) \cong (A/pA)^n$$

Proof. Obvious. □

5 Classification of Formal Group Laws

In this section we prove results that allows us to recombine the information about strict isomorphism classes of chunks into information about strict isomorphism classes of formal group laws.

Theorem 5.1.

$FG_{p,s}$ and $\lim(Ch^n_{p,s})$ are naturally isomorphic.

Proof. This result is intuitively obvious from representability and Corollary 3.8. □

Theorem 5.2. *$\pi_0 FG_{p,s}$ and $\lim(\pi_0\, Ch^n_{p,s})$ are naturally isomorphic.*

Proof. We already have natural transformations $\pi_0 FG_{p,s} \to \pi_0\mathrm{Ch}^n_{p,s}$, so by the universal property for limits we get a natural transformation $\pi_0 FG_{p,s} \to \lim(\pi_0\mathrm{Ch}^n_{p,s})$, so we just need to show that this is an isomorphism. Injectivity is easy to show. If $[F], [G] \in FG_{p,0}(A)$ map to the same class then we have a strict isomorphism between $\rho_*(F)$ and $\rho_*(G)$ for every $\mathrm{Ch}^n_{p,s}(A)$, which by theorem 5.1 means F and G are strictly isomorphic, so $[F] = [G]$. Showing surjectivity is slightly more complicated. Suppose we have $a \in \lim(\pi_0\mathrm{Ch}^n_{p,s})$, which is determined by its images $a_n \in \pi_0\mathrm{Ch}^n_{p,s}$. There is only 1 object in $\mathrm{Ch}^1_{p,s}(A)$, so set $F_1 = x + y \in \mathrm{Ch}^1_{p,s}(A)$. Now we proceed by induction. By lemma 2.2 we can choose an $F_n \in a_n$ such that $\rho_* F_n = F_{n-1}$. The choices of F_n determine an $F \in \lim(\mathrm{Ch}^n_{p,s}(R)) \cong FG_p(R)$ which means that $[F]$ maps to a as desired. □

Theorem 5.3. *Let A be a torsion-free $\mathbb{Z}_{(p)}$-algebra.*

a.
$$\pi_0 FG_{p,s}(A) \cong (A/pA)^{\mathbb{N}}.$$

b. Given a map of sets $s : A/pA \to A$ which is a section of $A \to A/pA$ and a $F \in FG_{p,s}(A/pA)$ let s_*F be such that $(s_*F)(v_i) = s(F(v_i))$. Then, $\pi_0 FG_{p,s}(A) \cong Ob(FG_p(A/pA))$.

Proof. a.) By Corollary 4.7 and Theorem 5.2,

$$\pi_0 FG_{p,s}(A) \cong \lim(A/pA)^n = (A/pA)^{\mathbb{N}}$$

b.) Let $S = \{s_*F \mid F \in Ob(FG_p(A/pA))\}$. The proof proceeds in two parts. First we show that no two distinct $F, G \in S$ are strictly isomorphic. Second we show that every $F \in Ob(FG_p(A))$ is strictly isomorphic to some $F' \in S$.

Suppose $F, G \in S$ are distinct. Let n be the smallest number such that $F(v_n) \neq G(v_n)$. Then, we know $F(v_n) \not\equiv G(v_n) \mod (p)$ by the definition of S. Thus, by Theorem 4.4, $\rho_{n*}(F)$ is not strictly isomorphic to $\rho_{n*}(G)$, so F and G are not strictly isomorphic.

For the second part, by Theorem 5.2 it is sufficient to show that if $\rho_{n*}(F) = \rho_{n*}(G)$ for $G \in S$ then there is $H \in S$ and F' such that $F \cong F'$ and $\rho_{n+1*}(F') = \rho_{n+1*}(H)$.

Let $a = s(F(v_{n+1}) + pA) - G(v_{n+1})$. Then, $F(v_{n+1}) \equiv G(v_{n+1}) + a \mod (p)$, so by Theorem 4.4, $\rho_{n+1*}(F) \cong a.(\rho_{n+1*}(G))$ and $(a.\rho_{n+1*}(G))(v_{n+1}) \in \mathrm{img}(s)$ so we are done. ◻

Theorem 5.4. *If R is torsion-free, then*

$$\pi_0 FG_s(R) \cong \prod_{p \text{ prime}} (A/pA)^{\mathbb{N}}.$$

Proof. This is proved by Corollary 2.13 in combination with Theorem 5.3 and the fact that,

$$(A \otimes \mathbb{Z}_{(p)})/p(A \otimes \mathbb{Z}_{(p)}) \cong A/pA.$$

◻

In the case where there is torsion we do not have the local global result. However, (at least in the p-typical case) we still do have an action of R on the p-typical chunks and the projection map is still a fibration. Thus, a classification of p-typical formal group laws requires only that we determine the fibers of this fibration. We may be able to do this by looking at the stabilizers of the induced action of A on strict isomorphism classes.

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