The Vertex Isoperimetric Problem on Kneser Graphs

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Abstract

For a simple graph $G = (V, E)$, the vertex boundary of a subset $A \subseteq V$ consists of all vertices not in A that are adjacent to some vertex in A. The goal of the vertex isoperimetric problem is to determine the minimum boundary size of all vertex subsets of a given size. In particular, define $\mu_G(r)$ as the minimum boundary size of all vertex subsets of G of size r. Meanwhile, the vertex set of the Kneser graph $KG_{n,k}$ is the set of all k-element subsets of $\{1, 2, \ldots, n\}$, and two vertices are adjacent if their corresponding sets are disjoint. The main results of this paper are to compute $\mu_G(r)$ for small and large values of r, and to prove the general lower bound $\mu_G(r) \geq \binom{n}{k} - \frac{1}{r}\binom{n-1}{k-1}^2 - r$ when $G = KG_{n,k}$. We will also survey the vertex isoperimetric problem on a closely related class of graphs called Johnson graphs and survey cross-intersecting families, which are closely related to the vertex isoperimetric problem on Kneser graphs.

1 Introduction

The classical isoperimetric problem on the plane asks for the minimum perimeter of all closed curves with a fixed area. The ancient Greeks conjectured that a circle achieves the minimum boundary, but this was not rigorously proven until the 19th century using tools from analysis. There are two discrete versions of this problem in graph theory: the vertex isoperimetric problem and the edge isoperimetric problem. In both problems, we wish to minimize the "boundary" of a vertex subset of a given size.

For a simple graph $G = (V, E)$ and vertex subset $A \subseteq V$, the *vertex boundary* is the set

$$\partial A = \{v \in V \setminus A : v \text{ adjacent to some } u \in V\}.$$

In other words, the vertex boundary of a vertex subset A is the set of vertices not in A that are adjacent to some vertex in A. The *edge boundary* of a vertex subset A consists of all edges with one vertex in A and one vertex in $V \setminus A$. In the *vertex isoperimetric problem*, we wish to minimize the size of the vertex boundary over all vertex subsets with a fixed size. Similarly, in the *edge isoperimetric problem*, we wish to minimize the size of the edge boundary over all vertex subsets with a fixed size. Bezrukov's [2] and Leader's [12] surveys summarize common techniques used and key results for both the vertex and edge isoperimetric problems.

In this paper, we will focus on the vertex isoperimetric problem. For convenience, define the *vertex isoperimetric function* for a graph G as

$$\mu_G(r) = \min\{|\partial A| : A \subseteq V, |A| = r\}.$$

(We will drop the G when the context is clear.) For example, $\mu(1)$ is equal to the minimum degree of the graph and $\mu(|V|) = 0$. The goal of the vertex isoperimetric problem on a specific graph is to compute $\mu(r)$ for all $1 \leq r \leq |V|$ and classify the subsets that produce the optimal boundaries.

In practice, computing all values of $\mu(r)$ is difficult and has only been achieved for a few classes of graphs. The study of vertex isoperimetric problems began in 1966 when Harper [10] solved the problem for the hypercube Q_n, which has the vertex set $\{0, 1\}^n$ and two vertices are adjacent if and only if they differ in one coordinate. Equivalently, the vertex set of Q_n is the power set of $[n] = \{1, ..., n\}$ and two vertices x, y are adjacent if and only $|x \triangle y| = 1$. Using this representation, Harper defined an order on the vertices of Q_n called the *simplicial order*. Under it, we have $x < y$ if $|x| < |y|$, or if $|x| = |y|$ and $\min(x \triangle y) \in x$. Harper proved that the initial segments I of the simplicial order are the optimal vertex subsets. That is, if I is an initial segment and A is a vertex subset with $|A| = |I|$, then $|\partial A| \geq |\partial I|$.

The vertex isoperimetric problem has also been solved for grids [5], the discrete torus [4], and Cartesian powers of a specific graph on five vertices called the diamond graph [3]. The optimal vertex subsets for these graphs are all nested, just like with the hypercube.

In this paper, we will consider the vertex isoperimetric problem on *Kneser graphs*. The vertex set of a Kneser graph is $[n]^{(k)}$, the set of all k-element subsets of $[n]$, and two vertices are adjacent if their corresponding sets are disjoint. The optimal vertex subsets for these graphs are not nested, so new techniques are needed. In Section 2, we will survey the vertex isoperimetric problem on Johnson graphs, which also have $[n]^{(k)}$ as a vertex set. This will give us insight for the same problem on Kneser graphs. In Section 3, we will survey intersecting

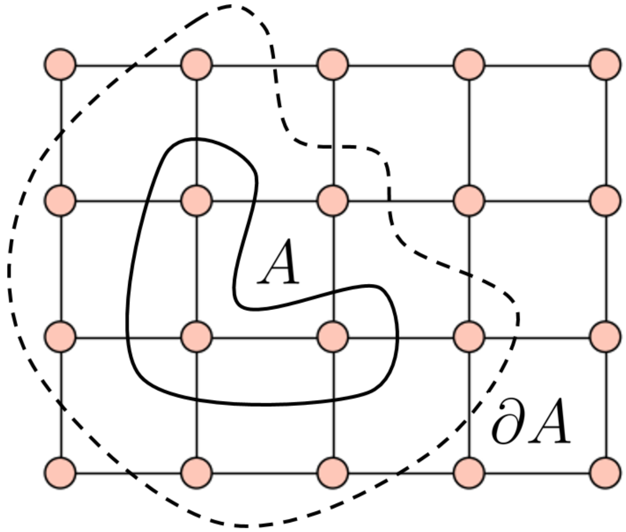

Figure 1: In this grid graph G, the vertex subset A consists of the vertices enclosed by the solid line, and the vertex boundary ∂A consists of the vertices between the solid and dashed lines. Note that by definition, the boundary is disjoint from the original subset. The vertex isoperimetric function satisfies $\mu_G(3) = 3$, since among all vertex subsets of size 3, taking a corner vertex and its two neighbors yields the minimum boundary.

and cross-intersecting families of sets. Finally, in Section 4 we will apply the tools described in Sections 2 and 3 to Kneser graphs. Specifically, we will compute the vertex isoperimetric function for the Kneser graph in special cases and bound the function in general.

2 Johnson Graphs

The vertex set of the *Johnson graph* $J(n,k)$ is $[n]^{(k)}$, and two vertices are adjacent if and only if they intersect in exactly $k-1$ elements. The vertex isoperimetric problem has been solved for certain values of r, and there are lower and upper bounds on $\mu_{J(n,k)}(r)$ in general. For convenience, in this section we will let $\mu(r) = \mu_{J(n,k)}(r)$.

2.1 Lower Bound of Isoperimetric Function

Christofides et al. [6] proved an "approximate" vertex isoperimetric inequality for Johnson graphs. They found a lower bound for boundary size but it is not necessarily the best possible bound.

Theorem 2.1. *(Christofides, Ellis, Keevash [6]) Let* $1 \leq k \leq n-1$*. Suppose that* $A \subseteq [n]^{(k)}$ *and* $|A| = \alpha\binom{n}{k}$*. Let* ∂A *be the vertex boundary of* A *in the Johnson graph* $J(n,k)$*. Then*

$$|\partial A| \geq c\sqrt{\frac{n}{k(n-k)}}\alpha(1-\alpha)\binom{n}{k}$$

for some absolute constant c *(taking* $c = 1/5$ *suffices).*

We can transform the previous inequality into a more familiar setting by plugging in $\alpha = \binom{n}{k}/r$. This yields the lower bound

$$\mu(r) \geq \frac{1}{5}\sqrt{\frac{n}{k(n-k)}} \cdot \frac{r(\binom{n}{k} - r)}{\binom{n}{k}}$$

for $1 \leq r \leq \binom{n}{k}$.

We will sketch a proof of the approximate vertex isoperimetric inequality. The main idea of the proof is to induct on n. To do so, we will need to relate the boundary of vertices whose elements are in $[n]$ with the boundary of vertices whose elements are in $[n-1]$. This motivates the following definition. Let $\mathcal{B} \subseteq [n]^{(k)}$. Then

$$\mathcal{B}_0 = \{x \in \mathcal{B} : n \notin x\}$$

$$\mathcal{B}_1 = \{x \in [n-1]^{(k-1)} : x \cup \{n\} \in \mathcal{B}\}.$$

are the *lower n-section* and *upper n-section* of $\mathcal{B}$, respectively. Every set in $\mathcal{B}$ either contains n or doesn't contain n, so $|\mathcal{B}| = |\mathcal{B}_0| + |\mathcal{B}_1|$. We also need the notions

$$\partial^- \mathcal{B} = \{y \in [n]^{(k-1)} : \text{there exists } x \in \mathcal{B} \text{ such that } y \subseteq x\}$$
$$\partial^+ \mathcal{B} = \{y \in [n]^{(k+1)} : \text{there exists } x \in \mathcal{B} \text{ such that } x \subseteq y\}$$

which are the *lower shadow* and *upper shadow* of $\mathcal{B}$, respectively. Finally, let $N(\mathcal{B}) = \mathcal{B} \cup \partial\mathcal{B}$ be the *neighborhood* of $\mathcal{B}$, which consists of all vertices in $\mathcal{B}$ or in its vertex boundary (with respect to the Johnson graph). Note that

$$|N(\mathcal{A})| = |(N(\mathcal{A}))_0| + |(N(\mathcal{A}))_1| = |N(\mathcal{A}_0) \cup \partial^+(\mathcal{A}_1)| + |N(\mathcal{A}_1) \cup \partial^-(\mathcal{A}_0)|.$$

Thus

$$|N(\mathcal{A})| \geq \max\{|N(\mathcal{A}_0)| + |\partial^-(\mathcal{A}_0)|, |N(\mathcal{A}_1)| + |\partial^+(\mathcal{A}_1)|, |N(\mathcal{A}_0)| + |N(\mathcal{A}_1)|\}.$$

Since $|N(\mathcal{A})| = |\mathcal{A}| + |\partial\mathcal{A}|$, we can rewrite the above inequality to contain no neighborhoods and only boundaries. We can then apply the inductive hypothesis to the right hand side because none of the sets in the families contains the element n. In particular, $N(\mathcal{A}_0)$ and $\partial^+(\mathcal{A}_1)$ are subsets of $[n-1]^{(k)}$, while $N(\mathcal{A}_1)$ and $\partial^-(\mathcal{A}_0)$ are subsets of $[n-1]^{(k-1)}$. The exact calculation of the bound is not very enlightening and we will omit the details.

2.2 Exact Values of Isoperimetric Function

The previous result relied on taking sections and shadows of set families. Meanwhile, Gutiérrez [9] adapted the shifting operation used in the proof of the vertex isoperimetric problem on the hypercube to the vertex isoperimetric problem on Johnson graphs. We will introduce some notation.

Associate each vertex in $V(J(n,k)) = [n]^{(k)}$ with its incidence vector $\mathbf{x} \in \{0,1\}^n$. The i-th entry of $\mathbf{x}$ is 1 if i is in the subset, and 0 otherwise. For example, the vertex $\{1,2,4\} \in [5]^{(3)}$ corresponds to the incidence vector $(1,1,0,1,0)$. For the rest of this section, each vertex of

$J(n,k)$ will be interpreted as an incidence vector. Define the XOR operation between two incidence vectors as

$$(\mathbf{x} \oplus \mathbf{y})_i = \mathbf{x}_i + \mathbf{y}_i \pmod 2$$

for $1 \le i \le n$. Next, let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ be the standard basis. That is, for $1 \le i \le n$, the vector $\mathbf{e}_i$ contains 1 in the i-th entry and is 0 elsewhere. Finally, let

$$|\mathbf{x}| = \sum_{i=1}^{n} x_i$$

be the *weight* of a vector.

This notation lets us compactly measure distances in the Johnson graph. Two adjacent vertices $\mathbf{x}, \mathbf{y}$ in the Johnson graph differ in two coordinates, so $|\mathbf{x} \oplus \mathbf{y}| = 2$. It can be shown by induction that the distance in the Johnson graph satisfies $d(\mathbf{x}, \mathbf{y}) = \frac{1}{2}|\mathbf{x} \oplus \mathbf{y}|$.

We are now ready to introduce *shifting*, the key concept used in the vertex isoperimetric problem on the Johnson graph.

Definition 1. Let $S \subseteq \{0,1\}^n$, and fix distinct $i, j \in [n]$. For $\mathbf{x} \in \{0,1\}^n$, define

$$T^*_{ij}(x) = \begin{cases} \mathbf{x} \oplus \mathbf{e}_i \oplus \mathbf{e}_j & \text{if } x_i = 0, x_j = 1 \\ \mathbf{x} & \text{otherwise} \end{cases}$$

This function switches the coordinates at i and j if $x_i = 0, x_j = 1$. Next, define

$$T_{ij}(S, \mathbf{x}) = \begin{cases} T^*_{ij}(x) & \text{if } T^*_{ij}(x) \notin S \\ \mathbf{x} & \text{otherwise} \end{cases}$$

Finally, let

$$T_{ij}(S) = \{T_{ij}(S, \mathbf{x}) : \mathbf{x} \in S\}.$$

By definition, the transformation T_{ij} preserves the size of the subset S. That is, $|T_{ij}(S)| = |S|$. A key fact about the transformation is that it cannot increase the boundary.

Lemma 2.2. *(Gutiérrez [9]) Let $S \subseteq V(J(n,k))$. Then*

$$|\partial T_{ij}(S)| \le |\partial S|.$$

The proof involves tedious casework and does not offer much insight. By Lemma 2.2, we can take an arbitrary vertex subset S and transform it to make it look closer and closer to the optimal vertex subset. For the special case $r = \binom{t}{k}$ with $k \le t \le n$, Gutiérrez was able to pin down the optimal vertex subset.

Theorem 2.3. *(Gutiérrez [9]) Fix the Johnson graph $J(n,k)$. Let $k \le t \le n$ and $r = \binom{t}{k}$. Then $[t]^{(k)}$ minimizes the vertex boundary among all vertex subsets of size r. In particular,*

$$\mu\binom{t}{k} = \binom{t}{k-1}(n-t).$$

This is Gutiérrez's main result. The main idea of the proof is to repeatedly apply the transformation T_{ij} to an arbitrary starting vertex subset S of size $r = \binom{t}{k}$. The transformations make S look more and more like $[t]^{(k)}$, until the transformed vertex subset eventually becomes $[t]^{(k)}$. Thus by Lemma 2.2, $|\partial S| \geq |\partial [t]^{(k)}|$. Since S was arbitrary, $[t]^{(k)}$ indeed has the minimum boundary. Finally, note that

$$|\partial [t]^{(k)}| = \binom{t}{k-1}(n-t)$$

because each vertex in the boundary has $k-1$ elements in common with some vertex of $[t]^{(k)}$, and the last element must be in $[n] \setminus [t]$.

3 Cross-Intersecting Families

Cross-intersecting families of sets have been studied in their own right as a branch of extremal combinatorics, and they have connections to the isoperimetric number of Kneser graphs. Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets. They are *cross-intersecting* if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Before summarizing results on cross-intersecting families of sets, we will examine the simpler notion of an intersecting family of sets.

A family of sets $\mathcal{A}$ is *intersecting* if $A \cap A' \neq \emptyset$ for all $A, A' \in \mathcal{A}$. Common problems in extremal combinatorics are to maximize the size of an intersecting family given certain constraints.

For example, fix a positive integer n, and suppose that $\mathcal{A} \subseteq \mathcal{P}([n])$ is intersecting. What is the maximum size of $|\mathcal{A}|$? Note that a set $A \in \mathcal{P}([n])$ and its complement cannot both be in $\mathcal{A}$, since they are disjoint. By pairing all sets in $\mathcal{A}$ with their complements, we see that $|\mathcal{A}| \leq 2^{n-1}$. The bound is tight as well. For example, we can take $\mathcal{A}$ to be all sets in $\mathcal{P}([n])$ that contain the element 1.

Suppose that we restrict ourselves to subsets of size k. That is, let $\mathcal{A} \subseteq [n]^{(k)}$ be intersecting. What is the maximum size of $|\mathcal{A}|$ now? If $n < 2k$, then every two subsets of size k intersect and we can take the entire subset family $\mathcal{A} = [n]^{(k)}$. Otherwise, if $n \geq 2k$, a natural construction is to fix a single point in $[n]$ and consider all subsets that contain this point, just like in the previous problem. We will call such set families of the form $\{A \in [n]^{(k)} : x \in A \text{ for some fixed } x \in [n]\}$ *dictatorships.* In a dictatorship, there are $k-1$ points required to complete the subset from the remaining $n-1$ points in $[n]$, so the family has size $|\mathcal{A}| = \binom{n-1}{k-1}$. The Erdős-Ko-Rado theorem asserts that this is the best bound. The original 1961 proof [7] used shifting techniques and kick-started the study of intersecting and cross-intersecting families. We will present an alternative and elegant proof of the Erdős-Ko-Rado theorem called the Katona cycle method.

Theorem 3.1. *(Katona [11]) Let $n \geq 2k$ be positive integers. Suppose that $\mathcal{A} \subseteq [n]^{(k)}$ is an intersecting family. Then*

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$

Proof. The core idea is to double count a cleverly constructed set. Let $\mathcal{A}$ be the intersecting family. Suppose that C is a cyclic order of the elements in $[n]$. An interval in C is some subset of consecutive elements in the cycle. We claim that for a fixed cycle C, there can be at most k subsets $A \in \mathcal{A}$ that are intervals in C. Fix any $A \in \mathcal{A}$ that is also an interval in C. Let the elements of A be $(a_1, \ldots, a_k)$ in clockwise order along the cycle. Every other subset $A' \in \mathcal{A}$ that is an interval in C intersects A. In particular, A' divides the interval A at some point into two parts. That is, for some $1 \leq i \leq k-1$, the subset A' contains either a_i or a_{i+1}, but not both. We say that A' splits at i. For a fixed i, at most one A' can split at i, because otherwise two subsets in $\mathcal{A}$ would be disjoint. This yields a maximum of $k-1$ subsets A' in addition to the original subset A. Thus the maximum number of subsets in $\mathcal{A}$ that are intervals in C is k.

We will now double count the number of pairs (C, A), where C is a cyclic order of $[n]$ and $A \in \mathcal{A}$ is an interval in C. Suppose we pick the subset $A \in \mathcal{A}$ first, which can be done in $|\mathcal{A}|$ ways. There are now $k!$ ways to permute the elements in the subset and $(n-k)!$ ways to permute the remaining elements, which determines the cycle containing A as an interval. We can also pick the cyclic order first in $(n-1)!$ ways. By the previous claim, there are at most k subsets $A \in \mathcal{A}$ that are intervals in C. This yields the inequality

$$|\mathcal{A}| \cdot k!(n-k)! \leq (n-1)! \cdot k$$

which proves the desired result. □

We now turn to cross-intersecting families of sets. Suppose that $\mathcal{A} \subseteq [n]^{(a)}$ and $\mathcal{B} \subseteq [n]^{(b)}$ are cross-intersecting families. How big can $|\mathcal{A}|$ and $|\mathcal{B}|$ be? There are two natural quantities to maximize: the sum $|\mathcal{A}| + |\mathcal{B}|$ and the product $|\mathcal{A}||\mathcal{B}|$. In both problems, we assume that $\mathcal{A}$ and $\mathcal{B}$ are nonempty to avoid trivial cases. Recall that the optimal intersecting family in the Erdős-Ko-Rado theorem consists of all subsets of a fixed size containing a certain point. This suggests that in the cross-intersecting problem, we should take $\mathcal{A}$ to be all a-subsets containing a certain point and $\mathcal{B}$ to be all b-subsets containing the same point. In other words, the families should be dictatorships. These families are optimal for maximizing the product $|\mathcal{A}||\mathcal{B}|$ under certain restrictions on n, a, b. However, the same configuration does not maximize the sum $|\mathcal{A}| + |\mathcal{B}|$ under certain restrictions on n, a, b, as the following theorem shows.

Theorem 3.2. *(Frankl, Tokushige [8]) Let $\mathcal{A} \subseteq [n]^{(a)}$ and $\mathcal{B} \subseteq [n]^{(b)}$ be nonempty cross-intersecting families. Suppose that $n \geq a + b$ and $a \leq b$. Then*

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{b} - \binom{n-a}{b} + 1.$$

The upper bound for the sum is achieved when $\mathcal{A}$ contains one a-subset, say $[a]$, and $\mathcal{B}$ contains all b-subsets that intersect $[a]$. The number of b-subsets disjoint from $[a]$ is $\binom{n-a}{b}$, so $|\mathcal{B}| = \binom{n}{b} - \binom{n-a}{b}$. A different configuration maximizes the product $|\mathcal{A}||\mathcal{B}|$.

Theorem 3.3. *(Matsumoto, Tokushige [13]) Let $\mathcal{A} \subseteq [n]^{(a)}$ and $\mathcal{B} \subseteq [n]^{(b)}$ be cross-intersecting families. Suppose that $n \geq 2a$ and $n \geq 2b$. Then*

$$|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{a-1}\binom{n-1}{b-1}.$$

The optimal configuration that achieves this maximum product is the one suggested earlier. We can take $\mathcal{A}$ to be all a-subsets containing 1 and take $\mathcal{B}$ to be all b-subsets containing 1. Then $|\mathcal{A}| = \binom{n-1}{a-1}$ and $|\mathcal{B}| = \binom{n-1}{b-1}$. Matsumoto and Tokushige actually proved a stronger result: unless $n = 2a = 2b$, we have $|\mathcal{A}||\mathcal{B}| = \binom{n-1}{a-1}\binom{n-1}{b-1}$ if and only if $\mathcal{A}$ and $\mathcal{B}$ are dictatorships.

The proofs of the previous two theorems both rely on the Kruskal-Katona theorem, which is proved using the same shifting technique as the original proof ofthe Erdős-Ko-Rado theorem. Thus shifting is a key technique used to study intersecting and cross-intersecting families.

4 Kneser Graphs

Recall that the vertex set of the Kneser graph $KG_{n,k}$ is $[n]^{(k)}$, and two vertices are adjacent if and only if they are disjoint subsets. This section contains the main results, namely values of the isoperimetric function of Kneser graphs in special cases and bounds on the function in general.

Definition 2. Let $\mu(n,k,r) = \mu_{KG_{n,k}}(r)$ be the isoperimetric function on the Kneser graph $KG_{n,k}$. That is,

$$\mu(n,k,r) = \min\{|\partial A| : A \subseteq [n]^{(k)}, |A| = r\}.$$

Next, define the dual function

$$f(n,k,r) = \max\{|[n]^{(k)} \setminus (A \cup \partial A)| : A \subseteq [n]^{(k)}, |A| = r\}.$$

Note that minimizing $|\partial A|$ is equivalent to maximizing $|[n]^{(k)} \setminus (A \cup \partial A)|$. Let A be this optimal subset. Every vertex is either in A, in the boundary of A, or neither, so

$$\mu(n,k,r) + f(n,k,r) + r = \binom{n}{k}.$$

Computing the isoperimetric function for Kneser graphs is straightforward for two cases. If $n < 2k$, then the Kneser graph $KG_{n,k}$ is the empty graph on $\binom{n}{k}$ vertices and $\mu(n,k,r) = 0$ for all r.

If $n = 2k$, then the Kneser graph $KG_{n,k}$ consists of $\frac{1}{2}\binom{2k}{k}$ disjoint edges. We have $\mu(n,k,r) = 0$ if r is even and $\mu(n,k,r) = 1$ if r is odd. For the rest of this paper, we will assume $n > 2k$.

To compute $\mu(n,k,r)$, rather than minimizing $|\partial A|$, we consider the dual problem of maximizing $|V \setminus (A \cup \partial A)|$. The dual problem is easier to analyze on the Kneser graph, since we will be able to use theorems about cross-intersecting families. We can reformulate the vertex isoperimetric problem on Kneser graphs as follows.

Lemma 4.1. *Fix $1 \le r \le \binom{n}{k}$. Then*

$$f(n,k,r) = \max\{|B| : A \subseteq [n]^{(k)}, |A| = r, A \text{ and } B \text{ are disjoint and cross-intersecting}\}$$

and $\mu(n,k,r) = \binom{n}{k} - f(n,k,r) - r$.

Proof. Pick any $A \subseteq [n]^{(k)}$, and suppose that $y \in V(KG_{n,k}) \setminus (A \cup \partial A)$. Then $y \notin A$ and $y \cap x \neq \emptyset$ for all $x \in A$. Equivalently, A and $V(KG_{n,k}) \setminus (A \cup \partial A)$ are disjoint cross-intersecting families. □

Theorem 4.2. *Let $G = (V, E)$ be a d-regular graph on m vertices. Define*

$$f(r) = \max\{|V \setminus (A \cup \partial A)| : A \subseteq V, |A| = r\}.$$

Then $f(r) = 0$ if and only if $m - d \leq r \leq m$. In particular, $\mu_G(r) = |V| - r$ if and only if $m - d \leq r \leq m$.

Proof. Suppose that $m - d \leq r \leq m$. Pick any $A \subseteq V$ such that $|A| = r$ and fix a vertex $y \in V \setminus A$. Since $|V \setminus A| \leq d$ and y has d neighbors, there exists some $x \in A$ such that x and y are adjacent. Thus every vertex not in A is adjacent to some vertex in A, and $f(r) = 0$.

For the other direction, suppose that $r \leq m-d-1$. Fix any vertex $y \in V$. Remove y and all of its neighbors from the vertex set, and pick any subset A of size r from the remaining $m - d - 1$ vertices. By construction, $y \in V \setminus (A \cup \partial A)$, so $f(r) \geq 1$.

Finally, note that minimizing $|\partial A|$ is equivalent to maximizing $|V \setminus (A \cup \partial A)|$. For an optimal subset A, the three sets $\{A, \partial A, V \setminus (A \cup \partial A)\}$ partition V. Thus

$$\mu_G(r) + f(r) + r = |V|$$

which shows that $f(r) = 0$ is equivalent to $\mu_G(r) = |V| - r$.

□

Applying Theorem 4.3 to Kneser graphs lets us compute the isoperimetric function for large values of r.

Corollary 4.3. *If $\binom{n}{k} - \binom{n-k}{k} \leq r \leq \binom{n}{k}$, then $\mu(n, k, r) = \binom{n}{k} - r$.*

We will now compute $\mu(n, k, r)$ for small values of r. Note that $\mu(n, k, 1) = \binom{n-k}{k}$, which is the degree of each vertex in $KG_{n,k}$.

Theorem 4.4. *We have*

$$\mu(n, k, 2) = \min \begin{cases} 2\binom{n-k}{k} - \binom{n-k-1}{k} \\ 2\binom{n-k}{k} - \binom{n-2k}{k} - 2. \end{cases}$$

Proof. Pick two distinct vertices $x, y \in V(KG_{n,k})$, and let $t = |x \cap y|$. There are two cases.

Case 1: $t \geq 1$. Each of x and y has $\binom{n-k}{k}$ neighbors. There are $2k - t$ elements of $[n]$ in $x \cup y$, so there are $\binom{n-(2k-t)}{k}$ vertices adjacent to both x and y. By overcounting, there are $2\binom{n-k}{k} - \binom{n-2k+t}{k}$ vertices adjacent to x or y, and this quantity is minimized when $t = k - 1$.

Case 2: $t = 0$. We use a similar overcounting argument as above. This case is different because x, y are adjacent and the vertex boundary of a subset does not include vertices in the subset. Each of x and y have $\binom{n-k}{k} - 1$ neighbors not in the set $\{x, y\}$, and there are $\binom{n-2k}{k}$ vertices adjacent to x and y. Thus the boundary has size $2\binom{n-k}{k} - \binom{n-2k}{k} - 2$.

Combining the above two cases yields the desired result.

□

We have computed exact values of $\mu(n,k,r)$ for small and large values of r. Now we will bound $\mu(n,k,r)$ for general r. The lower bound requires facts about cross-intersecting families from Section 4.

Theorem 4.5. *For all* $1 \leq r \leq \binom{n}{k}$*, we have*

$$\mu(n,k,r) \geq \binom{n}{k} - \frac{1}{r}\binom{n-1}{k-1}^2 - r.$$

Proof. Let $A \subseteq [n]^{(k)}$ where $|A| = r$, and let $B = [n]^{(k)} \setminus (A \cup \partial A)$. The families A and B are cross-intersecting, so by Theorem 3.3, we have

$$|B| \leq \frac{1}{r}\binom{n-1}{k-1}^2.$$

Taking the max of both sides over all $|A| = r$ yields

$$f(n,k,r) \leq \frac{1}{r}\binom{n-1}{k-1}^2.$$

Finally, Lemma 4.2 implies that

$$\mu(n,k,r) = \binom{n}{k} - f(n,k,r) - r \geq \binom{n}{k} - \frac{1}{r}\binom{n-1}{k-1}^2 - r$$

as desired. □

5 Conclusion

We have solved the vertex isoperimetric problem for special cases of Kneser graphs and bounded the vertex isoperimetric function for Kneser graphs in general. Future research would involve pinning down the exact values of the isoperimetric function or at least establishing tighter bounds. One way to do this is to adapt the proof of Theorem 3.3 to not only cross-intersecting families but also to disjoint cross-intersecting families. This bound could then be used to find a lower bound on $\mu(n,k,r)$, just like in the proof of Theorem 4.6.

Another possible direction of research is to study the edge isoperimetric problem on Kneser graphs. This problem is currently open. For motivation, one could first investigate the problem on Johnson graphs, which is known as the Kleitman-West problem. Kleitman conjectured a solution set for the latter problem, but it was disproved by Ahlswede and Cai [1].

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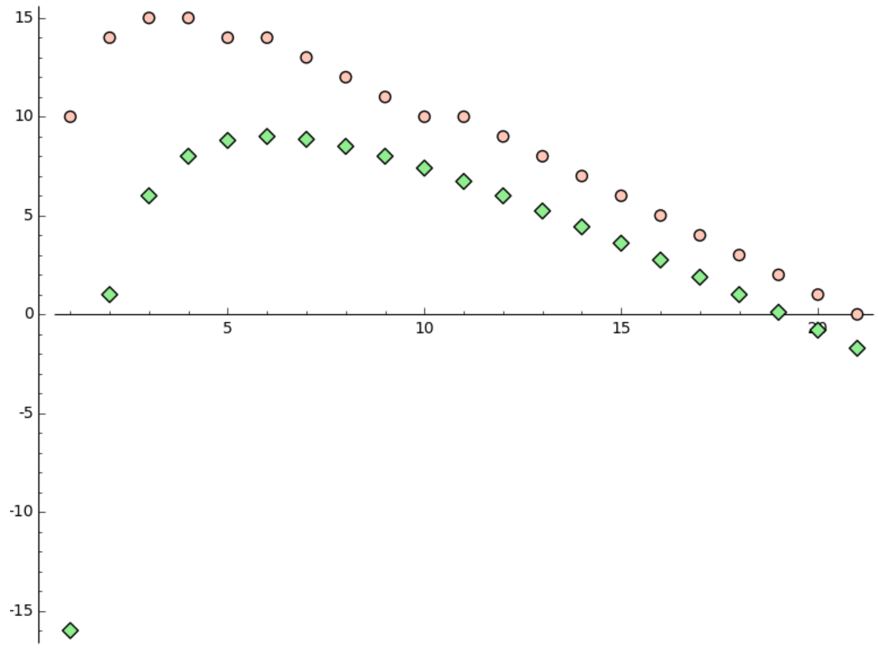

Figure 2: Vertex isoperimetric function of the Kneser graph $KG_{7,2}$ (red circles) as a function of r plotted against the lower bound from Theorem 4.6 (green diamonds). Observe that by Corollary 4.4, for $r \geq \binom{7}{2} - \binom{5}{2} = 11$, the isoperimetric function $\mu(r) = 21 - r$ is linear.

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