A Correspondence between Generic Alcoved Polytopes and Subdivisions of the Root Polytope of Lie Type A

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Abstract

Alcoved polytopes of Lie type A are polytopes whose facets are orthognal to the roots of root system A_{n-1}. An alcoved polytope of type A_{n-1} is generic if it has two facets orthogonal to each root in A_{n-1}. In this paper, we prove that there is one-to-one correspondence between equivalence classes of generic alcoved polytopes of type A_{n-1} and regular central subdivisions of the root polytope of A_{n-1}. We apply this result to give an explicit classification for small n.

1 Introduction

The notion of alcoved polytopes arises from affine Coexter arrangements of irreducible crystallographic root systems. Alcoved polytopes and their combinatorial properties were first introduced in [2] and [3]. Here we restrict to the *root system* A_{n-1} defined as the collection of vectors $\{e_i - e_j | i, j \in [n]\}$, where $[n] = \{1, 2, \ldots, n\}$. The *root polytope* Q_{n-1} of A_{n-1} is the convex hull of the roots of A_{n-1}, which lies in the hyperplane $x_1 + \ldots + x_n = 0$.

A convex polytope P is an *alcoved polytope of Lie type* A_{n-1} if each top-dimensional face of P is orthogonal to some root in A_{n-1}. In other words, the polytope P is bounded by the affine hyperplanes $H_{ij} = \{x \in \mathbb{R}^n | x_i - x_j = c_{ij}\}$. If P has a top-dimensional face contained in each H_{ij}, then P is said to be *generic*.

One may easily see that there is only one combinatorial type of generic alcoved polytopes of type A_2, namely the equiangular hexagon. However, it is not immediately clear which tool one may apply to study type A_3 and beyond.

In this paper, we show that there is a correspondence between generic alcoved polytopes of type A_{n-1} and regular central subdivisions of Q_{n-1}. Here we say that a subdivision of a polytope P is *regular* if it can be obtained by lifting the vertices of P and projecting the lower boundary. A subdivision of P is *central* if each cell of the subdivision contains the origin and the restriction of the subdivision to the boundary ∂P is a subdivision of ∂P. Using this correspondence, we give an explicit classification for the case A_3 up to combinatorial equivalence.

In Section 3, we give a description of the combinatorial structure of generic alcoved polytopes. The faces of a generic alcoved polytope P of type A_{n-1} can be represented by *face labels*, which are directed graphs on $[n]$. Let F_{ij} be the top-dimensional face of P that lies in the hyperplane H_{ij}. A face $F = \bigcap_{F_{ij} \supset F} F_{ij}$ is associated with the face label whose edges are $(i \to j)$ for all $F \subseteq F_{ij}$. It turns out that each node in a face label is either a source, a sink, or an isolated point. Hence the face labels of P can be viewed as directed graphs on partitions of $[n]$ into sources, sinks, and isolated points. We examine the conditions on the compatibility of face labels and explain the connection with triangulations of the root polytope of a complete bipartite graph, which is studied in [4, Sec 12]. In particular, a simplex in such a triangulation can be represented by a bipartite graph on $[n]$ whose edges, regarded as roots, are the vertices of this simplex.

We define two alcoved polytopes P_1, P_2 of type A_{n-1} to be *label-equivalent* if they have the same collection of face labels. In other words, we can obtain P_2 by moving the facets of P_1 parallel to themselves without changing the face structure. If the face lattices of two alcoved polytopes are isomorphic, then they are *combinatorially equivalent*. In Section 4, we prove the main result of the paper:

Theorem 1.1. *There is a one-to-one correspondence between label-equivalence classes of generic alcoved polytopes of type A_{n-1} and regular central subdivisions of the labeled root polytope Q_{n-1}.*

More explicitly, given a face $F = \bigcap_{(ij)} F_{ij}$ of a generic alcoved polytope P, there is a cell $\tau_F = \mathrm{conv}\{\mathbf{0}; e_i - e_j | F \subseteq F_{ij}\} \subset Q_{n-1}$. We show that the collection $\tau = \{\tau_F\}$ forms a regular central subdivision of Q_{n-1}.

Finally, we construct a bijection between combinatorial equivalence classes of generic alcoved polytopes of type A_3 and partial acyclic orientations of a hexagon modulo rotations, reflections, and the action of a *swap map*. Here a partial orientation is not acyclic if after contracting all undirected edges, we get a loop or a directed cycle. The swap map acts on an oriented hexagon by first reversing the orientations of a pair of opposite edges that have different orientations and then exchanging these two edges while preserving their clockwise or counterclockwise orientation. In Figure 1, for example, the swap map acts on the blue edge and the red edge.

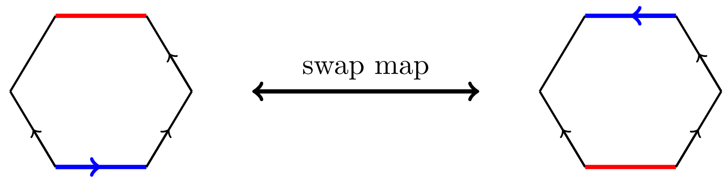


Figure 1: Example of the action of swap map.

In particular, Theorem 1.1 says that some of the $2^6 = 64$ central triangulations of a labeled Q_3 correspond to label-equivalence classes of *maximal* generic alcoved polytopes of type A_3 in the sense they have maximal number of vertices. By encoding generic alcoved polytopes with oriented hexagons and applying the swap map, we are able to show that there are only six combinatorial equivalence classes of maximal generic alcoved polytopes of type A_3.

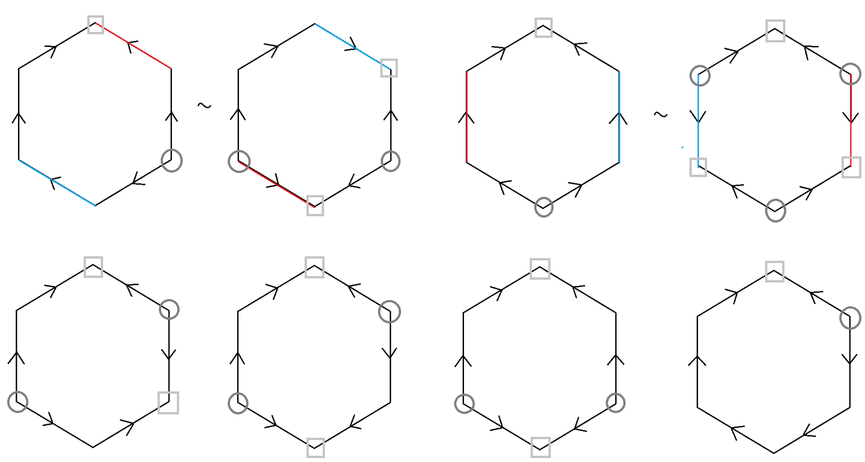


Figure 2: The six combinatorial equivalence classes of maximal types, encoded by orbits of oriented hexagons under the action of the swap map.

2 Preliminaries

This section serves as a more systematic formulation of the various concepts mentioned in the Introduction.

Recall that a *(convex) polytope* $P \subset \mathbb{R}^n$ is a bounded convex set of points in $\mathbb{R}^n$. Equivalently, it is the convex hull of all its vertices. A *face* of P is a set F of points in P such that there exists a linear function $y \in (\mathbb{R}^n)^*$ on P that is maximized exactly at all points in F. In particular, if F is a top-dimensional face of P, then we call it a *facet*.

2.1 Alcoved Polytopes of Lie Type A

Definition 2.1. A *root system* $\Phi \subset V$ is a collection of nonzero vectors called *roots* that satisfies the following conditions, cf. [1]:

i). The only scalar multiples of any root α in Φ are α and $-\alpha$.

ii). The set Φ is closed under reflection through the hyperplane orthogonal to any $\alpha \in \Phi$.

An *alcoved polytope* is a convex polytope bounded by some integer affine translations of the hyperplanes $H_\alpha = \{x \in \mathbb{R}^n | (x, \alpha) = 0\}, \alpha \in \Phi$. An alcoved polytope is *generic* if it has two facets orthogonal to each root $\alpha \in \Phi$.

The root system A_{n-1} is the collection of vectors $\{e_i - e_j | i \neq j \in [n]\}$, where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$. A generic alcoved polytope of (Lie type) A_{n-1} can be defined as the point set

$$P = \{x \in \mathbb{R}^n | x_i - x_j \leq c_{ij}, c_{ij} \in \mathbb{Z}, 1 \leq i \neq j \leq n\}.$$

Since the span of the roots in A_{n-1} is the hyperplane $H_0 = \{x \in \mathbb{R}^n | x_1 + \cdots + x_n = 0\}$, we will not distinguish between P and its intersection with H_0.

In this paper we only consider top-dimensional alcoved polytopes that contain the origin. Hence we assume that $c_{ij} > 0$ for all i, j.

We define an equivalence relation on alcoved polytopes as follows.

Definition 2.2. Given an alcoved polytope P of A_{n-1}, we label each face F by the roots corresponding to the hyperplanes that contain F. Define the *labeled face lattice* of P to be the face lattice of P with each element labeled. An alcoved polytopes of A_{n-1} are said to be *label-equivalent* if and only if they have the same labeled face lattice. If their unlabeled face lattices are isomorphic, then the two alcoved polytopes are *combinatorially equivalent*.

2.2 Subdivisions of Polytopes

Definition 2.3. A *subdivision* of a finite point configuration $A \subset \mathbb{R}^n$ is a collection τ of subsets $A_i \subsetneq A$ such that

i). The union of all $\mathrm{conv}(A_i)$ in τ is $\mathrm{conv}(A)$;

ii). The intersection of any two convex hulls is a common face of both, which is an element of τ.

We call the elements of τ its *cells*. A *triangulation* of A is a subdivision whose cells are simplicies.

In particular, a subdivision τ of A is a *(polytopal) subdivision* of the polytope $P = \mathrm{conv}(A)$ using the point set A. We define the *intersection lattice* $\mathcal{L}(\tau)$ of the subdivision τ to be the poset on $\tau \cup \{\mathrm{conv}(A)\}$ ordered by inclusion. It is known that this poset is indeed a lattice.

Let P be a polytope that contains the origin. We say that a subdivision τ of a polytope P is *central* if there is a subdivision $\tilde{\tau}$ of ∂P such that each cell of τ is the convex hull of the origin and a unique cell of $\tilde{\tau}$.

Another important class of polytopal subdivisons is *regular* subdivisions.

Definition 2.4. (cf. [6]) A subdivision τ of a polytope $P \subset \mathbb{R}^{n-1}$ is *regular* if and only if there exists a polytope $\tilde{P} \subset \mathbb{R}^n$ whose lower facets are in bijection with the top-dimensional cells of τ via the canonical projection $p : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$. Here a lower facet of $\tilde{P}$ refers to a facet that maximizes some linear function $y \in (\mathbb{R}^n)^*$ with $y_n < 0$.

In other words, we can lift each vertex of P such that the projections of faces on the lower boundary of the lifted polytope $\tilde{P}$ coincide with cells of τ.

More generally, we define the regular subdivision of a polytope P that lies in some hyperplane $H_\alpha = \{x \in \mathbb{R}^n | (x, \alpha) = 0\}, \alpha \in \mathbb{R}^n$ by replacing the projection p with the orthogonal projection onto H_α.

3 Face Lattices of Generic Alcoved Polytopes

In this section, we provide a description of the combinatorial structure of generic alcoved polytopes of type A_{n-1} using directed bipartite graphs on the vertex set $[n]$. This characterization gives us a clearer view of the face lattices of generic alcoved polytopes. It also helps us unveil the connection with subdivisions of the root polytope Q_{n-1}.

First, we describe a representation of a point in a generic alcoved polytope P of type A_{n-1}. We then provide a necessary and sufficient condition for an alcoved polytope of type A_{n-1} to be generic. Both results were communicated to us by Alexander Postnikov.

Proposition 3.1. *Let $x \in P$ be any point. We associate with x a directed graph G_x on the point set $[n]$ such that $(i \to j)$ is an edge of G_x if and only if $x_i - x_j = c_{ij}$.*

i). For any $i, j, k \in [b]$, at most one of $(i \to j)$ and $(j \to k)$ can be an edge of G_x.

ii). If x is a vertex, then the underlying undirected graph of G_x contains a spanning tree of the complete graph K_n.

Proof. If both $(i \to j)$ and $(j \to k)$ are edges of G_x, then $x_i - x_k = (x_i - x_j) + (x_j - x_k) = c_{ij} + c_{jk} > c_{ik}$ and thus $x \notin P$, a contradiction. $\square$

Lemma 3.2. *An alcoved polytope of type A_{n-1} is generic if and only if the collection of parameters $\{c_{ij}\}$ satisfies the triangular inequality $c_{ik} + c_{kj} > c_{ij}$ for all $i, j, k \in [n]$.*

Proof. Suppose that P is a generic alcove polytope of type A_{n-1} with parameters $\{c_{ij}\}$ and supporting hyperplanes $H_{ij} = \{x \in \mathbb{R}^n | x_i - x_j \leq c_{ij}\}$. Suppose to the contrary that $c_{ik} + c_{kj} \leq c_{ij}$ for some $k \neq i, j$. Note that any point $x \in P$ satisfies $x_i - x_k \leq c_{ik}$ and $x_k - x_j \leq c_{kj}$. It follows that $x_i - x_j = (x_i - x_j) + (x_j - x_k) \leq c_{ik} + c_{kj} \leq c_{ij}$. Therefore the intersection $P \cap H_{ij} \subseteq H_{ik} \cap H_{jk}$ has codimension greater than one, which implies that H_{ij} is not a facet of P, a contradiction. Hence the collection $\{c_{ij}\}$ satisfies the triangular inequalities.

Suppose that $\{c_{ij}\}$ satisfies all triangular inequalities. We can assume that $c_{ij} > 0$ for all i, j by translating the origin suitably. We want to show that the intersection of H_{12} (or any H_{ij}) and P has dimension $n-2$. Note that the point $x^1 \in H_{12}$ with $x^1_1 - x^1_k = c_{1k}$ for all k is a point in P since $x^1_i - x^1_j = c_{1j} - c_{1i} < c_{ij}$ for all $1 < i \neq j \leq n$. Now we consider the points $x_\varepsilon = (x_1 - \sum_{i=3}^n \varepsilon_i, x_2 - \sum_{i=3}^n \varepsilon_i, x_3 + 2\varepsilon_3, \dots, x_n + 2\varepsilon_n) \in H_{12}$. For small enough ε_i, the point x_ε lies in P. The convex hull of all such x_ε is spanned by $n-2$ vectors. Hence P is generic. $\square$

3.1 A Characterization by Face Labels

Now we slightly abuse the standard terminology of graph theory. Let G, G' be directed bipartite graphs on $[n]$. We mean by *subgraph* of G the graph obtained by removing some vertices of G and all their incident edges. The *intersection* $G \cap G'$ is a directed bipartite graph on $[n]$ whose edges belong to the intersection of the edge sets of G and G'. Similarly define the union $G \cup G'$. We say that G *spans* $[n]$ if G contains an undirected spanning tree on the point set $[n]$.

It follows from Proposition 3.1 that a node in the graph G_x associated with a vertex $x \in P$ is either a source or a sink, so G_x is a directed graph on a partition of $[n]$ into sources and sinks. We associate with each face $F \subset P$ a graph $G_F = \bigcap G_x$, where $x \in F$ is a vertex of P. Hence G_F is a directed graph on a partition of $[n]$ into sources, sinks, and isolated points. We call the graph G_F the *face label* of F.

As an example we will use face labels to show that there is only one combinatorial type of generic alcoved polytope of type A_2.

Example 3.3. Let P be a generic alcoved polytope of type A_2. Since the collection $\{c_{ij}\}$ satisfies the triangular inequalities, the six directed complete bipartite graphs on [3] all correspond to points in P. In particular, they represent the six vertices of P, as is shown in Figure 3. On the other hand, the root system A_2 consists of six roots, so a generic alcoved polytope of type A_2 has six top-dimensional faces (edges). Therefore P has to be a hexagon.

To sum up the previous discussions, we have the following proposition, which provides a combinatorial description of the faces of P.

Proposition 3.4. *The faces of P can be represented by face labels, which are directed bipartite graphs on $[n]$. A vertex of P corresponds to a face label that spans $[n]$. The least upper bound of two faces F_1, F_2 in $\mathcal{L}(P)$ corresponds to the intersection of the face labels of F_1 and F_2. The dimension of a face F is equal to the number of isolated points in the face label of F.*

3.2 Conditions on Face Labels of a Generic Alcoved Polytope

Now we want to investigate the conditions that determine whether a directed graph appears as a face label in a given generic alcoved polytope of type A_{n-1}.

Given a face label G, any two points i, j in a connected component of G are either in a cycle or connected by a path. Since G is a directed bipartite graph, it cannot have a

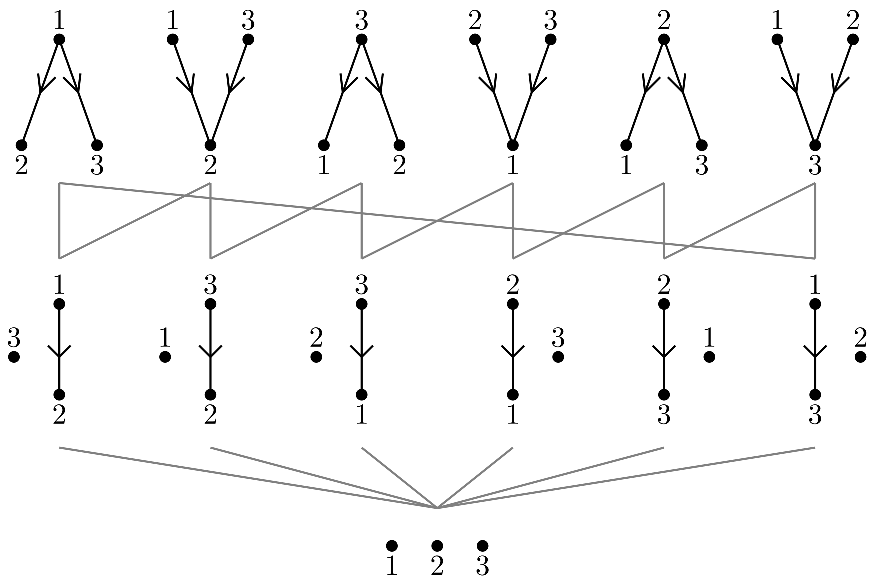


Figure 3: Labeled face lattice of a generic alcoved polytope of type A_2.

directed cycle of odd length. If $\{a_1 \to a_2 \leftarrow a_3 \to a_4 \leftarrow \ldots \leftarrow a_{2k-1} \to a_{2k} \leftarrow a_1\}$ is a cycle of length $2k$ in G with $k \geq 2$, then the collection of parameters $\{c_{ij}\}$ satisfies the equation

$$c_{a_1a_2} + c_{a_3a_4} + \cdots + c_{a_{2k-1}a_{2k}} = c_{a_3a_2} + c_{a_5a_4} + \cdots + c_{a_1,a_{2k}}.$$

If G contains the *alternating path* on any $(2k+1)$-tuple $(a_1, a_2, \ldots, a_{2k+1})$ with $k \geq 2$, which is the directed graph $\{a_1 \to a_2 \leftarrow a_3 \to a_4 \leftarrow \ldots \leftarrow a_{2k-1} \to a_{2k} \leftarrow a_{2k+1}\}$, then $\{c_{ij}\}$ satisfies the inequality

$$c_{a_1a_2} + c_{a_3a_4} + \cdots + c_{a_{2k-1}a_{2k}} < c_{a_3a_2} + c_{a_5a_4} + \cdots + c_{a_{2k-1}a-2k} + c_{a_{2k+1}a_{2k}} + c_{a_1a_{2k+1}}.$$

Similarly, we define the alternating path on $(a_1, a_2, \ldots, a_{2k})$. If G contains the alternating path on $(a_1, a_2, \ldots, a_{2k})$, then $\{c_{ij}\}$ satisfies

$$c_{a_1a_2} + c_{a_3a_4} + \cdots + c_{a_{2k-1}a_{2k}} < c_{a_3a_2} + c_{a_5a_4} + \cdots + c_{a_1a_{2k}}.$$

Next, we examine the relation among face labels of the same generic alcoved polytope.

Proposition 3.5. *Suppose G is the face label of a face F of P. Then any subgraph $G' \subset G$ is the face label of a face $F' \supset F$. In other words, if the collection $\{c_{ij}\}$ satisfies the inequalities corresponding to G, then $\{c_{ij}\}$ also satisfies the inequalities corresponding to G'. Furthermore, the generic polytope P is determined by the collection of even-term inequalities (or equalities) between $c_{a_1,a_2} + c_{a_3,a_4} + \cdots + c_{a_{2k-1},a_{2k}}$ and $c_{a_3,a_2} + c_{a_5,a_4} + \cdots + c_{a_1,a_{2k}}$ for all $2k$-tuples $(a_1, \ldots, a_{2k})$, where $k \leq n/2$.*

Proof. First let us consider the subgraphs with even number of vertices. The problem can be reduced to the cases where G and G' are the alternating paths on $[n]$ and $[2k]$ respectively with $2k < n$. Suppose to the contrary that $\{c_{ij}\}$ satisfies the inequality $c_{12} + c_{34} + \cdots +$

$c_{2k-1,2k} \geq c_{32}+c_{54}+\cdots+c_{2k-1,2k-2}+c_{1,2k}$. Let $x = (x_1, \ldots, x_n)$ be the vertex corresponding to G, then $c_{12} - c_{32} + c_{34} - c_{54} + \cdots + c_{2k-1,2k} = x_1 - x_2 + x_2 - x_3 + \cdots + x_{2k-1} - x_{2k} = x_1 - x_{2k} \geq c_{1,2k}$. Therefore $x \notin P$, a contradiction.

Then we want to show that any odd-term inequality can be deduced from some even-term inequality and triangular inequality. Without loss of generality, suppose that the alternating path on $[2k+1]$ is a face label of P. By the argument above, the subgraph on $[2k]$ is also a face label of P. Apply triangular inequality, we have $c_{12}+c_{34}+\cdots+c_{2k-1,2k} < c_{32} + c_{54} + \cdots + c_{2k-1,2k-2} + c_{1,2k} < c_{32} + c_{54} + \cdots + c_{2k-1,2k-2} + (c_{1,2k+1} + c_{2k+1,2k})$ as needed.

Hence the case where the subgraphs have odd number of vertices follows. □

Remark 3.6. It was suspected by Postnikov that the collection of deciding inequalities determines a hyperplane arrangement $\{x_{a_1,a_2} + x_{a_3,a_4} + \cdots + x_{a_{2k-1},a_{2k}} = x_{a_3,a_2} + x_{a_5,a_4} + \cdots + x_{a_1,a_{2k}} | (a_1, a_2, \ldots, a_{2k})$ is any $2k$-tuple in $[n]\} \subset \mathbb{R}^{n(n-1)}$. The faces of this arrangement are in bijection with label-equivalence classes of generic alcoved polytopes.

The following proposition serves as a criterion for deciding whether two face labels can appear in the same alcoved polytope.

Let G be a directed graph on $[n]$. Define G^{op} to be the directed graph on $[n]$ obtained by reversing the direction of all edges in G.

Proposition 3.7. *If two directed bipartite graphs G_1, G_2 on $[n]$ are the face labels of a generic alcoved polytope, then the union $G_1 \cup G_2^{op}$ contains no directed cycle of length ≥ 4 except for those in $G_1 \cap G_2^{op}$.*

Proof. Suppose to the contrary that $G_1 \cup G_2^{op}$ contains a minimal directed cycle of length ≥ 4 which is not contained in $G_1 \cap G_2^{op}$. Without loss of generality, label the directed cycle as $C = \{1 \to 2 \leftarrow 3 \to 4 \leftarrow \cdots \to 2k \leftarrow 1\}$. Set $E_1 = \{1 \to 2, 3 \to 4, \ldots, (2k-1) \to 2k\}$ and $E_2 = \{1 \to 2k, 3 \to 2, \ldots, (2k-1) \to (2k-2)\}$, then we have $C = E_1 \cup E_2^{op}$, $E_1 \subseteq G_1$ and $E_2 \subseteq G_2$. If $E_2 \subseteq G_1$, then $c_{12}+c_{34}+\cdots+c_{2k-1,2k} = c_{32}+c_{45}+\cdots+c_{1,2k}$, which implies that $E_1 \subseteq G_2$. But then $C = E_1 \cup E_2^{op} \subseteq G_1 \cap G_2^{op}$, a contradiction to our assumption. Suppose $E_2 \nsubseteq G_1$. Let $x = (x_1, \ldots, x_{2k})$ be the vertex corresponding to G_1. Then

$$\begin{aligned} c_{12} + c_{34} + \cdots + c_{2k-1,2k} =& (x_1 - x_2) + (x_3 - x_4) + \cdots + (x_{2k-1} - x_{2k}) \\ =& (x_3 - x_2) + (x_5 - x_4) + \cdots + (x_1 - x_{2k}) < c_{32} + c_{54} + \cdots + c_{1,2k}. \end{aligned}$$

Similarly, we have $E_1 \nsubseteq G_2$ and $c_{12} + c_{34} + \cdots + c_{2k-1,2k} > c_{32} + c_{54} + \cdots + c_{1,2k}$, which leads to a contradiction. □

At this point, one cannot help but notice the striking similarity of the criterion above and the condition on compatibility of simplicies in a triangulation of the root polytope $Q_{K_{m,n}}$ of the complete bipartite graph $K_{m,n}$. Here $Q_{K_{m,n}}$ is defined to be the convex hull $\mathrm{conv}\{e_i - \overline{e_j} | i \in [m], j \in [n]\} \subset \mathbb{R}^{m+n}$.

As is shown in [4, Sec12], the simplices in a triangulation of $Q_{K_{m,n}}$ can be represented by spanning trees of $K_{m,n}$ whose edges are oriented from $[m]$ to $[n]$. Suppose that T is a spanning tree of $K_{m,n}$. Two simplices can appear in the same triangulation if and only if the corresponding trees T_1 and T_2 satisfy the condition that $T_1 \cap T_2^{op}$ has no directed cycle of length greater than or equal to four.

Therefore, it is natural to ask if there is a connection between generic alcoved polytopes of type A_{n-1} and the subdivisions of the root polytope Q_{n-1}, which is the union of root polytopes $Q_{K_{|S|,|T|}}$ for all partitions (S,T) of $[n]$. We address this question in the following section.

4 Proof of the Main Theorem

Fix a labeling $\{(ij)\}$ of the vertices of Q_{n-1}. Let τ be a central subdivision of Q_{n-1}. We label a cell F of τ with a directed graph G on $[n]$ as follows: the edge $(i \to j)$ is an edge of G if and only if (ij) is a vertex of F. We call G the *cell label* of $F \in \tau$.

Theorem 4.1 (Restated)**.** *There is a one-to-one correspondence between label-equivalence class $[P]$ of generic alcoved polytopes of type A_{n-1}, and regular central subdivisions τ of the labeled Q_{n-1}. Given a face $F = \bigcap_{(ij)} F_{ij}$ of P, we take the cell $\tau_F = \operatorname{conv}\{\mathbf{0}; e_i - e_j | F \subseteq F_{ij}\}$. The collection $\tau_P = \{\tau_F\}$ is a regular central subdivision of Q_{n-1}. Conversely, given a regular central subdivision τ, the collection of cell labels is the collection of face labels of an label-equivalence class $[P]$.*

Proof. Pick any representative P of the label-equivalence class with parameters $\{c_{ij}\}$. Let $\tilde{P} \subset \mathbb{R}^n$ be the convex hull of $v_{ij} = e_i - e_j + c_{ij} \cdot (1, 1, \dots, 1)$ and $\mathbf{0} \in \mathbb{R}^n$. Let $y' \in (\mathbb{R}^n)^*$ be any linear function on $\tilde{P}$ such that $(y', (1, \dots, 1)) < 0$. Since P contains the origin, we can scale y' so that $y' = y - d \cdot (1, 1, \dots, 1)$, where $y = (y_1, \dots, y_n) \in \partial P$ and $d > 0$. Recall that a face $\tilde{F}$ of $\tilde{P}$ is a lower face if and only if there exists y' that reaches its maximum exactly on $\tilde{F}$, in which case we set $\tilde{P}_y = \tilde{F}$. Let π be the orthogonal projection onto the hyperplane $x_1 + \cdots + x_n = 0$.

Suppose that $d > 1$, then $(y', v_{ij}) = y_i - y_j - c_{ij} \cdot d < y_i - y_j - c_{ij} \leq 0$, so y' reaches its maximum 0 at $\mathbf{0}$ and $\pi(\tilde{P}_y) = \{\mathbf{0}\}$.

If $d = 1$, then $(y', v_{ij}) = y_i - y_j - c_{ij} \leq 0$. Hence y' reaches its maximum exactly on $\tilde{P}_y = \operatorname{conv}\{\mathbf{0};\ v_{ij}, y \in H_{ij} = \{x_i - x_j - c_{ij} = 0\}\}$. We call a face of $\tilde{P}$ that contains the origin a *central* face. If y is a vertex of P, then $\tilde{P}_y$ is a lower central facet of $\tilde{P}$. If F is a face of P, then $\tilde{P}_F = \bigcap_{y \in F} \tilde{P}_y$ is a lower central face of $\tilde{P}$. Suppose F_1, F_2 are two faces of P and F is the smallest face of P that contains both F_1 and F_2. In other words, F is the greatest lower bound of F_1 and F_2 in the face lattice of P^*. Then

$$\tilde{P}_{F_1} \cap \tilde{P}_{F_2} = (\bigcap_{y \in F_1} \tilde{P}_y) \cap (\bigcap_{y \in F_2} \tilde{P}_y) = \bigcap_{y \in F_1 \cup F_2} \tilde{P}_y = \tilde{P}_F.$$

Hence the central faces of $\tilde{P}$ are in bijection with faces of P. Furthermore, the cell labels of the central cells of τ is in bijection with the face labels of P.

Suppose that $d < 1$. If $y \in H_{ij}$, then $(y', v_{ij}) = y_i - y_j - c_{ij} \cdot d > 0$. Hence y' is not maximal at $\mathbf{0}$, so y' is maximized by some lower face that is not central.

We want to show that there exists a $P \in [P]$ such that $\tilde{P}$ has no lower facet that is not central.

Claim 4.2. *Suppose that all lower facets of $\tilde{P}$ are central. Then all lower facets of the polytope P_M with parameters $\{c_{ij} + M\}$ are central for all $M \geq 0$.*

Proof. Since a generic alcoved polytope of type A_{n-1} is determined by even term inequalities, the system of inequalities of P_M is the same as that of P. Thus $P_M \in [P]$. The lifted polytope $\tilde{P_M}$ is the convex hull of $\mathbf{0}$ and all $v'_{ij} = e_i - e_j + (c_{ij} + M) \cdot (1, 1, \ldots, 1)$. Denote the subdivision determined by $\tilde{P_M}$ as τ'.

Let F be a face of $\tilde{P}$ that is not central. Let y be any linear function that is maximized on $\tilde{P}$ by F, i.e., for all vertex v_{ij} of F, we have

$$(y', v_{ij}) = y_i - y_j - d \cdot c_{ij} > (y', v_{kl}) = y_k - y_l - d \cdot c_{kl}$$

for all $v_{kl} \notin F$. Since

$$(y', v'_{ij}) = y_i - y_j - d \cdot (c_{ij} + M) = (y', v_{ij}) - d \cdot M,$$

$(y', v'_{ij}) > (y', v_{kl}) - d \cdot M = (y', v'_{kl})$ for all $v'_{kl} \notin F'$. Hence y is maximized on $\tilde{P_M}$ by the face $F' = \mathrm{conv}\{v'_{ij}, v_{ij} \in F\}$ and $\pi(F') = \pi(F)$. Hence each noncentral cell of τ is a noncentral cell of τ'.

On the other hand, let $z' = z + (1/n, 1/n, \ldots, 1/n)$ be a linear function on $\tilde{P_M}$ such that $z \in \partial P_M$. Then $(z', v'_{ij}) = z_i - z_j - (c_{ij} + M) \leq 0$ and equality is achieved if and only if $z \in H'_{ij}$. Therefore z' is maximized by the central face $\mathrm{conv}\{\mathbf{0}, v'_{ij} \in H'_{ij}\}$. Note that the central faces of $\tilde{P_M}$ are in bijection with faces of P_M, which are in bijection with faces of P. Hence each central cell of τ is a central cell of τ', which means that $\tau \subseteq \tau'$. But τ is a subdivision already. Therefore $\tau = \tau'$ and all lower facets of $\tilde{P_M}$ are central. □

Claim 4.3. *Suppose that not all lower facets of P are central. Then there exists $M > 0$ such that all lower facets of P_M are central.*

Proof. Set $P' = \mathrm{conv}\{v_{ij}\}$ to be the convex hull of all lifted vertices. Then a face of $\tilde{P}$ is either central or a face of P'. Let F be a face in P'. A supporting hyperplane of F is the orthogonal complement to some linear function y' that is maximized by F. Note that there is at least one supporting hyperplane H_F that does not contain the origin, or F would be central. Furthermore, since $\tilde{P}$ is convex, it lies entirely on one side of H_F, say the half space H_F^+. Let v be the vector parallel to y' that points from the origin to H_F. Then $(v, y') > 0$ When we add M to each c_{ij}, we lift each vertex of P' by M in the direction of $(1, 1, \ldots, 1)$. Thus P'_M and $(H_F)_M$can be obtained by lifting P' and H_F in the direction of $(1, 1, \ldots, 1)$. Hence $(v_M, y') = (v + M \cdot (1, 1, \ldots, 1), y') = (v, y') + M \cdot y' \cdot (1, 1, \ldots, 1)$. Since $y' \cdot (1, 1, \ldots, 1) < 0$, there exist an M such that $(v_M, y') < 0$, which means that F_M is not a lower face of P_M. Note that P' has finitely many faces, hence there is an M such that P_M has no lower face that is also in P'_M. □

Therefore, given an equivalence class $[P]$, there exits a P_M that corresponds to a regular central subdivision τ_P of Q_{n-1} whose cell labels are in bijection with the face labels of P_M and thus P. In fact, the regular subdivision corresponding to any $P \in [P]$ is a refinement of τ_P, so τ_P is unique.

On the other hand, suppose that τ is a regular central subdivision of Q_{n-1}. Let $\{c_{ij}\}$ be a lifting such that the projections of the lower faces of $\tilde{P}$ are in bijection with cells of τ. We can add some large enough M to each c_{ij} such that the collection $\{c_{ij}\}$ satisfies the triangular inequalities. Hence $\{c_{ij}\}$ determines a generic alcoved polytope. Furthermore, the projection of P_M remains unchanged by Claim 4.2.

□

Corollary 4.4. *The maximum of the number of vertices of P is $\binom{2n-2}{n-1}$ and the minimum is $2^n - 2$. The maximum is achieved if and only $\mathcal{L}(P^*)$ is isomorphic to τ, where τ is a triangulation of Q_{n-1}. The minimum is achieved if and only if $\mathcal{L}(P^*)$ is isomorphic to $\mathcal{L}(Q_{n-1})$.*

Proof. By Lemma 12.5, [4], any triangulation of the root polytope $Q_{K_{m,n}}$ has $\binom{m+n-2}{m-1}$ simplices. Since the set of facets of Q_{n-1} is a disjoint union of facets of all $Q_{K_{m,n}}$, the number of simplices in a triangulation of Q_{n-1} is

$$\sum_{S\cup T=[n], S\cap T=\emptyset} \binom{|S|+|T|-2}{|T|-1} = \sum_{|S|} \binom{n}{|S|}\binom{n-2}{|T|-1} = \binom{2n-2}{n-1}.$$

□

Remark 4.5. One may have noticed in the proof of Theorem 1.1 that the bijection between faces of P and central cells of τ_P is independent of the nice properties of A_{n-1}. Let $\mathcal{V} = \{v_1, \dots, v_N\} \subset \mathbb{R}^n$ be a generic vector configuration whose convex hull Q contains the origin. We define an *alcoved polytope of type* $\mathcal{V}$ to be the point set

$$P = \{x \in \mathbb{R}^n | (x, v_i) = c_i, c_i \in \mathbb{Z}^+, i \in [N]\}.$$

Given a label-equivalence class $[P]$ of generic alcoved polytopes of type $\mathcal{V}$, consider the regular subdivision τ obtained by projecting the lower boundary of $\tilde{P} = \mathrm{conv}\{\mathbf{0}, v_i + c_i \cdot e_{n+1} | i \in [N]\}$. The regular subdivision τ may not be central, but the cells of τ that contain the origin are in bijection with the face labels of P.

It is clear that not every regular central subdivision of Q corresponds to a generic alcoved polytope of type $\mathcal{V}$. It would be interesting to see what kinds of vector configurations admit the one-to-one correspondence.

5 Application to Explicit Classification

Now that we have established a bijection between label-equivalence classes of generic alcoved polytopes of type A_{n-1} and central regular subdivisions of labeled Q_{n-1}, we wonder if the result can be bettered to a classification of the combinatorial types of generic alcoved polytopes. The classification of regular subdivisions remaining at large, we instead examine the system of inequalities that gives rise to an alcoved polytope.

5.1 Classification for n=4

A generic alcoved polytope P of type A_3 is determined by a system of four-term inequalities. Since we can always add a sufficiently large positive integer to all c_{ij} without changing the sign of any inequality, we assume that the triangular inequalities are satisfied automatically. We claim that P can be encoded by an acyclic orientation on the boundary of a hexagon.

Lemma 5.1. *The system of four-term inequalities on $\{c_{ij}\}$ (shown below on the left) has a solution if and only if the system of two-term inequalities obtained by removing the c_{i4} and c_{4j} in all four-term inequalities (shown below on the right) has a solution. Here $\varepsilon_i \in \{<,>,=\}$ for $i = 1, 2, \ldots, 6$.*

$$\begin{array}{ccc}
c_{12} + c_{34} \ \underline{\ \varepsilon_1\ }\ c_{14} + c_{32} & & c_{12} \ \underline{\ \varepsilon_1\ }\ c_{32} \\
c_{14} + c_{23} \ \underline{\ \varepsilon_2\ }\ c_{13} + c_{24} & & c_{23} \ \underline{\ \varepsilon_2\ }\ c_{13} \\
c_{13} + c_{42} \ \underline{\ \varepsilon_3\ }\ c_{12} + c_{43} & \rightarrow & c_{13} \ \underline{\ \varepsilon_3\ }\ c_{12} \\
c_{24} + c_{31} \ \underline{\ \varepsilon_4\ }\ c_{21} + c_{34} & & c_{31} \ \underline{\ \varepsilon_4\ }\ c_{21} \\
c_{21} + c_{43} \ \underline{\ \varepsilon_5\ }\ c_{23} + c_{41} & & c_{21} \ \underline{\ \varepsilon_5\ }\ c_{23} \\
c_{32} + c_{41} \ \underline{\ \varepsilon_6\ }\ c_{31} + c_{42} & & c_{32} \ \underline{\ \varepsilon_6\ }\ c_{31}
\end{array}$$

Proof. If the system of two-term inequalities has a solution, then we can set $c_{14} = c_{41} = c_{24} = c_{42} = c_{34} = c_{43}$ and obtain a solution for the system of four-term inequalities.

Now we represent the two-term inequalities by a partial orientation on a hexagon with vertices labeled by (ij) as follows: if $c_{ij} < c_{ik}$, then the edge between (ij) and (ik) points towards (ik); if $c_{ji} < c_{ki}$, then the edge between (ji) and (ki) points towards (ki); if $c_{ji} = c_{ki}$, then we will not orient the edge between (ji) and (ki).

Suppose that the system of two-term inequalities does not have a solution. Linear extension tells us that a partial ordering cannot be extended to a total ordering on the vertices of the oriented hexagon if and only if the partial ordering results in a directed cycle.([5, Sec 3.5]) Hence the orientation determined by the system of two-term inequalities after contracting all undirected edges is cyclic. If we recover all deleted terms in the two-term inequalities and add up all four-term inequalities, we get $\sum_{1\leq i\neq j\leq 4} c_{ij} < \sum_{1\leq i\neq j\leq 4} c_{ij}$. Therefore the system of four-term inequalities does not admit a solution. $\square$

In fact, the oriented hexagon of P has a cyclic orientation if and only if all ε_i are $<$ (respectively $>$) or $=$ with at least one $<$ (respectively $>$).

It follows from Lemma 5.1 that P can be encoded by a acyclically partially oriented hexagon in Q_3. Similarly, we get three more oriented hexagons by deleting the terms with subscript $1, 2$ and 3 in the system of inequalities respectively. In fact, each oriented hexagon is a copy of Q_2 with an orientation. As is shown in Figure 4, the hexagons are naturally embedded in the labeled root polytope Q_3. Furthermore, the edge set of Q_3 is a disjoint union of the edge sets of the four hexagons. Hence we can assign an orientation to the edges of Q_3 that agrees with the four embedded oriented hexagons.

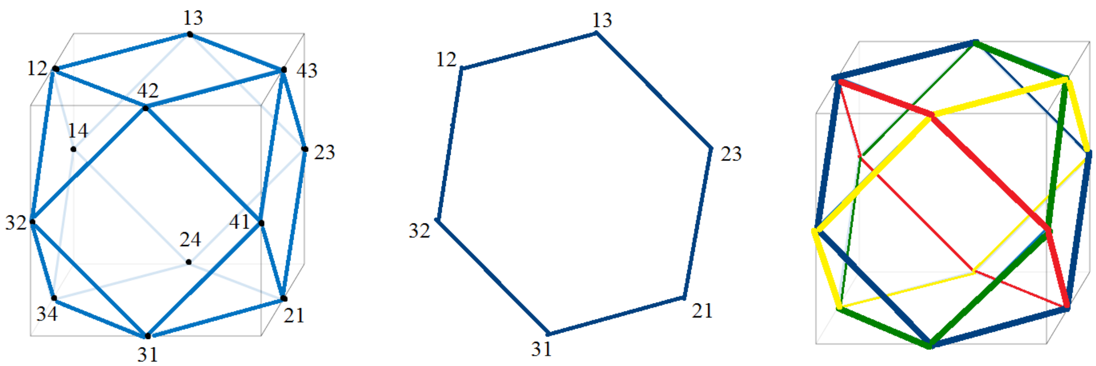


Figure 4: The root polytope Q_3 and the four embedded hexagons.

Remark 5.2. This phenomenon is not limited to A_3. Suppose we have a system of inequalities that determine a generic alcoved polytope of type A_{n-1}. It follows from Proposition 3.5 that if we restrict to the subsystem that does not involve the terms $c_{i,n}$ or $c_{n,i}$ for all i, then we get a system of inequalities that determine a generic alcoved polytope of type A_{n-2}. Hence the collection of faces of Q_{n-1} with codimension two can be decomposed into n disjoint sets, which form n copies of Q_{n-2} embedded in Q_{n-1}.

It is clear that the four oriented hexagons embedded in P encode the same system of inequalities. The question now arises as to whether there is a way to decide if two arbitrary oriented hexagons encode the same polytope. Observe that each square face of an oriented Q_3 represents a four-term inequality. Hence the orientation of one edge on a square decides the orientation of the other three edges, as is shown in Figure 5. In particular, opposite edges have opposite orientations and the two edges that share a vertex have the same orientation. On the other hand, any square face of Q_3 contains exactly one edge from each hexagon. Hence given any oriented hexagon, we can reconstruct the orientated Q_3 in which this hexagon is embedded.

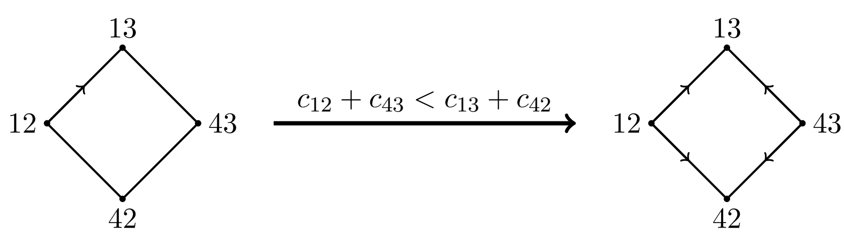


Figure 5: Deciding the orientation of a square face of Q_3.

Furthermore, we can give a combinatorial description of the operation, named *swap map*, that sends one oriented hexagon to the others in an oriented Q_3 up to reflection and rotation. The action of the swap map is captured in Figure 1. For the interest of space, we won't give a rigorous proof here. The reader can quickly verify this operation by trying out a few examples using the rightmost diagram in Figure 4. The underlying algebraic operation of the swap map is the following: it recovers the system of four-term inequalities, deletes all terms whose subscripts involve a certain index $i \in [n]$, and produces an oriented hexagon according to the two-term inequalities.

It remains to be shown that two combinatorially equivalent generic alcoved polytopes can be reduced to the same oriented hexagon modulo the action of swap map, reflections, and rotations. Recall that two polytopes are combinatorially equivalent if the planar graphs of their dual polytopes are isomorphic. Equivalently, we want to prove the following lemma:

Lemma 5.3. *Given a planar graph P^* of the dual polytope of a generic alcoved polytope P, there is a unique combinatorial equivalence class of oriented Q_3 embedded in P^*.*

Proof. By Theorem 1.1, the faces of P^* are in bijection with the cells of a subdivision $\tilde{\tau}$ of ∂Q_3. In particular, if the system of inequalities corresponding to P contains an equality, then $\tilde{\tau}$ has a cell that is a square face. It follows that P^* has 4 edges that forms a square face with no edges on its diagonal, which implies that they are the edges of a square face on ∂Q_3. Hence we can assume that the system of inequalities contains no equality.

In order to reconstruct the one-skeleton of Q_3 contained in P^*, it suffices to identify an edge in P^* that is not an edge of Q_3. We call such edges the *extra* edges in P^*. Note that each square face of the Q_3 embedded in P^* should have exactly one extra edge. On the other hand, each hexagon embedded in an oriented Q_3 has at least one node that is a source, while a node appears in three of the four oriented hexagons. Furthermore, if a node is a source in one hexagon, then it is a source in the two other hexagons. Hence there are at least two sources in an oriented Q_3, which are vertices of degree six in the planar graph P^*.

Suppose that P^* has two degree six vertices connected by an edge. Then this edge has to be an extra edge of P^*, or the square face of Q_3 containing these two vertices will have two extra edges.

Suppose there exist two vertices of degree six such that the shortest path between them has length three (i.e., the path contains three edges), then they are a pair of antipodal(opposite) vertices of the Q_3 (Figure 6 on the left). But the extra edges incident to these two vertices form a path of length two between them, which is a contradiction. In fact, in this case they are connected to two more degree six vertices by the incident extra edges. By the previous case, we are done.

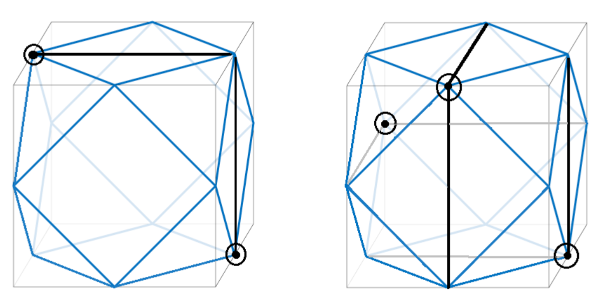


Figure 6: Vertices of degree six in Q_3. The black edges are the "extra" edges.

Otherwise, the shortest path between any pair of degree six vertices has length two. It is not hard to see that there is only one such planar graph (Figure 6 on the right) and it can be recovered uniquely to the one-skeleton of Q_3 with six extra edges. This completes the proof. □

Combining all the ingredients above, we've proven the following theorem on classification of generic alcoved polytopes of type A_3.

Theorem 5.4. *A combinatorial equivalence class of generic alcoved polytopes of type A_3 is encoded by a unique acyclic partial orientations of the boundary of a hexagon modulo the action of the swap map, reflections, and rotations. Conversely, given an acyclic partial orientation of a hexagon, we can reconstruct a unique combinatorial equivalence class of oriented Q_3.*

5.2 The Case n=5 and Beyond

Now we hope to generalize the techniques used in the case $n = 4$ to any n. Again we delete the terms $c_{i,n}$ and $c_{n,i}$ in all the even-term inequalities corresponding to a generic alcoved polytope of type A_{n-1}. It follows from Remark 5.2 that the resulting system of inequalities contains a subsystem that determines a generic alcoved polytope of type A_{n-2}. Furthermore, we obtain a collection of two-term inequalities of the form $c_{ij} < c_{ik}$ and $c_{ji} < c_{ki}$. In fact, we have the following proposition that has its counterpart in the case of A_3:

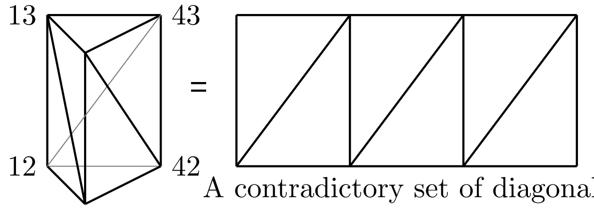
Proposition 5.5. *A generic alcoved polytope of type A_{n-1} corresponds to a regular central subdivision of Q_{n-2} with an acyclic partial orientation on the edges of Q_{n-2}, which are faces of dimension two.*

The idea of the proof is to recover the deleted terms in all the two-term inequalities of a cycle and sum up the four-term inequalities, which will result in a contradiction of the form $S < S$. The proposition follows from simple induction.

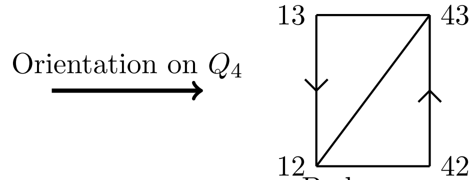
Note that given a regular central subdivision of Q_{n-2} with an orientation on the edges, we can recover the system of inequalities uniquely and deduce the other $n - 1$ subdivisions of Q_{n-2} with an orientation. Hence we can define the analog of the swap map for A_{n-1}. It follows from the proposition that there is an injective map from label-equivalence classes of generic alcoved polytopes of type A_{n-1} to the set modulo swap map of regular central subdivisions of Q_{n-2} with an acyclic partial orientation on the edges of Q_{n-2}.

In the case of $n = 5$, we have the advantage that facets of Q_4 are either tetrahedrons or prisms, which are three-dimensional objects. Theorem 1.1 tells us that a generic alcoved polytope of type A_4 corresponds to a subdivision of ∂Q_4. Hence we obtain one more condition on the admissible orientation of Q_4.

Notice that a subdivision of a prism is determined by the diagonals on the three square faces and vice versa. A contradiction arises if and only if the three diagonals are not connected, in which case they do not decide a subdivision. A prism with diagonals as such corresponds to a configuration of a "bad square" (shown below) on the labeld Q_4 obtained by deleting the terms with index 5 in the subscript.



A contradictory set of diagonals



Bad square

It turns out that the swap map sends a bad square to a cycle of length three. Therefore we have the following conjecture:

Conjecture 5.6. *There is a one-to-one correspondence between label-equivalence classes of generic alcoved polytopes of type A_4 and orbits of the swap map which do not contain an orientation of Q_3 that has a directed cycle.*

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