A Correlated Approach to the Hamburger-Cheeseburger Theorem

UROP+ Final Paler

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Abstract

This expository paper focuses on discrete and continuous models of random geometry. More specifically, we direct our attention to random planar maps in bijection with other discrete objects whose scaling limits have been well developed.

1 Introduction

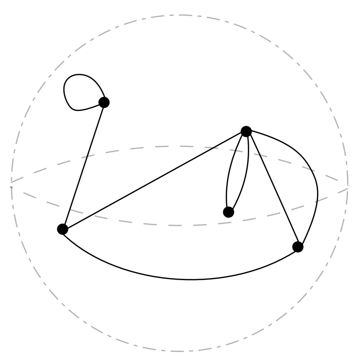
A planar map is a connected planar graph embedded in the sphere considered up to deformation. We consider these planar maps as equivalence classes of embedded graphs, where two graphs are equivalent if there exists an orientation preserving homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ that sends one graph to the other.

There has been great progress, in recent years, in the study of probabilistic aspects of large planar maps; many very different approaches have been used to understand the asymptotic behavior of random planar maps. In fact, this study has great motivation in statistical physics, as physicists believe that as these planar maps grow in size, the corresponding loop decorated surfaces converge in law to some universal random structure $\mathcal{M}$. One very famous approach has been to study the scaling limit of a sequence of random planar maps obtained by rescaling graph distances on the maps appropriately with their size and taking the limit as the size of the map tends to infinity. This approach requires that we view these maps as elements of the set of all compact metric spaces $\urcorner$ (up to isometry), equipped with the usual Gromov-Hausdorff metric d_{GH}. Le Gall showed that the scaling limit of uniform 2p-angulations (all faces of degree 2p) exists along a suitable subsequence and he furthermore showed that its topology is independent of the subsequence and proved that its Hausdorff dimension equals 4. In subsequent work, Miermont was able to show that uniform quadrangulations converge to the *Brownian map*- a limiting random metric space $(\mathbf{m}_\infty, D)$

Many of these discrete planar map can be tied together by the universality concept surrounding the Louiville Quantum Gravity (LQG), which provides a convenient way of describing random 2-Dimensional surfaces. The present study focuses on a LIFO (last-in-first-out) model; at each time step, a particular product can be ordered or produced. We observe that the inventory grows as a simple random walk on $\mathbb{Z}^2$. We present a bijection between the inventory projections and decorated planar maps due to Sheffield [She15], and hence we can talk about convergence of these planar maps in a particular topology.

We then discuss a small perturbation of the model presented in [She17]; we ask the question of convergence to Fractional Brownian Motion, implying that the inventory trajectories would this have correlated increments.

2 Mullin-Bernardi-Sheffield Bijection



Recall that a planar map is an embedding of a finite, connected graph (loops and multiple edges are allowed) in the plane $\mathbb{C} \cup \{\infty\}$(viewed as a Riemann sphere), considered up to deformation. A planar map determines faces, which are the connected components of the complementary of the union of edges. If we let $\mathbf{m}_n$ detnote a planar map with n edges, we denote by $E(\mathbf{m})$, $V(\mathbf{m})$ and

$F(\mathbf{m})$ the edge set, vertex set and set of faces of $\mathbf{m}_n$ respectively. Given a subset $\mathbf{t}_n$ of the edges of $\mathbf{m}_n$, we call the pair $(\mathbf{m}_n, \mathbf{t}_n)$ a ***decorated map***. We write $\mathbf{m}_n^\dagger$ for the corresponding *dual map* of $\mathbf{t}_n$

Now we write $\mathbf{t}_n^\dagger$ for the seubset of edges $\{e^\dagger : e \notin \mathbf{t}_n\}$. We fix an oriented edge of the map $\mathbf{m}_n$ and define it as the *root* edge of $\mathbf{m}$; from this point onward we shall assume that the maps in question are rooted. Let $\mathcal{Q} = \mathcal{Q}(\mathbf{m})$ be the map with vertex set $V \cup F$, and whose edge set is such that each $f \in F$ is connected to all its boundary vertices. We think of these new edges as *refinement edges*. Now suppose $\mathbf{t}_n$ is a subset of $E(\mathbf{m})$. In particular $\mathcal{Q}$ is bipartite indexed by distinct bipartitions V and F.

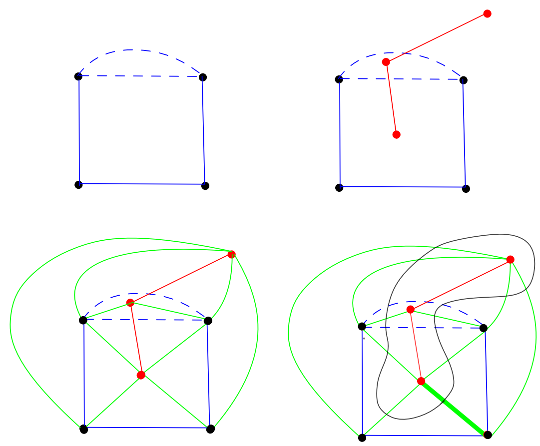


Figure 2. *Top left: Planar map with spanning tree highlighted by bold lines. Top right: Spanning tree with dual tree, which necessarily spans the vertices of the dual map. Bottom left: The map $\mathcal{Q}$. Bottom right: Loop (space filling path) separating the primal and dual spanning trees, to which a root is highlighted in bold.*

corresponding to a spanning tree of $\mathbf{m}_n$, and let $\mathbf{t}_n^\dagger$ be the corresponding dual edges of $\mathbf{t}_n$. The reader can easily check that $\mathbf{t}_n^\dagger$ is necessarily a spanning tree of $\mathbf{m}_n^\dagger$. Observe that the refinement edges split the map into triangles of two types: primal (meaning two refinement edges and one primal edge) and dual triangles (two refinement edges and a dual edge). Thus, we can think of the map as being split into quadrangles where one diagonal is either primal or dual.

We aim to 'reveal' the map, triangle by triangle, by exploring it with a space filling path that visits every triangle exactly once. This path goes through every triangle without crossing an edge of $\mathbf{t}$ or $\mathbf{t}^\dagger$. We will use the letters $\mathbf{h}$, $\mathbf{c}$ to indicate that a hamburger or cheeseburger has been produced resp. and the letters $\mathbf{H}$, $\mathbf{C}$ to indicate that a burger has been eaten (i.e. ordered and eaten immediately).

Now let $e_0, e_1, ..., e_{2n} = e_0$ be the sequence of edges of the triangulation hit by the path shown in Figure 2. For each e_i, let $d(e_i) := (d_1, d_2)$, where d_1 counts the number of edges in the tree $\mathbf{t}_n$ between the V endpoint of e_i and the root (e_0), and d_2 counts the number of edges in $\mathbf{t}_n^\dagger$ between the F endpoint of e_i and the root. The sequence $d(e_0), d(e_1), ..., d(e_{2n}) = d(e_0)$ defines a simple random walk on $\mathbb{Z}_+^2$. We obtain the corresponding word in the alphabet $\{\mathbf{c}, \mathbf{h}, \mathbf{C}, \mathbf{H}\}$ by writing $\mathbf{h}$ or $\mathbf{c}$ each time the first (resp. second) coordinate of $d(e_i)$ goes up, and $\mathbf{H}$ or $\mathbf{C}$ each time the first (resp. second) coordinate of $d(e_i)$ goes down.

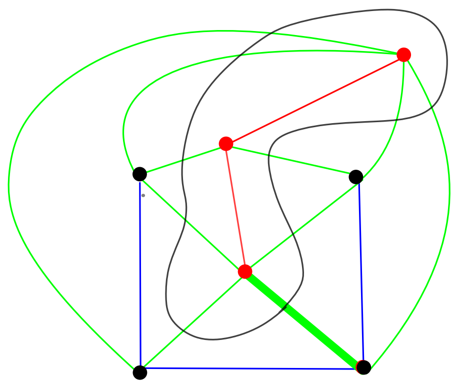


Figure 3. *The word associated with* $(\boldsymbol{m}_n, \boldsymbol{t}_n)$*:*
hccHhhCCHH

We introduce the quantities $(X_k, Y_k)_{1\leq k\leq 2n}$, which count the number of hamburgers and cheeseburgers in the stack at step k. It is easy to see that X and Y both stay non-negative throughout the exploration.Now $(X_k, Y_k)_{1\leq k\leq 2n}$ uniquely identify the word w encoding the decorated map $(\mathbf{m}_n, \mathbf{t}_n)$, and conversely, given such a process (X, Y), we can associate a unique word in Θ such that (X, Y) satisfies the above definition. Observe that X and Y can be interpreted as the corresponding contour functions of $\mathbf{t}_n$ and $\mathbf{t}_n^\dagger$ respectively, so that as $n \to \infty$,

$$\frac{1}{\sqrt{n}}(X_{\lfloor 2nt\rfloor}, Y_{\lfloor 2nt\rfloor})_{0\leq t\leq 1} \to (e_t, e_t^{'})_{0\leq t\leq 1}$$

where e, $e^{'}$ are independent, one-dimensional Brownian excursions. The following result thus summarizes our findings in the case of spanning trees.

Theorem 2.1[1][4]. *The set of spanning-tree decorated planar maps* $(\boldsymbol{m}_n, \boldsymbol{t}_n)$ *are in bijection with the contour functions* $(X_k, Y_k)_{1\leq k\leq 2n}$. *The pair of trees* $\boldsymbol{t}_n, \boldsymbol{t}_n^\dagger$ *converges to a pair of Continuum Random Trees, when scaled by* $\sqrt{n}$.

What if we remove the condition that $\mathbf{t}$ be a spanning tree? We now suppose that the collection of edges $\mathbf{t}_n$ is arbitrary. In this instance, we obtain a collection of loops through which our space-filling curve must traverse (touching only edges of $\mathcal{Q}$, none of $\mathbf{t}\cup\mathbf{t}^\dagger$. The general concept is that the space-filling path starts to explore the loop of the root edge, L_0. As the edges of $\mathbf{t}\cup\mathbf{t}^\dagger$ cannot be crossed, we must find a way to recursively *collapse* each loop so that the traversal may continue. To do this, we consider the last time L_0 is adjacent to some triangle in the complement of L_0 (i.e. those triangles that do not intersect L_0). Typically, this triangle is reached when we are about to close the loop L_0. The triangle sharing this edge with the boundary of L_0 is either an edge of $\mathbf{t}$ or of $t^\dagger$. Regardless, we now replace this edge with the opposite diagonal of the same quadrilateral: A dual edge is replaced with a primal edge and vice-versa. The effect of these flipped diagonals is to join one loop of the complement to the primary loop L_0. The reader can verify that when this procedure is iterated, we obtain a space-filling path that visits every quadrangle exactly twice (traversing some *virtual tree*). We thus associate to a decorated map $(\mathbf{m}_n, \mathbf{t}_n)$ a list of $2n$ symbols in the alphabet $\Theta = \{\mathbf{hcHCF}\}$, where each of the collapsing events is recorded by the symbol $\mathbf{F}$. [1] asserts that this list of symbols completely characterizes the decorated map $(\mathbf{m}_n, \mathbf{t}_n)$, such that each loop corresponds to a symbol $\mathbf{F}$.

We now describe how to reverse the bijection. We interpret the alphabet $\{\mathbf{hcHC}\}^{2n}$ as a LIFO model in hamburgers and cheeseburgers. The burgers are put into a single stack, and each time there is an order ($\mathbf{H}$ or $\mathbf{C}$, the freshest corresponding burger is removed from the stack ($\mathbf{h}$ or $\mathbf{c}$).

When the word w has no $\mathbf{F}$ symbols, the reversal is trivial so we proceed directly to the general case. The symbol $\mathbf{F}$ corresponds to a customer ordering the freshest available burger. So we can interpret the alphebet Θ a generators of a semi-group (as not every element has an inverse, per se), with the following relations:

- $\mathbf{cC} = \mathbf{cF} = \mathbf{hH} = \mathbf{hF} = \emptyset$
- $\mathbf{cH} = \mathbf{Hc}; \mathbf{hC} = \mathbf{Ch}$

Given a sequence of symbols X in Θ, we denote by $\bar{X}$ the reduced word formed by the above relations. Given such a sequence X with reduced word $\bar{X}$ satisfying $\bar{X} = \emptyset$, we make the following construction of $(\mathbf{m}_n, \mathbf{t}_n)$: Convert the $\mathbf{F}$ symbol to either $\mathbf{H}$ or $\mathbf{C}$ depending on its corresponding match (i.e. the burger to which it corresponds). Reversing the procedure in the spanning tree case, we construct a spanning tree decorated map (the condition $\bar{X} = \emptyset$ ensures that we can do this). Consider those quadrangles that have edges corresponding to the $\mathbf{F}$ symbols. We simply reassemble the loops by switching the *type* of the quadrangle: If a quadrangle corresponding to an $\mathbf{F}$has its triangles formed by primal edges, we simply replace that primal edge with its corresponding dual

edge, and vice versa. The map is now divided into several loops such that the number of loops is exactly $\#[\mathbf{F}]+1$.
Such a correspondence makes the study of limiting structures of large planar maps more feasible; a complicated concept such as 'number of loops' is reduced to a much simpler quantity to keep track of: number of **F** symbols in the given word.Theorem 2.1 describes a limiting law for a pair of trees, and the 'manner' in which they are glued. Of course, there are several other approaches to this universal convergence concept. Another model considers these discrete surfaces as random metric spaces. Miermont and Le Gall [2] so that these spaces converge in law to a limiting space called the *Brownian Map*; it was later shown by Sheffield and Miller that the Brownian Map is equivalently an LQG surface with $\gamma=\sqrt{\frac{8}{3}}$ (see appendix for more about LQG).

We aim to make a small perturbation to the hamburger-cheeseburger model; we introduce a sampling scheme for the addition of burgers to the end of the stack. This model exploits the idea that increments are now correlated, so we would expect our limiting distribution (of the corresponding random walk, say) to have the distribution of some *Fractional Brownian Motion* with Hurst parameter H.

3 Fractional Brownian Motion

Before we introduce formal definitions, we shall first paint an intuitive picture of the 'tweak' to the original model. Given μ- a law on the natural numbers- we associate the random walk with sequence of increments $\{X_k : k \in \mathbb{N}\}$ equal to -1 or 1. We attach independent samples k_n (taken to have a power law distance in time from the present) of μ to the vertices of $\mathbb{Z}$. The sequence of increments X_k is such that given the values $\{X_k : k < n\}$, X_n is set to be the increment X_{n-k_n}. In a similar manner, given a semi-infinite sequence of letters from our alphabet Θ, the symbol at step n is simply set to be the symbol that occurred k_n steps in the past, and so on. In this manner, we expect a corresponding planar map model whose structure is strongly dependent on its so-called 'history'.

3.1 Preliminaries

We can extend the idea of semi-random walks to that of correlated random walks, i.e., the walks in which the steps are not independent, but each step depends on all previous steps instead. Again, the limit of such a type of biased random walk introduces a generalization of Brownian motion–the so-called fractional Brownian motion- a one-parameter family of stochastic processes, mapping the real line to itself. They are the only stationary-increment Gaussian processes that, for some fixed $H > 0$, are invariant under space-time rescalings.

Motivated from some applications in hydrology, telecommunications, queueing theory and mathematical finance, there has been a recent interest in input noises without independent increments and possessing long-range dependence and self-similarity properties. Long-range dependence in a stationary time series occurs when the covariances tend to zero (like a power function) and so slowly that their sums diverge.

Definition 3.1: A Gaussian process is called a *fractional Brownian motion* of Hurst parameter $H \in (0,1)$ if it has mean zero and covariance function

$$\mathbb{E}(B_s^H B_t^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

or equivalently,

$$\mathbb{E}(|B_t^H - B_s^H|^2) = |t-s|^{2H}, S_0^H = 0$$

This process was introduced by Kolmogorov and studied by Mandelbrot and Van Ness, where a stochastic integral representation in terms of a standard Brownian motion was established. The parameter H is called Hurst index from the statistical analysis, developed by the climatologist Hurst. Fractional Brownian motion has the following properties:

- *Self-Similarity*: For any $a > 0$, $\{a^{-H} B_{at}, t \geq 0\}$ and $\{B_t^H, t \geq 0\}$ have the same probability distribution.

- *Stationary Increments*: The increments of the process in $[s,t] \sim N(0,|t-s|^{2H}$. It follows that for any integer $k \geq 1$

$$\mathbb{E}(B_t^H - B_s^H)^{2k}) = \frac{(2k)!}{k!2^k}|t-s|^{sHk}$$

- $\forall\, \epsilon > 0$ and $T > 0$, there exists a non-negative random variable $G_{\epsilon,T}$ such that $\mathbb{E}(|G_{\epsilon,T}|^p) < \infty$ for all $p \geq 1$ and almost surely

$$\left| B_t^H - B_s^H \right| \leq G_{\epsilon,T}|t-s|^{h-\epsilon}$$

In other words, the Hurst parameter H controls the *regularity* of the trajectories of the Fractional Brownian Motion.

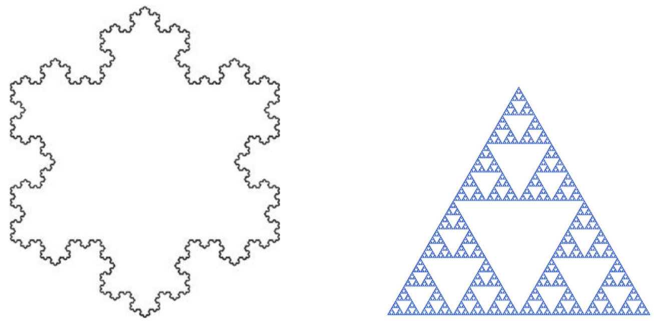


Figure 4: A visual representation of self- similarity as displayed by the Koch Curve and the Sierpinski Triangle. 'Zooming in' on any part of these objects displays an object that is similar to the 'bigger picture'.

In[3], we discover that the above walk does in fact converge to Fractional Brownian motion, in the sense of finite dimensional distributions. Essentially, it asserts that the above walk associated to an extremal μ-Gibbs measure (See [3]) for a choice of $\mu \in_\alpha$, the time-scaled process $S(nt)$, further rescaled by subtracting its mean a multiplying by some n-dependent factor, converges in law to Fractional Brownian motion with Hurst parameter $\alpha + \frac{1}{2}$ (we say that $\mu \in_\alpha$ if there exists a slowly varying function $L : (0, \infty) \to (0, \infty)$ for which $\mu\{n, ..., \infty\} = n^{-\alpha}L(n)$).

3.2 A similar Bijection

Given the correlated random walk outlined in the previous section, we can associate a random planar map quite naturally. We consider here the case of the spanning trees as outlined in section 2 (i.e. we anticipate words in Θ composed of $\{\mathbf{h}, \mathbf{c}, \mathbf{H}, \mathbf{C}\}$).

We consider the event that our space-filling path has already traversed infinitely many triangles in the semi-infinite canonical triangulation $\mathcal{Q}$. We consider this triangulation as a collection of *blocks* of trees; our focus here is the final n primal (and dual) edges of this triangulation, and we consider the *history* of the space-filling curve as groups of n primal edges (with corresponding dual edges), each block with its distinct root edge e_0. the d_i's are simply redefined as follows: if e_i is the i^{th} edge to be traversed by the loop, and $d = (d_1, d_2)$, we visit the block k_i units in the past, and we observe the *discrepancy* at that step in the past (that is, we record which coordinate of the corresponding (d_1, d_2) changed) and we attach this discrepancy to the edge e_i. A little bit of thought shows that this is essentially sampling the burger type (or order type) attached to the i^{th} edge of the block k_i units in the past.

It's difficult to say (from this picture) whether or not this modified picture converges to some non-degenerate random metric space or universal structure. Our work here (in the case of spanning trees) is closely related to some of the work done by Le Gall and Miermont in [2]. Their approach in the characterization of the *Brownian Map* considers a model without loops (as above) and to interpret the discrete surfaces as random metric spaces. They were able to show that these spaces converge in law to a limiting random metric space. The method utilizes the powerful Cori-Vauquelin-Shaeffer (CVS) Bijection, which exhibits a bijection between well-labelled trees and quadrangulations. It's thus far easier to understand the asymptotic of these large random quadrangulations, as they coincide with the study of $\mathbb{R}$-trees, which we will discuss later.

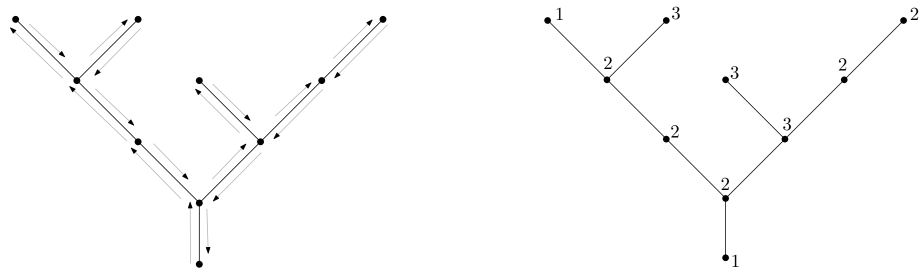


Figure 5: *On the left, we have a plane tree τ, and its contour illustration is shown by the arrows.The contour function provides some intuition behind the enumeration of plane trees: One can intuit that these trees are in bijection with non-negative lattice paths (Dyck Paths), which is another catalan object. On the right, we see a labelling of τ according to the following scheme: the root is labelled 1, and the absolute value of the difference between the labels of any two adjacent vertices is at most 1.*

Another reasonable model for our 'tweak' is this metric space model; considering $\mathbb{R}$-trees encoded by Fractional Brownian Motion. In this manner, we can say more about what expect our limiting surface to be (in particular, whether or not it is degenerate).

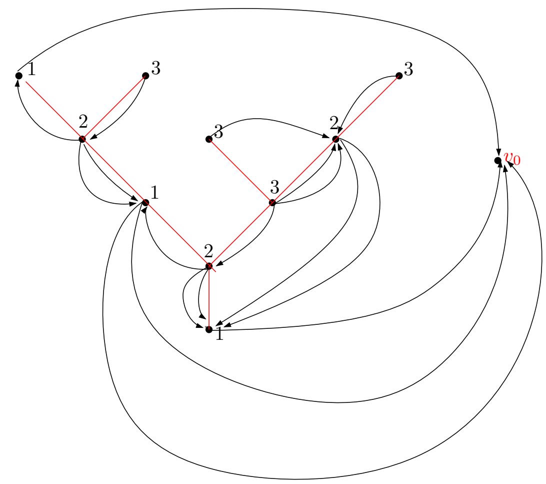


Figure 2: An example of the CVS bijection. The original tree is highlighted in red, and the quadrangulation is done according to the following algorithm: We add an isolated vertex v_0, and connect each corner labelled 1 to v_0. We then connect each corner labelled i to $i-1$. One obtains a quadrangulation with the labels indicating the respective graph distances from the vertices to the root [5]

3.3 ℝ-trees

Recall that a metric space $(\mathcal{T}, d)$ is an $\mathbb{R} - tree$ if the following hold for each $a, b \in \mathcal{T}$

- $\exists!$ isometric map $f_{a,b} : [0, d(a,b)] \to \mathcal{T}$ such that $f_{a,b}(0) = a$ and $f_{a,b}(d(a,b)) = b$

- For continuous, injective $g : [0,1] \to \mathcal{T}$ with $g(0) = a$ and $g(1) = b$, it holds that
$$g([0,1]) = f_{a,b}(0, d(a,b))$$

Now for every $s, t \in [0,1]$ we set

$$m_g(s,t) = \underset{r \in s \wedge t, s \vee t}{inf} g(r)$$

and

$$d_g(s,t) = g(s) + g(t) - 2m_g(s,t)$$

The reader can easily verify that d_g satisfies the triangle inequality (and is clearly variable symmetric), thus is a metric on $[0,1]$. We now introduce the equivalence relation $s \sim t$ iff $d_g(s,t) = 0$. We write the quotient space $\mathcal{T}_g := [0,1]/\sim$. Clearly d_g induces a distance on the space $\mathcal{T}_g$. We denote by $p_g := [0,1] \to \mathcal{T}_g$

the canonical projection onto $\mathcal{T}_g$. This coding induces a cyclic ordering on the tree $\mathcal{T}_g$ and we note the following:

Theorem 3.2[2]: The metric space $(\mathcal{T}_g, d_g)$ is an $\mathbb{R}$-tree with root $\rho = p_g(0) = p_g(1)$ We thus view this space as a rooted $\mathbb{R} - tree.$.

Definition 3.3. The CRT is the $\mathbb{R}$-tree $(\mathcal{T}_e, d_e)$ encoded by the Brownian excursion **e**

It is well known that the CRT has Hausdorff dimension 2, and is homeomorphic to $\mathbb{S}^2$. [5] asserts that the $\mathbb{R}$-tree encoded by the Fractional Brownian bridge has Hausdorff dimension $\frac{1}{H}$, and if we consider the hamburger-cheeseburger model as a 'mating of trees', we see that when the trees are 'glued', the quotient is a closed topological relation, and should be homeomorphic to $\mathbb{S}^2$. In particular, we believe that the limiting surface of such trees should be decorated by a variant of a space filling SLE (see appendix).

Theorem 3.4[5]: The hausdorff dimension d_{Haus} of the $\mathbb{R}$-tree encoded by the Fractional Brownian bridge has hausdorff dimension $\frac{1}{H}$ a.s.

Conjecture 3.5. The quotient, after gluing two $\mathbb{R}$-trees encoded by Fractional Brownian motion, is homeomorphic to $\mathbb{S}^2$.

4 Further Discussion

A lot can said about the universality concept introduced earlier. In particular, there have been significant developments in the study of these random surfaces.

4.1 FK Models

A (critical) Fortuin-Kasteleyn (FK) planar map of size $n \in \mathbb{N}$ and parameter $q > 0$ is a pair $(\mathbf{m}, S)$ consisting of a planar map $\mathbf{m}$ with n edges and a subset S of the set of edges of m, sampled with weight $q^{K(S)/2}$ where $K(S)$ is the number of connected components of S plus the number of complementary connected components of S. This model is critical in the sense that its partition function has power law decay as $n \to \infty$. If $(\mathbf{m}, S)$ is a critical FK planar map of size n and parameter q, then the conditional law of S given $\mathbf{m}$ is that of the uniform measure on edge sets of $\mathbf{m}$ weighted by $q^{K(S)/2}$. The critical FK planar map is conjectured to converge in the scaling limit to a conformal loop ensemble (CLE_κ) with $\kappa \in (4, 8)$ satisfying $q = 2 + 2cos(\frac{8pi}{\kappa})$ on top of an independent Liouville quantum gravity (LQG) surface with parameter $\gamma = \frac{4}{\sqrt{\kappa}}$ [1].

4.2 SLE_κ

The Schramm-Loewner evolution (SLE) is a measure on continuous curves that is a candidate for the scaling limit for discrete planar models in statistical physics. It is difficult to speak about SLE without introducing the discrete models that motivate it, as well as the concepts needed to make its definition clear. We think of them as the scaling limit of some Self Avoiding Walks (SAW) and informally, the index κ indicates 'winding activity' of said paths.[6]

5 Acknowledgements

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