IN SEARCH OF A MAHLER-TYPE INEQUALITY FOR THE EXTERNAL CONFORMAL RADIUS
SPUR FINAL PAPER, SUMMER 2015

JINGWEN CHEN
MENTOR: NICK STREHLKE

1. ABSTRACT

If K is a convex set in the plane that contains the origin, let K° be its polar body. We attempt to minimize the product $c(K)c(K^\circ)$, where c denotes the external conformal radius or logarithmic capacity of K. This follows the classical Mahler inequality, which concerns extremizing the analogous volume product $\mathrm{vol}(K)\,\mathrm{vol}(K^\circ)$. The volume product is, in the plane, minimized by the triangle and maximized by the disk. Recently, Bucur and Fragalá worked on the analogous problem for the functional $\lambda_1(K)\lambda_1(K^\circ)$, where λ_1 is the first Dirichlet eigenvalue for K, showing that this quantity is minimized when K is a disk, and that, modulo invertible linear transformations, it is maximized among axisymmetric planar convex sets by the square. It turns out that among the regular polygons the conformal radius product is minimized by the pentagon, and we give some ideas about minimizing it more generally.

Date: July 29 2015.

2. INTRODUCTION

For convex body K, the polar body is defined by

$$K^\circ = \{y : \sup_{x \in K}(x, y) \leq 1\}.$$

Mahler considered the volume product $\mathrm{vol}(K)\,\mathrm{vol}(K^\circ)$, which is invariant under non-singular linear transformations. He showed that among symmetric bodies in the plane, the product is minimized by the square and maximized by the disk. With the symmetry condition removed, the triangle with centroid the origin minimizes the product. Santaló later proved using Steiner symmetrization that the ellipsoids maximize the volume product in higher dimensions: the inequality $\mathrm{vol}(K)\,\mathrm{vol}(K^\circ) \leq \mathrm{vol}(B)\,\mathrm{vol}(B^\circ)$ for origin-symmetric convex bodies is known as the Santaló inequality. However, the corresponding lower bound, which among origin-symmetric convex bodies is conjectured to be attained by the cube, is unproved in higher dimensions.

Recently, Dorin Bucur and Ilaria Fragalá treated a similar problem for the first Dirichlet eigenvalue $\lambda_1(K)$ of a convex body K. The eigenvalue product $\lambda_1(K)\lambda_1(K^\circ)$ is not invariant under nonsingular linear transformations, though it is invariant under scaling. By applying the Santaló inequality and the Faber–Krahn inequality, Bucur and Fragalá showed that the eigenvalue product $\lambda_1(K)\lambda_1(K^\circ)$ is minimized in all dimensions when K is a ball. Formulating the maximization problem is troublesome, because along a sequence of long, thin rectangles, for instance, the eigenvalue product will become infinite. To get around this, Bucur and Fragalá show that the

$$\sup_K \inf_T \lambda_1(TK)\lambda_1((TK)^\circ),$$

the supremum being taken over all origin-symmetric convex bodies K and the infimum over all non-singular linear transformations T. They also consider a more specialized problem, where the supremum is taken over axisymmetric convex bodies K (convex bodies symmetric with respect to each of the coordinate hyperplanes) and the infimum over all non-singular diagonal linear transformations. Using a combination of theoretical and numerical tools, Bucur and Fragalá were able to prove that, in the planar case, the axisymmetric problem is solved by the square.

Following Bucur and Fragalá, we consider the problem of extremizing the logarithmic capacity in the plane. The logarithmic capacity $c(K)$ of K is defined by

$$\log c(K) = \lim_{|z|\to\infty} \log|z| - G_K(z, \infty),$$

where $G_K(z, \infty)$ is the Green's function for K with pole at infinity. It is the same as the transfinite diameter of the set K, which can be defined as the limit τ as $n \to \infty$ of the (decreasing) sequence of geometric means

$$\tau_n = \sup_{z_1,\dots,z_n \in K} \prod_{j\neq k} (z_j - z_k)^{1/(n(n+1))}.$$

The capacity gives a measure of a set that is useful in potential theory, and as the second definition suggests it behaves very loosely like diameter. It is difficult to compute in general. Another name for logarithmic capacity when K is simply connected is external conformal radius, and this comes from the fact that, if f is the conformal map from the exterior of the unit disk to the exterior of K that takes infinity to itself, then

$$f(z) = (c(K))z + O(1)$$

as $|z| \to \infty$. We study the product $c(K)c(K^\circ)$ of the capacity of a convex planar set with the capacity of its polar body.

Like the eigenvalue product, this is not invariant under non-singular linear transformations, though it is invariant under scaling. Although the capacity is monotone increasing under inclusion ($c(K) \leq c(K')$ if $K \subset K'$), opposite the situation with the Dirichlet eigenvalue, the minimization problem is easier to formulate here as well. Thus we focus our attention on this problem, looking for

$$\inf_K c(K)c(K^\circ).$$

It is not necessary to restrict to origin symmetric convex bodies for the minimization problem, although this may make the proof easier. The maximization problem can be formulated in the same way as for the eigenvalue product as

$$\sup_K \inf_T c(T(K))c((T(K))^\circ),$$

the infimum taken over non-singular linear transformations T and the supremum over origin-symmetric convex bodies K.

We show, using the Schwarz–Christoffel formula for conformal mapping to the exterior of a polygon, that among the regular polygons the pentagon solves the minimization problem. The capacity product decreases from the regular triangle to the pentagon, and then increases monotonically as $n \to \infty$ to the capacity product for the disk.

An outline of the report is as follows. First the polar body is introduced and basic properties are proved, and then some properties of the capacity are established. Finally, we use conformal mapping to compute the capacity for the regular polygons and for some other polygonal regions.

3. The polar body

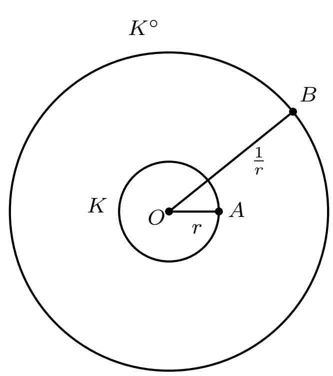
Definition 3.1. For a set of points $K \in \mathbb{R}^d$, we define its polar body K° by

$$K^\circ = \{y : (x, y) \leq 1, \forall x \in K\}$$

Here $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^d$.

Next we will show some examples of the polar body.

Example 3.1 (Polar body of a ball)**.** Let $B_r = \{x \in \mathbb{R}^d \colon |x| \leq r\}$. Then the polar body B_r° is easily seen to be the ball $B_{1/r}$. The volume product is therefore equal to $|B_1|^2$. In $\mathbb{R}^2$, for instance, it is π^2.

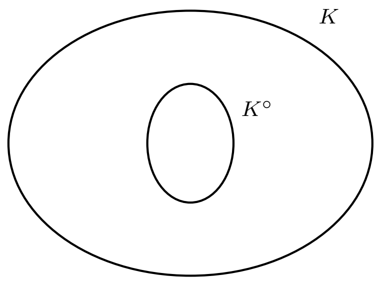


Example 3.2 (Polar body of an ellipse)**.** The polar body of the ellipsoid K given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1,$$

where $ab \neq 0$, is the ellipse $K^\circ = \{(x, y) : a^2x^2 + b^2y^2 \leq 1\}$.

Generally, let an ellipsoid $E \subset \mathbb{R}^d$ be given by the equation $\sum_{i=1}^d \left(\frac{x_i}{a_i}\right)^2 \leq 1$. Then E° is the ellipsoid given by the equation $\sum_{i=1}^d (a_i x_i)^2 \leq 1$.

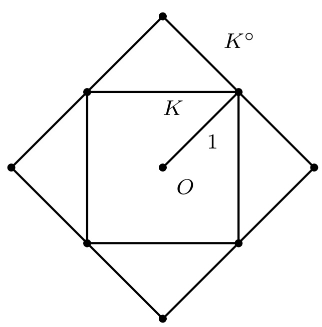


Example 3.3 (Polar body of a square)**.** The polar body of the square

$$K = \{(x, y) : -1 \leq \max\{|x|, |y|\} \leq 1\}$$

is a square

$$K^\circ = \{(x, y) : -2 \leq |x| + |y| \leq 2\}.$$



Example 3.4 (Polar body of a half-space)**.** We consider the half-space

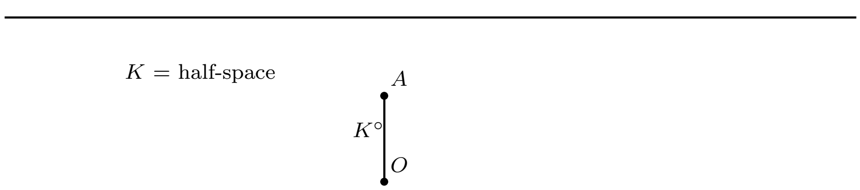
$$H_1 = \{z = (x_1, \dots, x_d) : x_1 \geq 1\}.$$

From the definition, we know that

$$H_1^\circ = \left\{ z = (y_1, \dots, y_d) : \sum_{i=1}^d x_i y_i \leq 1, \forall x_1 \geq 1 \right\}.$$

Since we can choose arbitrary $x_2, \dots, x_d$, we know that $y_2 = \cdots = y_d = 0$ for any $(y_1, \dots, y_d) \in H_1^\circ$. And we know that the equality $x_1 y_1 \leq 1, \forall x_1 \geq 1$ is equal to $y_1 \leq 1$, thus $H_1^\circ = \{z = (y_1, \dots, y_d) : y_1 \leq 1, y_2 = \cdots = y_d = 0\}$.

Generally the polar body of a half-space is contained in a line. For instance, in R^2, the polar body of the half-plane $\{(x,y) : ax+by \le 1\}$, where $a,b>0$, is the line segment $\{(x,y) : bx = ay,\ 0 \le x \le a\}$.



The product $\mathrm{vol}(K)\,\mathrm{vol}(K^\circ)$ of volumes is invariant under nonsingular linear transformations. This is because $(TK)^\circ = (T^*)^{-1}K^\circ$, where T^* is the adjoint of T.

Next we will use a notation for the half-plane of a point. On $\mathbb{R}^2$, for any point $A \ne O$, where O is the origin, we define the half-plane H^A as the polar body of the line segment OA. For instance, when $A=(a_1,a_2)$, where $|a_1|+|a_2| \ne 0$, then $H^A = \{(x,y) : a_1x + a_2y \le 1\}$.

Using this definition we can easily express the polar body of convex set. We begin with a lemma.

Lemma 3.1. If $A \subset B \subset \mathbb{R}^n$, then $B^\circ \subset A^\circ$.

Proof. From the definition, we know that for any $x \in B^\circ$, then $y \in B$ for any $y \in A$, thus $(x,y) \le 1$, so $x \in A^\circ$. As a result, $B^\circ \in A^\circ$. □

Lemma 3.2. The polar body of any set K is always convex.

Proof. Since if $a, b \in K^\circ$, and $0 \le \lambda < 1$, we know that for any $x \in K$, then $(a,x) \le 1$ and $(b,x) \le 1$. Thus $(\lambda a + (1-\lambda)b, x) = \lambda(a,x) + (1-\lambda)(b,x) \le 1$. So $\lambda a + (1-\lambda)b \in K^\circ$. Thus K° is convex. □

Theorem 3.1. For a convex set K, if the origin $O \in K$, then the polar body of K is the intersection of all half-plane corresponding to the point contained in K. We can express as $K^\circ = \cap_{A\in K} H^A = \cap_{A \in \partial K} H^A$.

Proof. Since K is convex, and $O \in K$, we know that $OA \subset K$ for any $A \in K$. Thus by using Lemma 3.1, we know that $K^\circ \subset H^A$, thus $K^\circ \subset \cap_{A\in K} H^A$.

For the other direction, assume there exists a point $P \in \cap_{A\in K} H^A$ such that $P \notin K^\circ$. Then from the definition of polar body, we know that there exists a point $Q \in K$, such that $(P,Q) > 1$. But $P \in H^Q$, thus $(P,Q) \le 1$, contradicting $(P,Q) > 1$. So $\cap_{A\in K} H^A \subset K^\circ$.

This proves that $K^\circ = \cap_{A\in K} H^A$.

The equation $\cap_{A\in K} H^A = \cap_{A\in\partial K} H^A$ is true because $\cap_{A\in K} H^A \subset \cap_{A\in\partial K} H^A$ when K contains the origin and the other inclusion $\supset$ is always true. □

We next show that $K^{\circ\circ} = K$ when K contains the origin, beginning with a lemma.

Lemma 3.3. Any convex body K in $\mathbb{R}^d$ can be expressed as the intersection of some half-spaces.

Proof. Denote by $\mathcal{H}(K)$ the set of all half-spaces containing K. Then $\cap_{H\in\mathcal{H}(K)} H$ is an intersection of half-spaces and it contains K.

Assume for a contradiction that there exists a point $x \in \cap_{H\in\mathcal{H}(K)} H$ with $x \notin K$. Then we choose a point y such that $|xy| = \inf_{z\in K}\{|xz|\}$ (where $|xz|$ means the length of the line segment between the points x and z). Then we denote the plane that passes through y and is perpendicular to xy by P, and denote the space on the opposite side with x respect to the plane P by Q.

Since $x \in \cap_{H\in A} H$, and $x \notin Q$, thus $Q \notin A$, so there is a point $w \in K$ such that $w \notin Q$. Make a coordinate system with the origin $y = 0$, the plane P by $(a_1, \dots, a_n) : a_1 = 0$, and point $x = (1, 0, \dots, 0)$. Then we know that $w = (w_1, \dots, w_n)$ such that $w_1 > 0$. Since K is convex, and $y, w \in K$, thus for any $0 \le \lambda \le 1$, $(\lambda w_1, \dots, \lambda w_n) \in K$. Let $\lambda = \frac{w_1}{w_1^2+\cdots+w_n^2}$, and denote the point $(\lambda w_1, \dots, \lambda w_n)$ by u, then $|xu| = \sqrt{\frac{(w_1^2+\cdots+w_n^2)^2-w_1^4}{(w_1^2+\dots w_n^2)^2}} < 1 = |xy|$, and this contradicts the definition of y.

Thus $\cap_{H\in A} H \in K$, and we know $\cap_{H\in A} H = K$. □

Theorem 3.2. For any convex set K, if the origin O is contained in K, then the polar body of its polar body is K itself. We can express as $(K^\circ)^\circ = K$

Proof. Since for any point $x \in K$, then for any $y \in K^\circ$, $(x, y) \le 1$, as the definition, we know that $x \in (K^\circ)^\circ$. Thus $K \subset (K^\circ)^\circ$.

On the other hand, by using theorem 3.1, we know that $(K^\circ)^\circ = \cap_{A\in K^\circ} H^A$. If there exists $x \in (K^\circ)^\circ$, but $x \notin K$.then there exists a real number $0 \le c < 1$, such that $cx \in \partial K$, and this means that for any a such that $c < a < 1$, $ax \notin K$. Since K is convex, by using lemma 3.3 we can denote $K = \cap_{H\in P} H$, which P is a set of half-space. Since $x \notin \cap_{H\in P} H$, so there exists a half-space $H \in P$ such that $x \notin H$. Then from example 3.4, we know that the polar body of the half-space is a line segment OL, with one point is the origin.

Since $K \in H$, we know that $OL \in K^\circ$. Since $x \notin H$, we know that $(x, L) > 1$ (this is easy to show if we make the coordinate system and choose the half-space to be $z = (z_1, \dots, z_n) : z_1 \le 1$). Then since $L \in K^\circ$ and $x \in (K^\circ)^\circ$, $(x, L) > 1$, there is contradiction. We know that $(K^\circ)^\circ \subset K$. Thus $(K^\circ)^\circ = K$. □

Example 3.5 (Polar body of a polygon)**.** The polar body of a n-gon that contains the origin is also a n-gon. This is easy to prove by using theorem 3.1. The polar body of a n-gon is actually the intersection of n half-planes corresponding to the n vertices, which is also a n-gon.

4. Properties of logarithmic capacity

There are many definitions of capacity, and we give two equivalent definitions here. The definition in higher dimensions ($d \ge 3$) is slightly different. We use the notation K' to express the complement of the compact set K.

Definition 4.1. For a compact set $K \subset \mathbb{C}$, the logarithmic capacity $c(K)$ is defined by

$$-\log c(K) = \inf \left\{ \iint_{\mathbb{C}^2} \log \frac{1}{|z-w|} \, d\mu(w) \, d\mu(z), \mu \in \mathcal{M}(K) \right\},$$

where $\mathcal{M}(K)$ is the set of Borel probability measures supported on K.

If $c(K) = 0$, then K is said to be a polar set. If K is not polar, then it has a Green's function with pole at infinity, and the logarithmic capacity can be defined in terms of this Green's function as well. Any convex set other than the empty set and a set with one point is non-polar.

Proposition 4.1. If $K \subset \mathbb{C}$ is a non-polar compact set, then there exists a unique function u on $K^{'} = \mathbb{C} \setminus K$ such that

(i) $u = 0$ on ∂K.
(ii) $\triangle u = 0$ on $K^{'}$.
(iii) $u(z) = \log|z| + O(1)$ as $|z| \to \infty$ through the unbounded component of K'.

The logarithmic capacity $c(K)$ then satisfies

$$u(z) = \log|z| - \log c(K) + O(1/z)$$

as $|z| \to \infty$.

We can easily get four properties of logarithmic capacity from the definition above:

1. Capacity is invariant under translation and rotation.
2. If two compact sets A and B satisfy $A \subset B$, then $c(A) \leq c(B)$.
3. When we consider the change of capacity under scaling, we know that if K amplifies to rK, then the capacity $c(K)$ will amplify to $rc(K)$.
4. The capacity of any compact set is equal to the capacity of the boundary of the compact set.

The fundamental solution to Laplace's equation on the complex plane $\mathbb{C}$ is the function

$$\Gamma(z) = \frac{1}{2\pi}\log|z|.$$

Then for the disk $D_r = \{z : |z| \leq r\}$, the unique function u from the proposition is $\log|z/r|$. Thus the logarithmic capacity $c(D_r)$ of the disk is equal to r.

Next we will state a theorem about conformal mapping.

Theorem 4.1 (Invariance of capacity under conformal mapping)**.** For two convex body $K_1, K_2 \in \mathbb{R}^2$, if there exists a conformal (bijective holomorphic) map $f : \mathbb{C} \setminus K_1 \to \mathbb{C} \setminus K_2$, with the property that

$$f(z) = z + O(1),$$

as $|z| \to \infty$ in the unbounded component of $\mathbb{C} \setminus K_1$, then

$$c(K_1) = c(K_2).$$

If we consider the function f of the form

$$f(z) = z + \frac{c}{z},$$

where c is a real number, and think about the image of the unit circle $|z| = 1$ under f, we can learn the capacity for some special cases.

Example 4.1 (Capacity of a segment)**.** When $c = 1$, then the unit circle will map to a line segment $-2 \leq \mathrm{Re}(z) \leq 2$. When $c = -1$, then the unit circle will map to a line segment $-2 \leq \mathrm{Im}(z) \leq 2$. And the point outside the unit circle will map to all points except the line segment.Thus we know that the line segment with length 4 will have capacity 1. Using scaling, we know the capacity of a line segment with length L is $L/4$.

Example 4.2 (capacity of an ellipse)**.** When $0 \leq c < 1$, the unit circle will map to an ellipse

$$\frac{x^2}{(1+c)^2} + \frac{y^2}{(1-c)^2} = 1,$$

and the points outside the unit circle will map to all points outside this ellipse. Since the capacity of the circle is 1, we know that the capacity of the ellipse is 1. Generally, using scaling, we find that the capacity of the ellipse

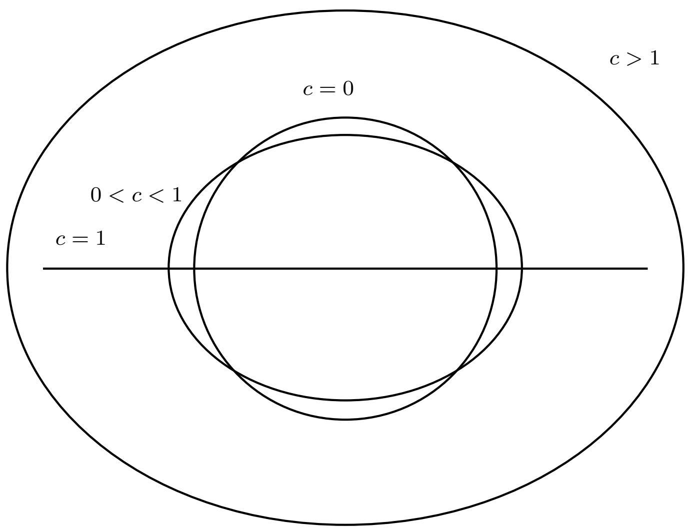
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with $ab \neq 0$, is $(a+b)/2$.

When $|c| > 1$, the unit circle will map to an ellipse

$$\frac{x^2}{(1+c)^2} + \frac{y^2}{(1-c)^2} = 1,$$

but this ellipse does not have capacity 1. The problem is that the points outside the unit circle will not all map to points outside the ellipse. For instance, $\sqrt{c}i$ will map to 0 under the map $f = z + \frac{c}{z}$.



We state another theorem about the change of capacity after mapping.

Theorem 4.2. Let K be a compact set, and let f be a polynomial

$$f(z) = \sum_{i=0}^{n} a_i z^i,$$

where $a_n \neq 0$. Then

$$c(f^{-1}(K)) = \sqrt[n]{\frac{c(K)}{|a_n|}}$$

It is not true that for any two set A, B, $c(A) + c(B) \geq c(A \cup B)$. Actually if A and B have non-zero capacity and we translate them to be very far from each other, the capacity of $A \cup B$ will tend to ∞. We will show an example for this.

Example 4.3 (Capacity of two disjoint segments)**.** The disjoint union $[-b,-a]\cup[a,b]$, for $b>a\geq 0$, is the inverse image under $f(z)=z^2$ of the segment $[a^2,b^2]$. Its capacity is therefore

$$\frac{\sqrt{b^2-a^2}}{2}.$$

From this we see that, for any positive real number d,l, we define $A=\{(x,y): -l-d\leq x\leq -l, y=0\}$, and $B=\{(x,y): l\leq x\leq l+d, y=0\}$, actually two collinear segments of length d are separated by a distance $2l$ then the capacity of A and B is both $\frac{d}{4}$, and the capacity of their union $A\cup B$ is

$$\sqrt{\frac{d(2l+d)}{4}}.$$

Example 4.4. More generally, the inverse image of a segment $[a^n,b^n]$ on the nonnegative real axis under the polynomial $f(z)=z^n$ has capacity

$$\left(\frac{b^n-a^n}{4}\right)^{1/n}.$$

As $n\to\infty$, this approaches b, which is the capacity of the circle with radius b. The inverse image consists of n regularly spaced and rotated segments placed around this circle.

5. RESULT

In this section, we will consider only compact convex sets in the plane. We wish to minimize the product $c(K)c(K^\circ)$ of the logarithmic capacity of a set with the logarithmic capacity of its polar body. First, we say something about why the minimization problem is easier to state. Let $a>0$ and let E_a be the ellipse with equation

$$\frac{x^2}{a^2}+y^2=1.$$

The polar body E_a° of E_a is the ellipse with equation

$$a^2x^2+y^2=1.$$

The logarithmic capacity of E_a is $(a+1)/2$ and the logarithmic capacity of E_a° is $(1+1/a)/2$. So the capacity product is

$$c(E_a)c(E_a^\circ)=\frac{(1+a)(1+1/a)}{4}.$$

This product tends to infinity if $a\to 0$ or $a\to\infty$. Thus

$$\sup_K c(K)c(K^\circ)=\infty,$$

even if we take the supremum over convex bodies symmetric in the origin.

As in [2], we can formulate a maximization problem for the capacity product as follows:

$$\sup_K \inf_T c(T(K))c(T(K^\circ)),$$

where the infimum is taken over all non-singular linear transformations T and the supremum over all convex bodies K symmetric in the origin. However, we will just consider the problem of minimizing the product of capacity of a body and its polar body.

We first show a lower bound for the capacity product. We also use the following theorem, which says that with fixed area the disk minimizes logarithmic capacity ([1], Theorem 5.3.5):

Theorem 5.1. If $K \subset \mathbb{C}$ has area A, then

$$c(K) \geq \sqrt{A/\pi}.$$

Theorem 5.2. The product of capacity of a body and its polar body is bigger than $\frac{3\sqrt{3}}{2\pi}$

Proof. From Theorem 5.1, we know that for any body with area $a \geq 0$, it has at least capacity $\sqrt{\frac{a}{\pi}}$ (the capacity for circle with area a).

Recall that for Mahler problem, the equilateral triangle minimizes the product of the area of a body and its polar body. Thus we know that for any body K,

$$\operatorname{area}(K)\operatorname{area}(K^\circ) \geq \frac{27}{4},$$

which is reached when K is an equilateral triangle.

From the above two conclusion, we can easily get a lower bound of the product of the capacity of a body and its polar body, $\frac{3\sqrt{3}}{2\pi}$. □

According to [1] the capacity of a regular n-gon with side length L is

$$\frac{\Gamma(1/n)}{2^{1+2/n}\pi^{1/2}\Gamma(1/2+1/n)}L.$$

Let P_n be the regular n-gon inscribed in the unit circle, and let P_n° be the polar body, which is the regular n-gon circumscribing the unit circle. The side length of P_n is $2\sin(\pi/n)$, and the side length of P_n° is $2\tan(\pi/n)$. Therefore, the product of the capacities is

$$\frac{1}{2^{4/n}\pi}\left(\frac{\Gamma(1/n)}{\Gamma(1/2+1/n)}\right)^2 \sin(\pi/n)\tan(\pi/n).$$

The capacity product for the circle is 1. For the equilateral triangle it is slightly larger than 1, and for the square it is slightly smaller (a little bigger than 0.98). It is smallest for a pentagon (between 0.97 and 0.98) and then it increases to 1 as $n \to \infty$.

This means that even though regular n-gon will minimize the capacity with same area among all n-gons (see [3]), the regular n-gon will not minimize the capacity product among all n-gons. For instance, we can regard the regular pentagon as a degenerate hexagon, then, since regular pentagon has smaller capacity product compared with regular hexagon, there are hexagons with capacity product smaller than the regular hexagon.

However, the equilateral triangle with centroid the origin will minimize the capacity product among all triangles. Also, the square centered at the origin will minimize the capacity product among all parallelograms. We prove this below. We use [3], which shows that with fixed area, among all n-gons, the regular n-gon P_n will minimize the logarithmic capacity. So for any n-gon A_n,

$$\frac{c(A_n)}{\sqrt{\operatorname{area}(A_n)}} \geq \frac{c(P_n)}{\sqrt{\operatorname{area}(P_n)}}.$$

Proposition 5.1. Among all triangles, the equilateral triangle with centroid the origin will minimize the capacity product. Among all parallelograms, the square with center the origin minimizes the capacity product.

Proof. The equilateral triangle P_3 with centroid the origin minimizes the volume product among all convex sets, so it minimizes the volume product among all triangles. By [3], the equilateral triangle minimizes capacity among all triangles with fixed area. So if T is a triangle, then

$$c(T)c(T^\circ) \geq \sqrt{\operatorname{area}(T)\operatorname{area}(T^\circ)}\frac{c(P_3)c(P_3^\circ)}{\sqrt{\operatorname{area}(P_3)\operatorname{area}(P_3^\circ)}} \geq c(P_3)c(P_3^\circ).$$

Since the square P_4 with center the origin minimizes the volume product among all parallelograms (they all have the same area, actually) and minimizes capacity among all parallelograms with the same volume, the same argument shows that this square minimizes the capacity product among all parallelograms. □

Now we show a lower bound on the capacity product among n-gons with a certain property. Generally it is not true that the regular n-gon minimizes the capacity product among all n-gons, as we show afterward. For the theorem we use [3], that the regular n-gon minimizes capacity among n-gons with fixed area, and [4] (Theorem 3), which shows that if K is a convex set with $P_n \subset K \subset P_n'$, where P_n and P_n' are regular n-gons ($n \geq 3$) centered at the origin and each vertex of P_n lies on a side of P_n', then

$$\operatorname{area}(K)\operatorname{area}(K^\circ) \geq \operatorname{area}(P_n)\operatorname{area}(P_n^\circ) = n^2 \sin^2(\pi/n).$$

Theorem 5.3. Let A_n be an n-gon with $P_k \subset A_n \subset P_k'$, where P_k and P_k' are regular k-gons ($3 \leq k \leq n$) centered at the origin and each vertex of P_k lies on a side of P_k'. Then

$$c(A_n)c(A_n^\circ) \geq \frac{k\sin(\pi/k)}{n\sin(\pi/n)}c(P_n)c(P_n^\circ).$$

Proof. From [3] and [4], we know that

$$\begin{aligned}
c(A_n)c(A_n^\circ) &\geq \sqrt{\operatorname{area}(A_n)\operatorname{area}(A_n^\circ)}\frac{c(P_n)c(P_n^\circ)}{\sqrt{\operatorname{area}(P_n)\operatorname{area}(P_n^\circ)}} \\
&\geq \sqrt{\operatorname{area}(P_k)\operatorname{area}(P_k^\circ)}\frac{c(P_n)c(P_n^\circ)}{\sqrt{\operatorname{area}(P_n)\operatorname{area}(P_n^\circ)}} \\
&= \frac{k\sin(\pi/k)}{n\sin(\pi/n)}c(P_n)c(P_n^\circ).
\end{aligned}$$

□

As mentioned, we can get some formulas from [1] to calculate the capacity for some bodies with specific shapes. Now we want to show the explicit computation for capacity. We can use Schwarz-Christoffel mapping to calculate the capacity product for polygon inscribe the unit circle. The Schwarz-Christoffel mapping from the interior of the unit disk to the exterior of the polygon with interior angles $\pi a_1, \ldots, \pi a_n$ is given by

$$f(z) = \int_{z_0}^{z} w^{-2} \prod_{k=1}^{n} \left(1 - \frac{w}{z_k}\right)^{1-a_k} dw,$$

where the points z_k on the unit circle are the prevertices and the point z_0 is any point in the unit disk except 0. The function f is a conformal map, and it has a pole at $z = 0$. Since $f(z) = 1/z + O(1)$ as $z \to 0$, the function $f(1/z)$ gives a conformal map from the exterior of the unit disk to the exterior of the polygon with $f(1/z) = z + O(1)$ as $|z| \to \infty$. This means the capacity of the polygon is 1. We try to find the size of the polygon when it is a triangle and when it is a regular polgyon.

First, we do this for a regular polygon. The interior angles are $\pi(n-2)/n$, so $a_k = 1-2/n$ and $1-a_k = 2/n$. The prevertices must be at $e^{2\pi ki/n}$, so the product is

$$\prod_{k=1}^{n}\left(1-\frac{w}{z_k}\right)^{1-a_k} = (1-w^n)^{2/n}.$$

To find the side length, we look at the integral of $|f'(z)|$ between two prevertices:

$$\int_0^{2\pi/n} |1-e^{in\theta}|^{2/n}\, d\theta = \frac{2}{n}\int_0^{\pi} |1-e^{i2\theta}|^{2/n}\, d\theta = \frac{2^{1+2/n}}{n}\int_0^{\pi} |\sin\theta|^{2/n}\, d\theta.$$

We can use a formula to show that this gives the same result as the formula from earlier in this section.

Next, we do this for a triangle. Considering when $z_1 = 1$, $z_2 = e^{2i\pi\alpha}$, $z_3 = e^{2i\pi(\alpha+\beta)}$, then the map

$$f(z) = \int^z w^{-2}(1-w)^{1-\beta}(1-e^{-2i\pi\alpha}w)^{\alpha+\beta}(1-e^{-2i\pi(\alpha+\beta)}w)^{1-\alpha}dw$$

will map the unit disk to the outside of a triangle with angle $\pi\alpha, \pi\beta, \pi(1-\alpha-\beta)$. The circumradius of the image triangle is

$$\frac{1}{2\sin(\pi\alpha)}\int_0^{2\pi\alpha}\left|\left(1-e^{i\theta}\right)^{1-\beta}\left(1-e^{i\theta-2i\pi\alpha}\right)^{\alpha+\beta}\left(1-e^{i\theta-2i\pi(\alpha+\beta)}\right)^{1-\alpha}\right| d\theta$$

Recall that $|1-e^{i\alpha}| = 2|\sin\frac{\alpha}{2}|$. Thus (1) can be rewritten as

$$\frac{2}{\sin(\pi\alpha)}\int_0^{2\pi\alpha} |(\sin\theta/2)^{1-\beta}(\sin(\theta/2-\pi\alpha))^{\alpha+\beta}(\sin(\theta/2-\pi\alpha-\pi\beta))^{1-\alpha}|d\theta.$$

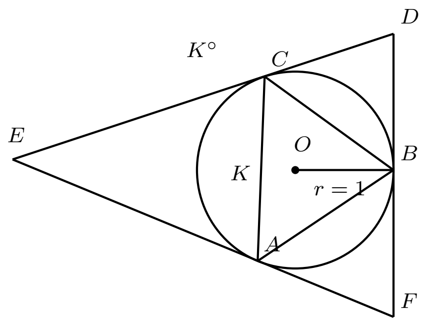
Then we know that if a triangle with angle $\pi\alpha, \pi\beta, \pi(1-\alpha-\beta)$, with circumradius r, then its capacity is

$$c(r,\alpha,\beta) = \frac{r\sin\pi\alpha}{2\int_0^{2\alpha}|(\sin\frac{\theta}{2})^{1-\beta}(\sin(\frac{\theta}{2}-\pi\alpha))^{\alpha+\beta}(\sin(\frac{\theta}{2}-\pi\alpha-\pi\beta))^{1-\alpha}|d\theta}$$

Next we will show a lemma to caculate the capacity product for the triangle. We all choose the circumcircle center of the triangle as the origin. If the triangle is obtuse triangle, or the right trianlge, then in this situation the polar body is unbounded, has infinite capacity, so we just consider acute triangle.

Lemma 5.1. For a triangle with 3 angle $\alpha, \beta, \pi-\alpha-\beta$, then the product of the circumradius of the triangle and its polar body is $\frac{1}{4\cos\alpha\cos\beta\cos(\pi-\alpha-\beta)}$.

Proof. Because when we scale the origin trianlge by $k > 0$, then its polar body scale by $\frac{1}{k}$, this does not change the product of circumradius. Now we assume the origin trangle has circumradius 1.



We assume the origin triangle $\triangle ABC$ has angle α, β, γ corresponding to A, B, C, and has circumradius 1. Then its polar body is the triangle $\triangle DEF$ as shown above, has angle $\pi - 2\alpha, \pi - 2\beta, \pi - 2\theta$. Then length

$$\begin{aligned} EF &= AE + AF \\ &= \tan \angle EOA + \tan \angle FOA \\ &= \tan(\pi - \alpha - \beta) + \tan \beta \\ &= \frac{\sin \alpha}{\cos \beta \cos(\pi - \alpha - \beta)}. \end{aligned}$$

Thus the circumradius of $\triangle DEF$ is

$$\frac{\sin \alpha}{2 \cos \beta \cos(\pi - \alpha - \beta) \sin(\pi - 2\alpha)} = \frac{1}{4 \cos \alpha \cos \beta \cos(\pi - \alpha - \beta)}.$$

□

Then we know that for any triangle with angle $\pi\alpha, \pi\beta, \pi(1 - \alpha - \beta)$, then the capacity product of this trianlge is $c(r, \alpha, \beta)c(r^\circ, 1 - 2\alpha, 1 - 2\beta)$, where r denotes the circumradius of origin triangle and r° denotes the circumradius of its polar body. From lemma 5.1 we know that

$$rr^\circ = \frac{1}{4 \cos \pi\alpha \cos \pi\beta \cos \pi(1 - \alpha - \beta)}.$$

From this we know that the capacity product of triangle is a function only related with α, β, and we give the function for the capacity product of triangle.

I also have a conjecture about choosing the origin inside a convex body to minimizes the capacity of its polar body. We only consider the situation that the origin inside the body, because when the origin is outside the convex body, its polar body is unbounded.

Conjecture 5.1. The circumcircle center of an acute triangle is the unique point as the origin that minimizes the capacity of its polar body. And generally, for any body K, the center of the smallest circle contains K is the unique point as the origin that minimizes the capacity of its polar body.

If we assume the conjecture 5.1 is true, then we know that for any triangle T that minimizes the capacity product among all triangles,then the origin is its circumcircle center. Then since $(T^\circ)^\circ = T$, so T° also minimizes the capacity product, thus the circumcircle center of T is also the circumcircle center of T°, this means that T is equilateral triangle, and equilateral triangle will minimizes the capacity product among triangles.

6. Acknowledgements

This project was done in the Summer Program in Undergraduate Research (SPUR) of the Massachusetts Institute of Technology Mathematics Department. I thank Nicholas Brian Strehlke for being my mentor and helping me throughout the project. I thank Ankur Moitra and David Jerison for giving valuable suggestions. I thank Slava Gerovitch for organizing the SPUR program.

References

[1] Thomas Ransford, *Potential Theory in the Complex Plane* Publisher: Cambridge University Press, Print Publication Year:1995

[2] Dorin Bucur, Ilaria Fragala; *Blaschke-Santalo and Mahler Inequalities for the first eigenvalue of the Dirichlet Laplacian* April 15, 2015, ArXiv preprint

[3] Alexander Yu. Solynin and Victor A. Zalgaller*, *An isoperimetric inequality for logarithmic capacity of polygons* 2004

[4] K. J. Brczky, E. Makai Jr., M. Meyer, S. Reisner *Volume product of planar polar convex bodies — lower estimates with stability* July 6 2015