EXTENSION OF K_4-INTERSECTING FAMILIES OF GRAPHS

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Abstract

The problem on the largest triangle-intersecting family of graphs on n labeled vertices was proposed by Simonovits and Sós in 1976. They conjectured the upper bound to be $2^{\binom{n}{2}-3}$, which is obtained by taking all graphs containing some fixed triangle. This was proven by Ellis, Filmus, and Friedgut, who also conjectured that their results extend to the K_t-intersecting families. Berger and Zhao recently proved the case for $t = 4$. We present current progress to prove the case for $t = 5$, including reductions and optimizations in the computational process. Moreover, we extend these conjectures to hypergraph $K_4^{(3)}$-intersecting families of graphs.

1 INTRODUCTION

A triangle-intersecting family of graphs is a family $\mathcal{F}$ of graphs on n labeled vertices such that each pair of graphs in $\mathcal{F}$ intersects at a triangle. Simonovits and Sós in 1976 posed the question of the largest triangle-intersecting family of graphs on n labeled vertices and conjectured a tight bound of $2^{\binom{n}{2}-3}$. In 1986, Chung, Graham, Frankl, and Shearer [5] proved that every triangle-intersecting family of graphs on n labeled vertices has a size of less than $2^{\binom{n}{2}-2}$ using Shearer's entropy theorem. The tight bound was proven by Ellis, Filmus, and Friedgut [6] using Fourier analysis methods and reduction to a linear program. They also proved the uniqueness and stability of the maximizer. In particular, a family of graphs with a size close to the tight upper bound will also be close to a triangle-umvirate. They also conjectured an extension of their results for K_t-intersecting families of graphs and the tetrahedron-intersecting family of hypergraphs.

Definition 1.1. *A hypergraph $\mathcal{H}$ is a pair $(V, \mathcal{E})$ where V is a finite set of vertices and $\mathcal{E}$ is a collection of subsets of V, called* ***edge set****. We denote the number of edges in graph $\mathcal{H}$ as $e(\mathcal{H})$ and the number of vertices as $v(\mathcal{H})$. An* ***r-uniform*** *hypergraph or* ***r-graph*** *is a hypergraph such that all its edges have size r. The 2-graphs are called* ***graphs****.*

2^V is the set of all subsets of V. $\binom{V}{r}$ denotes the set of all r-subsets of the set V. $\binom{V}{r}$ is called the *complete r-graph* over V, and is abbreviated to $K_n^{(r)}$, where $|V| = n$. $K_n^{(2)}$ is the complete graph on n vertices, also denoted by K_n. $K_4^{(3)}$ is complete 3-graph on 4 vertices, also called *tetrahedron* in this paper.

A *complete t-partite r-graph*, denoted by $\mathcal{T}_n^{(r)}(t)$, has a t-partition of its vertices $V = \bigcup_{i\in[t]} V_i$. The edge set contains exactly all r-subsets of V where all members of the subset are in distinct parts. Complete t-partite 2-graphs are abbreviated as $\mathcal{T}_n(t)$. A *complete t-equipartite r-graph*, denoted by $\mathcal{P}_n^{(r)}(t)$, is a complete r-graph such that all parts have almost equal size, i.e. $|V_i| = \lfloor (n + i - 1)/t \rfloor$ [9]. Complete t-equipartite 2-graphs are abbreviated as $\mathcal{P}_n(t)$.

Definition 1.2. *For an unlabeled hypergraph $\mathcal{H}$, a family $\mathcal{F}$ of hypergraphs on n common labeled vertices is* ***$\mathcal{H}$-intersecting*** *if for every pair of $G_1, G_2 \in \mathcal{F}$, $G_1 \cap G_2$ contains a copy of $\mathcal{H}$. A family of hypergraphs is called* ***$\mathcal{H}$-umvirate*** *if all hypergraphs contain some fixed copy of $\mathcal{H}$.*

Here, we describe the conjecture on the upper bound, as well as the stability and uniqueness of the maximizer in a general form.

Conjecture 1.3. *Let $\mathcal{H}$ be a complete r-graph on k vertices and $\mathcal{F}$ be an $\mathcal{H}$-intersecting family of r-graphs on n vertices. Let $N := \binom{n}{r}$ be the number of edges in a complete r-graph on n vertices and $M := \binom{k}{r}$ be the number of edges in $\mathcal{H}$. Then,*

- $|\mathcal{F}| \le 2^{N-M}$.
- *Equality is achieved if and only if $\mathcal{F}$ is a $\mathcal{H}$-umvirate.*

- *There is an absolute constant $C > 0$ such that for all $\epsilon > 0$, if $|\mathcal{F}| \geq (1-\epsilon)2^{N-M}$, then there exists a $\mathcal{H}$-umvirate $\mathcal{U}$ such that $|\mathcal{U} \triangle \mathcal{F}| \leq C\epsilon 2^N$.*

Recently, Berger and Zhao [2] gave a tight solution for the case when $\mathcal{H} = K_4$ using a similar Fourier analysis as in [6] but with a simplified verification of dual constraints. Our paper presents the progress on the cases of $\mathcal{H} = K_5$ and $\mathcal{H} = K_4^{(3)}$.

Orgnization. In Section 2, we explain the reduction to a linear program introduced in [6] and the framework to reduce the verification of dual constraints to a finite computation introduced in [2]. In Section 3, we introduce further computational optimization to verify the case for K_5. In Section 4, we introduce the extension of the conjecture to hypergraphs. Then, we conclude with future steps in Section 5.

2 REDUCTION TO A LINEAR PROGRAM AND VERIFICATION OF DUAL LINEAR CONSTRAINTS

In this section, we briefly present the framework introduced in [6] to reduce the problem to one of a linear program. It was also explained carefully in [2]. We also show the methods introduced in [2] to reduce the verification of dual linear constraints to a finite computation.

A hypergraph $\mathcal{H}$ is $\mathcal{F}$-free if no subgraph of $\mathcal{H}$ is a member of the hypergraph family $\mathcal{F}$. The *Turán* number of $\mathcal{F}$, denoted by $ex(n, \mathcal{F})$, is the maximum number of edges of an $\mathcal{F}$-free r-graph on n vertices. An $\mathcal{F}$-free r-graph $\mathcal{H}$ on n vertices such that $e(\mathcal{H}) = ex(n, \mathcal{F})$ is called an *extremal hypergraph*, denoted by $ex(\mathcal{H})$. Tuŕan proved that $ex(n, K_t) = (1 - 1/(t-1))\binom{n}{2} + O(n)$, which is achieved by complete $(t-1)$-equipartite graphs on n vertices, $\mathcal{P}_n(t-1)$ [14]. He also conjectured that $ex(n, K_4^{(3)})$ is attained in the 3-graph on n vertices defined as follows. Let $V = V_0 \cup V_1 \cup V_2$ be an equipartition of V. The edge set contains all possible edges that either intersect all V_i's or contain two vertices of $V_{i+1 (mod\ 3)}$ and one in V_i. The notation above is taken from [9]. The exact value of $ex(n, K_4^{(3)})$ is not yet proven but several papers proved upper bounds. The best-known upper bound of $0.593592\ldots$ was given by Chung and Lu in [4]. Another likely correct upper bound was given by Razborov in [13]. Another hypergraph that we will consider in Section 4 is $K_4^{(3)-}$, or tetrahedron with one edge removed. A family of dense graphs that is $K_4^{(3)-}$-free is complete 3-equipartite 3-graphs [3]. The Turán number for this graph is also not proven exactly. The current best upper bound is 0.2871 by Baber and Talbot [1] and the best known lower bound is $0.2857\ldots$ by Frankl and Füredi [7]. A good summary of Turán type problems can be found in [10].

Following the conjecture on the extremal hypergraph of $K_4^{(3)}$, we define *complete 3-proper* 3-graphs, denoted by $\mathcal{T}_n^*$ as follows. For a 3-graph, Let $V = V_0 \cup V_1 \cup V_2$ be a partition (not necessarily equipartition) of V. We call an edge *proper* if it either intersects all V_i's or contains two vertices of $V_{i+1 (mod\ 3)}$ and one in V_i. Otherwise, we call the edge improper. We call a graph *3-proper* if there is a 3-partition of its vertices such that all edges are proper. A complete 3-proper graph is a 3-proper graph whose vertices can be partitioned into three independent sets such that the edge set contains all 3-subset of V that could form a proper edge.

Degree of a hypergraph is defined as the maximum degree of all vertices in the hypergraph. *Co-degree* of two vertices u, v is defined as the number of distinct edges containing both u and v. The co-degree of a hypergraph is defined as the maximum co-degree of all pairs of vertices.

Below we see how we can reduce the problem into one of a linear program by considering the intersection of hypergraphs and extremal hypergraphs for $\mathcal{H}$. The reason why we consider intersections with the densest graphs that are $\mathcal{H}$-free is to have a rich set of resulting subhypergraphs such that the linear program has a rich set of coefficients and is more likely to be solvable.

2.1 REDUCTION TO A LINEAR PROGRAM

Each hypergraph G on n labeled vertices is indicated by a $\mathbb{F}_2^N$ vector where each dimension is an indicator for the existence of an edge and N is the number of edges in a fully connected graph on n vertices.

Proposition 2.1 ([2], Proposition 2.1)**.** *Let f be a 0/1 valued Boolean function and ν be a real valued Boolean function satisfying $\mathbb{E}[\nu] = 1$ and $\langle f * f, \nu \rangle = 0$. Let $m = \max_{\lambda \neq 0} |\hat{\nu}(\lambda)|$ Then,*

- *(Upper Bound)* $\mathbb{E}[f] \leq \frac{m}{1+m}$.
- *(Maximum) If equality is achieved, then $\hat{f}(\lambda) = 0 \forall \lambda \neq 0$ with $|\hat{\nu}(\lambda)| < m$.*

- *(Stability)* $\forall \epsilon, \delta \in (0,1], \mathbb{E}[f] \geq \frac{m}{1+m} - \epsilon$, *then*

$$\sum_{\lambda \neq 0, |\hat{\nu}(\lambda) \leq (1-\delta)m|} \hat{f}(\lambda)^2 \leq \frac{(1+m)\varepsilon}{m(\frac{1-2m}{1+m} + \varepsilon)\delta}$$

The proof of the proposition follows from Proposition 2.1 in [2].

Definition 2.2. *The* ***support distribution*** *of $\mathcal{H}$, denoted by $\mathcal{T}_{\mathcal{H}}$, is a uniform distribution over a set of dense $\mathcal{H}$-free graphs. In particular, for $\mathcal{H} = K_t$, $\mathcal{T}_{\mathcal{H}}$ will be a uniform distribution over all complete $(t-1)$-partite graphs on n vertices, or $\mathcal{T}_n(t-1)$. For $\mathcal{H} = K_4^{(3)}$, $\mathcal{T}_{\mathcal{H}}$ will be a uniform distribution over all complete 3-proper 3-graphs on n vertices, $\mathcal{T}_n^*$.*

In order to apply Proposition 2.1 to a $\mathcal{H}$-intersecting family of graphs, we need to find a suitable ν. It is hard to directly find a ν that satisfies the constraints, so we will reduce the search space to find ν of a specific form. We restate the linear program (Proposition 2.3 in [2]) as follows.

Lemma 2.3. *Let $f : \mathbb{F}_2^N \to \mathbb{F}_2$ be an indicator function of $\mathcal{H}$-intersecting families of hypergraphs on n labeled vertices. Also, let $\{f_G\}$ be a set of functions indexed by hypergraphs on n vertices. If $\nu : F_2^N \to \mathbb{R}$ satisfies:*

$$\hat{\nu}(G) = (-1)^{e(G)} \mathbb{E}_{T \sim \mathcal{T}_{\mathcal{H}}} f_T(T \cap G)$$

then $\langle f * f, \nu \rangle = 0$.

Proof. We can show that ν is supported on graphs whose complements are $\mathcal{H}$-free following the proof of Lemma 2.3 in [2]. Thus, for any x, y such that $f(x) = f(y) = 1$, $x + y$'s complement must contain a copy of $\mathcal{H}$ where x and y intersect. Thus, $\langle f * f, \nu \rangle = \mathbb{E}[f(x)f(y)\nu(x+y)] = 0$. $\square$

Theorem 2.4. *Fix an $\mathcal{H}$ with M edges. If there exists a set of unlabeled graphs $\{H\}$, corresponding coefficient $\{c_H\}$ and $\delta > 0$ such that for any G on n labeled vertices, we have*

$$\mu(G) := (-1)^{e(G)} \sum_H c_H \mathbb{P}_{T \sim \mathcal{T}_{\mathcal{H}}}[G \cap T \cong H]$$

that satisfies

(a) $\mu(0) = 2^M - 1$,

(b) $|\mu(G)| \leq 1$ *for all* $G \neq \emptyset$,

(c) $|\mu(G)| \leq 1 - \delta$ *whenever G has more than M edges,*

then Conjecture 1.3 holds for $\mathcal{H}$. Here $A \cap B \cong C$ means the resultant subgraph from the intersecting edge sets of A and B is isomorphic to C.

Proof. The proof follows Section 2.3 in [2]. The uniqueness claim is a special case of Lemma 2.8 in [8] and the stability follows from a result of Kindler and Safra [11]. $\square$

Definition 2.5. *Define a k-coloring of a hypergraph with vertex set V as a mapping $\varphi : V \to \{0, 1, \ldots, k-1\}$.*

Remark. *Proposition 2.5 in [2] expressed the probability as $\mathbb{P}[G_{t-1} \cong H]$. Where G_{t-1} is the subgraph of G formed by uniformly randomly coloring vertices of G with $t-1$ colors and deleting all monochromatic edges. This is equivalent to $\mathbb{P}_{T \sim \mathcal{T}_{\mathcal{H}}}[G \cap T \cong H]$ for the case $\mathcal{H} = K_t$, where we consider intersections with a uniformly distributed complete $(t-1)$-partite graph since all edges in $G \cap T$ will contain vertices from distinct independent sets. It will be used interchangeably in this paper.*

Berger and Zhao found a valid μ for the case of $t \in \{3, 4\}$ in [2] by reducing the verification of dual linear constraints to a finite computational problem. We will briefly present the main idea behind the reduction.

2.2 VERIFICATION OF THE DUAL LINEAR CONSTRAINTS

In order to apply Theorem 2.4 to prove Conjecture 1.3, we need to come up with a valid set of $\{H\}$ and $\{c_H\}$. In order to do so, we need to verify that $\mu(G)$ satisfies the constraints for all graphs, so we need to bound $\mu(G)$ for large graphs. The intuition behind why we can find an upper bound for large graphs based on small graphs is that for G larger than H, $\mathbb{P}[G \cap T \simeq H]$ is decreasing with the number of edges in G, so the contribution to the sum $(-1)^{e(G)} \sum_H c_H \mathbb{P}[G \cap T \simeq H]$ will decay.

Below we show how we can verify the validity of μ using a finite number of checks given an upper bound on μ for large graphs.

Proposition 2.6. *Fix a hypergraph $\mathcal{H}$ with M edges. Given a set of hypergraphs $\{H\}$ with coefficients $\{c_H\}$, suppose there is an $n_H \geq v(\mathcal{H})$ such that*

$$\forall G, v(G) > n_H, |\mu(G)| \leq F(\{G : v(G) \leq n_H\}),$$

where F is some function on all hypergraphs with at most n_H vertices. Then, we can verify that $\{H\}$ and $\{c_H\}$ satisfy the constraints on μ with $\mathcal{O}(2^{poly(n_H)})$ number of checks, where $poly(x)$ means polynomial in x.

Proof. Given the upper bound on $|\mu|$ for large hypergraphs, we just need to verify that constraints on μ in 2.4 are satisfied for small hypergraphs (with at most n_H vertices) and that the upper bound is bounded away from 1. In particular, we need to check that

(a) $\mu(0) = 2^M - 1$.

(b) $|\mu(G)| \leq 1$ for all G with at most M edges.

(c) $|\mu(G)| \leq 1 - \delta$ for all G with more than M edges but at most n_H vertices.

(d) $F(\{G : G \subseteq K_{n_H}\}) \leq 1 - \delta$.

Step $(b), (c)$ involves checking all subhypergraphs of K_{n_H}, so only $\mathcal{O}(2^{poly(n_H)})$ checks are required. Since the upper bound is only a function of small hypergraphs, step (d) can also be computed in finite time. ☐

3 K_5-INTERSECTING FAMILIES OF GRAPHS

The verification for K_5 is implemented using codes in Python. We used graph data taken from [12] and a valid upper bound function F for complete graphs K_t introduced in [2]. Since computation time increases exponentially with the n_H, the computational power quickly becomes the limiting factor. In this section, we present and introduce several optimizations to further simplify the verification of dual constraints. We also give the coefficients for subgraphs of K_5.

Berger and Zhao [2] introduced an upper bound function $F(\{G : G \subseteq K_{n_H}\})$ for the cases $\mathcal{H} = K_t$ and found valid pairs of n_H and $\{H\}$ for the cases $\mathcal{H} = K_3$ and $\mathcal{H} = K_4$.

Proposition 3.1 ([2], Proposition 3.5). *For each $\mathcal{H} = K_t, t \geq 3$, given a list $\{H\}$ of unlabeled graphs on at most n_H vertices with corresponding coefficients $\{c_H\}$, for any G on $n > n_H$ labeled vertices, we have*

$$\sum_H |c_H| \cdot \mathbb{P}_{T \sim \mathcal{T}_\mathcal{H}}[G \cap T \cong H] \leq \max_{G \subseteq K_{n_H}} \left[\frac{1}{(t-1)^{\kappa(G)-1}} \sum_H \widetilde{c_H} \cdot \mathbb{P}[G \cap T \cong H] \cdot DC_{t,n_H}(v(H)) \right]$$

where $\kappa(G)$ is the number of connected components in G.
The coefficient DC_{t,n_H} is defined as

$$DC_{t,n_H}(x) := \max_{n \in \mathbb{Z}, n > n_H} \frac{\binom{n}{n_H}}{(t-1)^{n-n_H} \binom{n-x}{n_H - x}} := F(\{G : G \subseteq K_{n_H}\}).$$

Since $\frac{\binom{n}{n_H}}{\binom{n-x}{n_H-x}}$ starts decreasing when $n \geq 2x+1$, we only need to numerically check the maximum over $n_H < n \leq 2x$. Denote $S(H)$ as the set of graphs that can be transformed to H by repeatedly identifying pairs of disconnected vertices. Then $\widetilde{c}_H$ is defined as

$$\widetilde{c}_H := \max_{H' \in S(H)} |c_{H'}|.$$

Proof. Proof can be seen in Section 3 of [2]. ☐

We will also use this

$$F(\{G : G \subseteq K_{n_H}\}) = \max_{G \subseteq K_{n_H}} \left[\frac{1}{(t-1)^{\kappa(G)-1}} \sum_{H} \widetilde{c_H} \cdot \mathbb{P}[G \cap T \cong H] \cdot DC_{t,n_H}(v(H)) \right]$$

for the K_5 case. We used a list of G up to 9 vertices (there are 274668 non-isomorphic graphs on at most 9 vertices), so computational power is a great constraint. Therefore, we introduce below further optimizations. These optimizations are briefly mentioned in [2] but we give more detailed proofs below.

Proposition 3.2. *Fix an $\mathcal{H}$ with M edges. In order to satisfy the constraints in 2.4, c_H are uniquely determined for all subhypergraphs $H \subset \mathcal{H}$. In particular, $\mu(G) = -1$ for all $G \subset \mathcal{H}$.*

Proof. Construct a matrix A of dimension $2^M - 1$ by $2^M - 1$ where the rows are indexed by $H' \subset \mathcal{H}$ and the columns are indexed by $G' \subset \mathcal{H}$. The entry indexed by (G', K') has a value

$$(-1)^{e(G')} c_{H'} \mathbb{P}_{T \sim \mathcal{T}_{\mathcal{H}}}[G' \cap T \cong H'].$$

This matrix is lower triangular with no zeros on the diagonal. Thus, it is full rank. Now, augment A to a matrix B with an additional column of -1 except for the first entry with value $2^M - 1$ and an additional row of $(-1)^{e(\mathcal{H})} c_{H'} \mathbb{P}_{T \sim \mathcal{T}}[\mathcal{H} \cap T \cong H']$ for $H' \subset \mathcal{H}$ with the last entry set to -1.

$$B = \left[\begin{array}{cc:c} & & 2^M - 1 \\ & A & -1 \\ & & \vdots \\ \hdashline (-1)^{e(\mathcal{H})} c_{\emptyset} \mathbb{P}_{T \sim \mathcal{T}}[\mathcal{H} \cap T \cong \emptyset] & \dots & -1 \end{array} \right]$$

Note that $\{c_{H'}\}$ is a set of valid coefficients if and only if it is in the null space of B. To show that there is a unique set of $\{c_{H'}\}$, we just need to show that B has nullity equal to 1, i.e. rank equal to $2^M - 1$. First note that the rank of B is at least the rank of the submatrix A, i.e. $2^M - 1$. Now, we want to show that the rank of B is at most $2^M - 1$ by showing that rows are not linearly independent.

To show that rows of B are linearly dependent, we just need to show that sum of each column of B is 0. The last column is trivial. For any column indexed by $H' \subset \mathcal{H}$, we have

$$(B^T \mathbf{1})_{H'} = \sum_{G' \subseteq \mathcal{H}} (-1)^{e(\mathcal{H})} c_{H'} \mathbb{P}_{T \sim \mathcal{T}}[G' \cap T \cong H'].$$

Since H' are strict subsets of $\mathcal{H}$, there is some edge $e \in G \backslash H'$. If we pair up G' and $G' \oplus e \subseteq G$, we have $\mathbb{P}_{T \sim \mathcal{T}}[G' \cap T \cong H'] = \mathbb{P}_{T \sim \mathcal{T}}[G' \oplus e \cap T \cong H']$, but $(-1)^{e(G' \oplus e)} = -(-1)^{e(G')}$, so they cancel out in the sum. This shows that the sum of each column of B is 0. ☐

Lemma 3.3. *For two disconnected components U, V of G, if we identify one pair of vertices $u \in U, v \in V$ and call the resulting hypergraph G_1, then $\mathbb{P}[G \cap T \cong H] = \mathbb{P}[G_1 \cap T \cong H]$.*

Proof. Consider $G = U \sqcup V$. Since U, V are disjoint, $U \cap T \cong H$ and $U \cap T \cong H'$ are independent events, we have

$$\#\{T : G \cap T \cong H\} = \#\{T : U \cap T \cong H\} \#\{T : V \cap T = \emptyset\} + \#\{T : U \cap T = \emptyset\} \#\{T : V \cap T \cong H\}.$$

Recall that if T follows a uniform distribution over complete k-partite graphs or 3-proper graphs ($k = 3$ in this case), it is equivalent to considering a uniformly random k-coloring $\varphi : V \to \{0, 1, \dots k-1\}$ of vertices of G. Also, if $G \cap T \cong H$ for some G and H, then rotations ($\varphi(v) \to \varphi(v) + 1 (mod\ k)$) of the colors of $v \in V$ preserve this property. When we identify vertices u and v, we added the constraints that u and v must be of the same color. We can think of this as uniformly randomly coloring vertices in U without constraints, then for any coloring of vertices of V, if the color of v is not equal to the color of u, rotate the colors of vertices in V until they are equal. Thus,

$$\mathbb{P}[G_1 \cap T \cong H] = \frac{\#\{T : G_1 \cap T \cong H\}}{\#\{T\}} = \frac{\#\{T : G \cap T \cong H\}/k}{\#\{T\}/k} = \mathbb{P}[G \cap T \cong H].$$

☐

Recursively applying the above lemma, we can derive similar results for an arbitrary number of disconnected components. From this, we can just consider isoclasses of graphs. Namely, consider the equivalence relation $G \sim G'$. Assuming $v(G') > v(G)+1$, $G \sim G'$ if G can be transformed to G' by repeatedly identifying one pair of vertices from two disconnected components in the current graph. In all below, H denotes equivalence classes of graphs. As an example, below three graphs on the left are all in the same equivalence class, but K_3 is not in the same equivalence class.

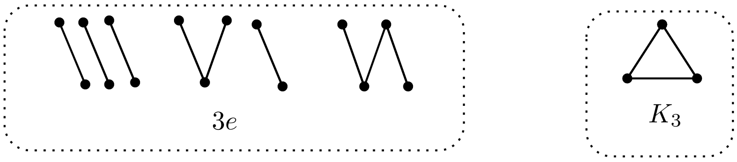


Figure 1: Examples of equivalence classes for K_t

Note that for hypergraphs, since we can only identify one pair of vertices from two disconnected components, the graph on the right with two pairs of identifying vertices won't be in the equivalence class $2e$.

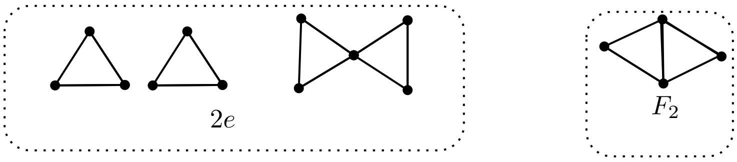


Figure 2: Examples of equivalence classes for tetrahedron

We also made optimizations in terms of code implementation. Since checking isomorphism is time-consuming, TBD. Our computational procedure follows steps in Algorithm 3.

Algorithm 1 Computational Steps to Generate Coefficients Set for K_5

$allG \leftarrow$ all edge vectors of subgraphs of K_5, namely, all binary strings of length $2^{\binom{9}{2}}$
$allColors \leftarrow$ all strings of length 9 consisting of 0,1,2,3, indicating partitions of all vertices
$allT \leftarrow$ convert each coloring into an edge list of edges that are not monochromatic
$allSubgraphs \leftarrow$ zeros(allG.length, allT.length)
for $G \in allG$ **do**
  for $T \in allT$ **do**
    $allSubgraphs[G, T] \leftarrow G \cap T$
  end for
end for
$allH \leftarrow$ for each H, generate edge list of all graphs on at most 9 vertices isomorphic to H
$P \leftarrow$ zeros(allG.length, allH.length)
for $G \in allSubgraphs$ **do**
  for $subgraph \in G$ **do**
    for $H \in allH$ **do**
      if $subgraph$ in H **then**
        $P[G, H]$ += 1
      end if
    end for
  end for
end for
for $row \in P$ **do**
  $row \leftarrow row \cdot (-1)^{e(G)}$
end for
$A \leftarrow$ P/allT.length
$A_{small} \leftarrow$ rows of A corresponding to G with at most 10 edges at most 9 vertices.
$A_{large} \leftarrow$ rows of A corresponding to G with at least 11 edges at most 9 vertices.

for $col \in P$ **do**
 $col \leftarrow col \cdot DC_H$
end for
$A_{extra} \leftarrow P$
Use a convex programming solver to solve the optimization problem

 $maximize$ $x[0]$ subject to

1. $abs(A_{small} \cdot c) \leq 1$
2. $abs(A_{large} \cdot c) \leq 1 - x[0]$
3. $x[1] = 2^M - 1$
4. $x[0] > 0$

In the resulting convex programming problem, $x[0]$ represents δ in Theorem 2.4, $x[1]$ represents coefficient on $\emptyset$. We want to maximize δ, by bounding $|\mu(G)|$ away from 1 for large G. We can additionally set coefficients for all subgraphs of K_5 to the precomputed value to speed up the convex programming solver. The set of precomputed coefficients for all subgraphs of K_5 can be found here.

4 $K_4^{(3)}$-INTERSECTING FAMILIES OF HYPERGRAPHS

In this section, we extend the results on graphs to 3-graphs. The framework of reduction to linear program introduced in Section 2 still applies. We show that the conjecture 1.3 is likely to not hold for $\mathcal{H} = K_4^{(3)-}$, which is $K_4^{(3)}$ with one edge removed, but present some evidence to support why the conjecture is likely to work for $K_4^{(3)}$.

As shown in Figure 4, on the left is an illustration of a 3-partite 3-graph. For $\mathcal{H} = K_4^{(3)-}$, $\mathcal{T}_{\mathcal{H}}$ in Definition 2.2 is a uniform distribution over all complete 3-partite 3-graphs $\mathcal{T}_n^{(3)}(3)$. On the right is an illustration of a 3-proper 3-graph on n vertices. The black edge is an edge with all three vertices in different positions. Let the three vertex sets be labeled $0, 1, 2$, then the blue edges are edges with one vertex in the set i and two in the set $i+1 \pmod 3$. The densest graphs on n vertices that are $K_4^{(3)}$-free are complete 3-proper 3-graphs.

Notice if there are two vertices in a 3-graph G being partitioned into the same vertex set, then any edge containing these two vertices can never be in $G \cap T$, where T is a complete 3-partite 3-graph. Thus, if there are two vertices with high degrees in G and in the same partition, then all edges containing those two vertices cannot appear in $G \cap T$. Intuitively, we cannot find an upper bound on $|\mu(G)|$ as in Proposition 2.6 for $\mathcal{H} = K_4^{(3)-}$ since there are bad cases of G such that $|\mu(G)|$ does not decrease as G gets larger. Below we will give a more detailed example of the above argument and explain why this is not of concern when $\mathcal{H} = K_4^{(3)}$.

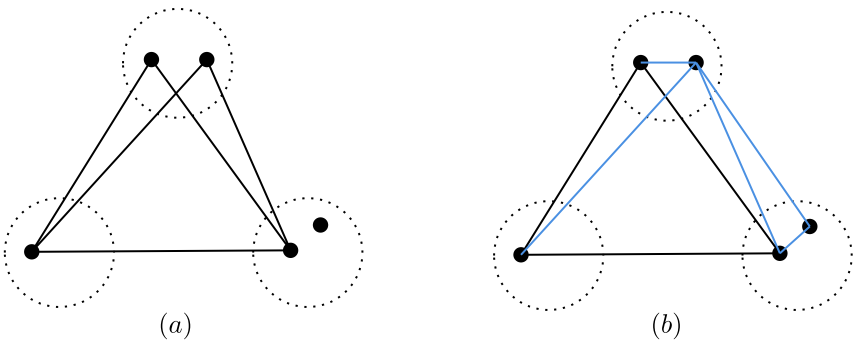


Figure 3: An illustration of (a) a 3-partite 3-graph and (b) a 3-proper graph

Recall that the feasibility of finding an upper bound in Proposition 2.6 relies on the fact that for G larger than H, $\mathbb{P}[G \cap T \simeq H]$ converges to 0 as $e(G) \to \infty$, so the contribution to the sum $(-1)^{e(G)} \sum_H c_H \mathbb{P}[G \cap T \simeq H]$ will decay. We show how the argument fails to work for $\mathcal{H} = K_4^{(3)-}$.

Proposition 4.1. *Fix $\mathcal{H} = K_4^{(3)-}$. Given a set of hypergraphs $\{H\}$, there is a sequence of hypergraph G with an increasing number of edges such that $G \cap T \cong H$ does not converge to 0 as $e(G) \to \infty$.*

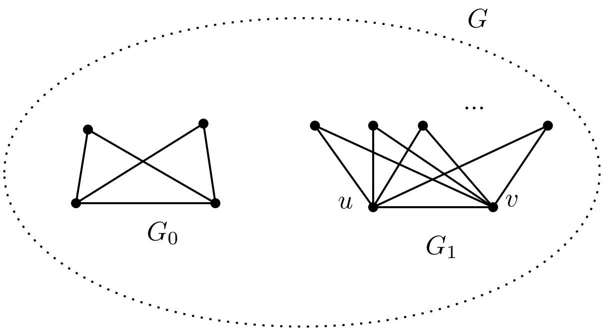


Figure 4: A plot of G with high degree

Proof. Consider the graph in Figure 4. First notice that for a 3-graph T to be $K_4^{(3)-}$ free, it needs to be a 3-partite 3-graph, such that we can color the vertices of T using three colors where the vertices in each edge all have distinct colors. Let $\mathcal{T}$ be a uniform distribution over all 3-partite 3-graphs.

Consider $G_k = G \sqcup W_k$ where G is a graph chosen such that the probability that $G \cap T \cong H$ is positive for all H in the set. W_k is a graph with all edges containing two fixed vertices u, v. Consider a 3-coloring of vertices, φ. Observe that as long as u and v in the graph is mapped to the same color, all edges in W_k cannot appear in $G \cap T$. Then, W_k (thus G_k) can have arbitrarily many number of edges but $\mathbb{P}[G \cap T \simeq H]$ does not decay to 0, since

$$\begin{aligned}\mathbb{P}(G_k \cap T \simeq H) &\geq \mathbb{P}[\varphi(u) = \varphi(v)] \cdot \mathbb{P}[G \cap T \cong H] \\ &= \frac{1}{3} \cdot \mathbb{P}[G \cap T \cong H]\end{aligned}$$

where $\mathbb{P}[G \cap T \cong H]$ is a constant as G remains intact as we add edges to W_k. □

The argument for $\mathcal{H} = K_4^{(3)}$ does not have this concern because the support distribution is complete 3-proper 3-graphs. Consider a 3-coloring of vertices in a 3-graph. Given the colors of two vertices of an edge in a 3-graph, no matter what those two colors are (can be equal), there is still a positive probability that this edge is proper. Therefore, given a fixed set of $\{H\}$, $\mathbb{P}[G \cap T \cong H]$ decreases as $e(G)$ increases since it is less likely to have all the extra edges in $G \backslash H$ to be improper. Below we will provide some more detailed evidence.

From Proposition 3.2, we immediately have coefficients for subgraphs of $K_4^{(3)}$ as shown below.

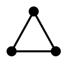
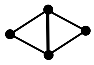
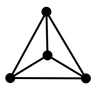
	empty	$1e$	F_2	S_3
H				
coefficients	15	-10.2	0.6	-6.6

Figure 5: Coefficients for subgraphs of $K_4^{(3)}$

Notice that given a fixed set of H, it is harder for graphs with vertices of high degree or co-degree to satisfy the constraints on μ. This is because the probability of $G \cap T \cong H$ becomes more concentrated on H with a degree close to G, which necessarily needs a graph with a high degree (thus a large number of edges) to be included in the set of H. Before we present some empirical evidence for this, let's define some classes of graphs.

Definition 4.2. *Here we want to define 5 specific classes of 3-graphs.*

1. *If a 3-graph is a **fan graph** with k edges (denoted by F_k) is a 3-graph on $k+2$ vertices, then there is a labeling of vertex as $u, v_0, ...v_k$ such that the edge set is $\mathcal{E} = \{\{u, v_i, v_{i+1}\} \mid i \in \{0, ..., k-1\}\}$.*

2. *If a 3-graph is a **wheel graph** with k edges (denoted by W_k) is a 3-graph on $k+1$ vertices, then there is a labeling of vertex as $u, v_0, ...v_{k-1}$ such that the edge set is $\mathcal{E} = \{\{u, v_i, v_{i+1 (mod\ k)}\} \mid i \in \{0, ..., k-1\}\}$.*

3. *If a 3-graph is a **star graph** with k edges (denoted by S_k) is a 3-graph on $k+1$ vertices, then there is a labeling of vertex as $u, v_0, ...v_{k-1}$ such that the edge set is $\mathcal{E} = \{\{u, v, v'\} \mid (v, v') \in \binom{\{v_0, ..., v_{k-1}\}}{2}\}$.*

4. *If a 3-graph is a **wedge graph** with k edges (denoted by Wd_k) is a 3-graph on $k+2$ vertices, then there is a labeling of vertex as $u_0, u_1, v_0, ...v_{k-1}$ such that the edge set is $\mathcal{E} = \{\{u_0, u_1, v_i\} \mid i \in \{0, ..., k-1\}\}$.*

5. *If a 3-graph is a **double wedge graph** with $2k$ edges (denoted by Wdd_k) is a 3-graph on $k+3$ vertices, then there is a labeling of vertices as $u, w_0, w_1, v_0, ...v_{k-1}$ such that the edge set $\mathcal{E} = \{\{u, w_j, v_i\} \mid j \in \{0,1\}, i \in \{0, 1, ...k-1\}\}$.*

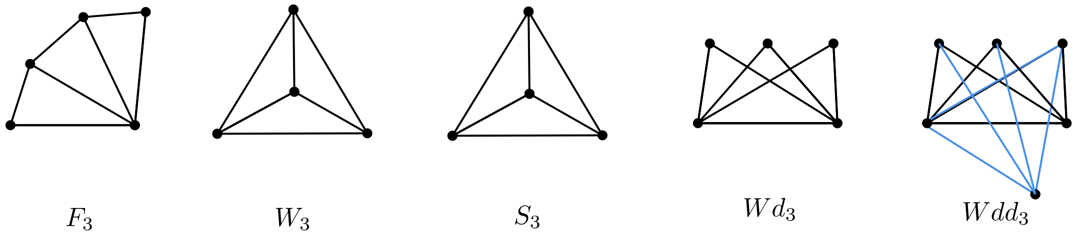


Figure 6: Illustration of high degree/ co-degree 3-graphs

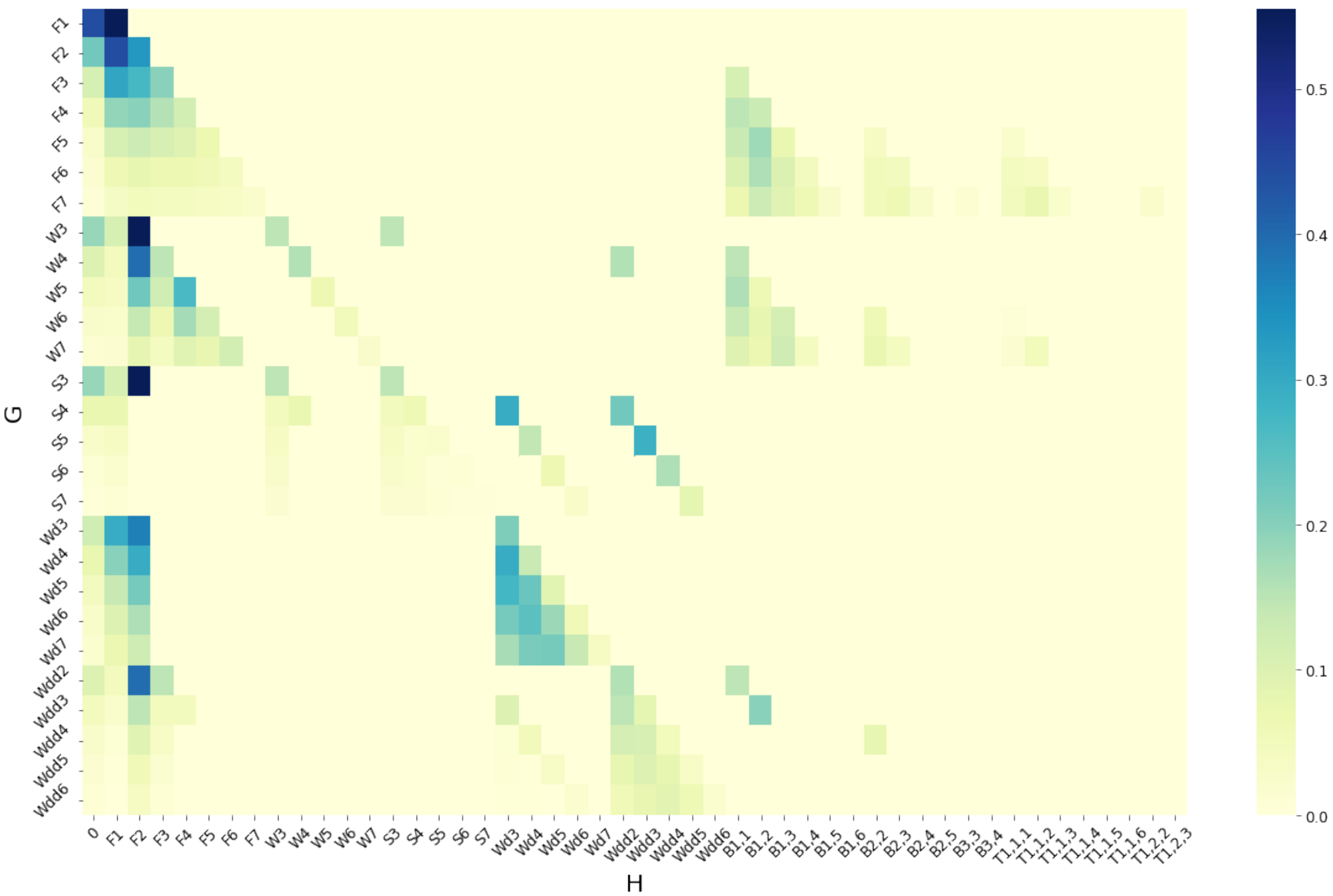


Figure 7: A heatmap of probabilities that each G intersects a 3-proper graph being isomorphic to H

Figure 7 is a heatmap of probabilities $\mathbb{P}[G \cap T \cong H]$ for a set of G labeled on the y-axis and a set of H labeled on the x-axis. We have new notations for subgraphs of G. Namely, Bi, j is a union of two fan graphs F_i, F_j with the vertex with the highest degree as the only intersection. Ti, j, k is a union of three fan graphs F_i, F_j, F_k with the vertex with the highest degree as the only intersection.

As evident from Figure 7, the probability $\mathbb{P}[G \cap T \cong H]$ for G with a high degree/ co-degree is concentrated on increasingly large H with similar degree/codegree as G. We want to calculate these probabilities and show that they decrease with k.

Below, we calculate probabilities $\mathbb{P}[G \cap T \cong H]$ of a different set of H that probability is concentrated in for each G of high degrees. Namely, for F_k or W_k, probabilities are concentrated in subgraphs of $F_s, s \le k$. For S_k, probabilities are concentrated in Wd_{k-1} and Wdd_{k-2}. For Wd_k, probabilities are concentrated in $Wd_s, s \le k$. For Wdd_k, probabilities are concentrated in $Wdd_s, s \le k$. We also calculate the probability of $\mathbb{P}[G \cap T \cong H]$ when H is empty or a single edge. This is because coefficients $c_\emptyset$ and c_{1e} have high magnitude (Figure 4) so we want to ensure that these probabilities also decay with k.

Below, we calculate the probability that $G \cap T \cong H$ for several cases of G and H and show that the probability converges to 0 as $k \to \infty$.

$H = \emptyset$ or $1e$, $G = F_k$

WLOG, let the node with the highest degree be u. Let $\varphi(u) = 0$.

First, consider a fan graph G with vertices $u, v_0, ...v_k$. Let's consider the probability that $G \cap T$ is empty for each possible color of v_0.

To calculate the probability that $G \cap T$ is empty when $\varphi(v_0) = 0$, we first want to show the below lemma.

Lemma 4.3. *The number of distinct binary sequence of length n that does not contain consecutive 1's is $F(n+2)$, where $F(i)$ is the ith Fibonacci number (F(0) = 0 and F(1) = 1).*

Proof. To see this, notice that if the sequence starts with 1, then the second bit must be 0 and we have a problem with $n-2$ length. If the sequence starts with 0, then we are left with a problem of $n-1$ length. Thus, let the number of binary sequences of length n that does not contain consecutive 1's be $f(n)$. We have $f(n) = f(n-1) + f(n-2)$. Also, $f(1) = 2, f(2) = 3$. □

When $\varphi(v_0) = 0$, for $\{u, v_0, v_1\}$ to be improper, $\varphi(v_1) \in \{0, 2\}$. When $\varphi(v_0) = 2$, $\varphi(v_1)$ can only be 2. When $\varphi(v_0) = 1$, $\varphi(v_1)$ can only be 1. Inducting on $\varphi(v_i), i \ge 1$, we know that when $\varphi(v_0) = 0$, all remaining $\varphi(v)$ can be any sequence of 0 or 2, but there can not be $i \in [k-1]$ such that $\varphi(v_i) = \varphi(v_{i+1}) = 2$. On the other hand, when $\varphi(v_0) = 1$ (or 2), all remaining $\varphi(v_i), i \ge 1$ must be 1 (or 2).

Thus, from Lemma 4.3, the probability that $G \cap T$ is empty when $\varphi(v_0) = 0$ is $\frac{F(k+2)}{3^k}$ and probability that $G \cap T$ is empty when $\varphi(v_0) = 1$ or 2 is $\frac{1}{3^k}$. The total probability of a fan graph with k edges having an empty $G \cap T$ is $\frac{F(k+2)+2}{3^{k+1}}$.

We can similarly calculate that the probability of a fan graph with k edges having $G \cap T \cong 1e$ (where $1e$ is 1 edge) is

$$\frac{1}{9}(2 \cdot \sum_{j=0}^{k-1} \frac{F(j+2)}{3^j} \cdot \frac{1}{3^{k-1-j}} + 3 \cdot \sum_{j=0}^{k-1} \frac{1}{3^{k-1}})$$

To see this, we know that an edge $\{u, v_j, v_{j+1}\}$ is proper with $\varphi(u) = 0$ if and only if $\{v_j, v_{j+1}\}$ has colors $\{0,1\}, \{1,0\}, \{1,2\}, \{2,1\}$, or $\{2,2\}$. Then, we can calculate the probability that all edges $\{u, v_i, v_{i+1}\}$ are improper for $i < j$ and $i > j$ respectively using results from the previous part. Then, summing over all possible positions of j gives us the result as claimed.

Using the estimation of Fibonacci numbers, we have

$$F(n) \sim \frac{\varphi^n}{\sqrt{5}} + O(\varphi^{-n})$$

where $\varphi = \frac{\sqrt{5}+1}{2}$.

Thus,

$$\mathbb{P}[F_k \cap T \cong \emptyset] \sim \frac{\varphi^2}{3\sqrt{5}}(\frac{\varphi}{3})^k,$$

$$\mathbb{P}[F_k \cap T \cong 1e] \sim \frac{2\varphi}{\sqrt{5}(\varphi-1)}(\frac{\varphi}{3})^{k+1} + \frac{k}{3^k} \sim \frac{2\varphi^2}{3\sqrt{5}(\varphi-1)}(\frac{\varphi}{3})^k.$$

Both converge to 0 as $k \to \infty$ since $\frac{\varphi}{3} < 1$.

$H = \emptyset$ or $1e$, $G = W_k$

Consider a wheel graph with vertices $u, v_0, ...v_{k-1}$. Then, for $G \cap T$ to be empty, if $\varphi(v_0) = 1$ or 2, then all remaining vertices need to be the same color as v_0. Else, if $\varphi(v_0) = 0$, then all remaining vertices need to form a $(k-1)$-length sequence with no consecutive 2's. Thus, $\mathbb{P}[G \cap T \cong \emptyset] = \frac{F(k+1)+2}{3^k}$

For $G \cap T$ to contain only 1 edge, WLOG, let $\{u, v_0, v_1\}$ form a proper edge. Then, the only possibility of the colors for v_0, v_1 are $\{2, 2\}$, which forces v_{k-1} and v_2 to have color 0 and the remaining vertices form a sequence of length $k-4$ with no consecutive 2's. Otherwise, if v_0, v_1 have colors $\{0, 1\}$ or $\{1, 2\}$, then WLOG, let v_1 have color 1. It will force all remaining vertices to have color 1 and result in an extra edge $\{u, v_{k-1}, v_0\}$. Thus, probability of $G \cap T$ to contain only 1 edge is $\frac{F(k-2)}{3^k}$.

We can also get the asymptotic behavior of probabilities as

$$\mathbb{P}[W_k \cap T \cong \emptyset] \sim \frac{\varphi}{\sqrt{5}}(\frac{\varphi}{3})^k$$

and

$$\mathbb{P}[W_k \cap T \cong 1e] \sim \frac{1}{\varphi^2\sqrt{5}}(\frac{\varphi}{3})^k.$$

Both converge to 0 as $k \to \infty$ since $\frac{\varphi}{3} < 1$.

$H = F_k$, $G = F_k$

WLOG, let the color of the vertex with the largest degree u be 0. Then $\varphi(v_0), ...\varphi(v_{k-1})$ can be any sequence of $\{0, 1, 2\}$ such that there is no $i \in \{0, 1, ...k-2\}$ such that $\{\varphi(v_i), \varphi(v_{i+1})\} \in \{\{0, 0\}, \{1, 1\}, \{0, 2\}\}$. To count the total number of valid sequences, let the number of valid sequences of length k be $f(k)$. Let the number of valid sequences of length n and starting with i be $f_i(k)$ for $i \in \{0, 1, 2\}$. Then, we have

$$f_2(k) = f_1(k-1) + f_2(k-1)$$

$$f_0(k) = f_1(k-1)$$

$$f_1(k) = f_0(k-1) + f_2(k-2)$$

or

$$\begin{pmatrix} f_0(k) \\ f_1(k) \\ f_2(k) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} f_0(k-1) \\ f_1(k-1) \\ f_2(k-1) \end{pmatrix}$$

However, notice that $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is not diagonalizable, so we want to prove an upper and lower bound on the probability. First, notice that the number of sequences of $\{0, 1, 2\}$ such that there is no $i \in \{0, 1, ...k-2\}$ such that $\{\varphi(v_i), \varphi(v_{i+1})\} \in \{\{0, 0\}, \{1, 1\}, \{0, 2\}\}$ is less than or equal to the number of sequences such that there is no $i \in \{0, 1, ...k-2\}$ such that $\{\varphi(v_i), \varphi(v_{i+1})\} \in \{\{0, 0\}, \{1, 1\}\}$. The latter can be solved by considering

$$\begin{pmatrix} f_0(k) \\ f_1(k) \\ f_2(k) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} f_0(k-1) \\ f_1(k-1) \\ f_2(k-1) \end{pmatrix}$$

where the coefficient matrix can be factorized and using $f_i(1) = 1$ for all i, we have

$$f(k) = f_0(k) + f_1(k) + f_2(k) = (-1)^k + (1-\sqrt{2})^k + (1+\sqrt{2})^k$$

where the last term is dominating when k becomes large. Thus, $\mathbb{P}[F_k \cap T \cong F_k] = \frac{f(k)}{3^k}$ converges to 0 as $k \to \infty$.

$H = Wd_{k-1}$ or Wdd_{k-2}, $G = S_k$

WLOG, let the color of the vertex with the highest degree u be 0.

First, let's consider the first equality. If the color of the vertex v_i with the same codegree of u is 0. Then all remaining vertices need to be 1 to form an edge with u, v_i. If the color of v_i is 2, then the color of the remaining vertices can be either 2 or 1. However, they cannot contain both 2 and 1, since otherwise, those two vertices will form an edge with u. Also, there cannot be two remaining vertices both be 2 since otherwise, they can form an edge with u. Thus, all remaining vertices need to be 1. If v_i has color 1, then the remaining vertices can have colors 0 or 2. However, as argued before, there cannot be two remaining vertices with color 2. Thus, either all remaining vertices are 0 or at most one remaining vertex is 2. Thus, there are k possible colorings. Thus, in total, there are $k+1+1$ valid colorings out of the 3^{k+1} possible colorings of $v_0, ...v_{k-1}$. Also, the vertex v_i with same co-degree as u can be any of $v_i, i \in \{0, 1, ...k-1\}$. Thus, we have $k(k+2)$ valid colorings in total, so $\mathbb{P}[S_k \cap T \cong Wd_{k-1}] = \frac{k(k+2)}{3^{k+1}}$.

Now, for the second equality, WLOG, let v_0, v_1 be two vertices with co-degree $k-2$ and u has degree $2(k-2)$. Then, (u, v_0, v_1) can be of colors $(0,0,0), (0,0,2), (0,2,0)$ or $(0,1,1)$. In each case, the total number of valid colorings for the remaining $k-2$ vertices is $1, 1, 1$ and $k-1$. Also, we have $\binom{k}{2}$ choices for v_0, v_1. Thus, we have the total probability $\mathbb{P}[S_k \cap T \cong Wdd_{k-2}] = \frac{\binom{k}{2}(2+k)}{3^{k+1}}$. Both these probabilities converge to 0 as $k \to \infty$.

General 3-uniform Graphs

The results above are for G with high degrees/ co-degrees. When G is a general 3-graph with a bounded degree, for a fixed H, we can also give an upper bound on the probability of $G \cap T \cong H$ as a function of $e(G)$ and show that it decreases with $e(G)$.

Lemma 4.4. *When $\Delta(G) \leq k$,*

$$\mathbb{P}[G \cap T \cong H) \leq \mathbb{P}[e(G \cap T) \leq e(H)] \leq \frac{2(1+k)e(G)}{9(2/9e(G) - e(H))^2}$$

Proof. Consider a uniformly random 3-coloring of vertices of G. Define $X = \sum_{e \in \mathcal{E}(G)} X_e$, then

$$EX = \frac{2}{9}e(G)$$

$$\begin{aligned} EX^2 &= \sum_{e \in \mathcal{E}(G)} X_e + \sum_{e \cap e' \leq 1} \mathbb{E}[X_e]\mathbb{E}[X_{e'}] + \sum_{e \cap e' = 2} \mathbb{E}[X_e X_{e'}] \\ &\leq \frac{2}{9}e(G) + \frac{4}{81}e(G)^2 + \frac{2}{27} \cdot 3ke(G) \end{aligned}$$

$$Var[X] \leq \frac{2}{9}e(G)(1+k)$$

Then, by Chebyshev's inequality, we have

$$\begin{aligned} P[e(G \cap T) \leq e(H)] &= P[X \leq e(H)] \\ &\leq \mathbb{P}[|X - EX| \geq \frac{2}{9}e(G) - e(H)] \\ &\leq \frac{Var[X]}{(2/9e(G) - e(H))^2} \\ &\leq \frac{2(1+k)e(G)}{9(2/9e(G) - e(H))^2} \end{aligned}$$

□

5 FURTHER WORK

Using the introduced reduction framework and computation optimization, we have the following conjecture.

Conjecture 5.1. *Conjecture 1.3 holds for $\mathcal{H} = K_t$ for all positive integers t and $\mathcal{H} = K_4^{(3)}$.*

The intersecting family of hypergraphs for these two families have remained open questions for a long time and was also proposed in [2, 6]. Below we provide some potential directions to prove the above conjecture.

First, we are optimistic that we can prove the case for $\mathcal{H} = K_5$ by expanding the list of graph G and running computational verification on larger graphs (graphs with more than 9 vertices), such that we can find a set of unlabelled graphs H and coefficients c_H that satisfies all the constraints in Theorem 2.4. Towards proving the cases for general K_t, we want to derive a general form for the set of H and coefficients c_H in terms of t that will allow us to construct a μ that satisfies all constraints in Theorem 2.4 for the case when $\mathcal{H} = K_t$. For the case $\mathcal{H} = K_4^{(3)}$, following the intuitions in Section 4, our next step is to find an n_H and an exact expression for the upper bound $F(\{G : v(G) \leq n_H\})$ on $|\mu(G)|$ for large 3-graphs G. We can then use computational verification to find a valid μ.

We also conjecture that the argument only holds for $\mathcal{H}$-intersecting families when $\mathcal{H}$ is a complete graph.

Conjecture 5.2. *If $\mathcal{H}$ is not a complete graph, there does not exist a function μ that satisfies the constraints in Theorem 2.4.*

For example, suppose $\mathcal{H} = K_t^-$, which is a complete graph K_t with one edge removed. From Proposition 3.2, we will be able to find the unique set of pre-determined c_H for $H \subset K_t^-$. However, when calculating $\mu(K_t)$, we cannot use additional H other than the set of subgraphs of K_t^- because any subgraph of K_t that is not a subgraph of K_t^- would contain a copy of K_t^-. Therefore, we have more variables than linear constraints and we conjecture that given the set of pre-determined c_H,

$$|\mu(K_t) = |(-1)^{\binom{t}{2}} \sum_{H:H \subset K_t^-} c_H \mathbb{P}_{T \sim \mathcal{T}_\mathcal{H}}[G \cap T \cong H]| > 1.$$

6 ACKNOWLEDGEMENT

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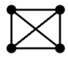
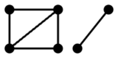
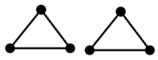
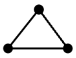
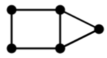
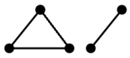
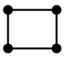
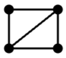
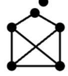
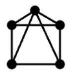
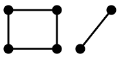
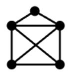
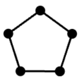
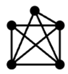
H	name	c_H	H	name	c_H
	0e	1,023.00		K4	1,396.93
	1e	-339.67		dia+	34.98
	2e	111.00		K3,K3	64.78
	K3	-334.33		house	-103.92
	3e	-33.30		V3	-40.67
	K3+	102.11		K4+	-211.58
	C4	-108.85		wedge3	40.22
	4e	4.93		fan3	-74.17
	dia	-224.70		C5x	218.81
	K3++	-18.48		C4inner	-314.37
	C4+	44.16		houx	24.39
	C5	-43.61		K5-	429.13

Figure 8: Coefficients c_H for all subgraphs of K_5