COMPARING TWO BRAID GROUP ACTIONS ON DERIVED CATEGORIES OF FLAG VARIETIES AND PROJECTIVE SPACE

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Abstract. In this paper we seek to compare two actions of braid groups on two similar categories arising from flag varieties. One is an action of the annular braid group by *spherical twists*, from [1], on the derived category of coherent sheaves on $\omega_{\mathbb{P}^n}$ set-theoretically supported at zero. The other, from [2] is an action of the extended affine braid group on the the derived category of coherent sheaves on $\tilde{\mathfrak{g}}$ (the Grothendieck resolution) set-theoretically supported at zero. In low-dimensional cases, these two categories coincide and the actions are known to be related. In this paper, we investigate a possibility for generalizing this relationship to higher dimensions, but conclude that this direction will likely not succeed.

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1. Introduction

This paper investigates two braid group actions on derived categories of coherent sheaves on two different, but related, algebraic varieties.

The first action, which we will call the *action by spherical twists*, comes from [1], and was originally motivated by the study of stability conditions. Let Z be a Fano variety and ω_Z be the total space of its canonical bundle. Then this is an action of the annular braid group on the set of *spherical collections* in the derived category of coherent sheaves set-theoretically supported on the zero section of ω_Z. This action on the set of spherical collections induces an action of a subgroup of the annular braid group on the category itself. We will examine in particular the Fano varieties $Z = \mathbb{P}^1, \mathbb{P}^2$.

The other action, which we will call the *affine braid group action*, comes from [2]. Let G be a semisimple algebraic group and $\tilde{\mathfrak{g}}$ be its Grothendieck (or alternatively Springer) resolution. Then this is an action of the extended affine braid group of G on the derived category of coherent sheaves set-theoretically supported on the zero section of $\tilde{\mathfrak{g}}$. The action is defined via the pushforward-pullback adjunction for the projections $G/B \to G/P$ associated to each simple root.

The two actions are known to coincide in certain low-dimensional cases. As we will compute in this paper, when $G = SL_2$ and $Z = \mathbb{P}^1$, we have $G/B = Z$, and moreover the Springer resolution of G is exactly the canonical bundle of Z. Thus in this case the two actions are the same category, and in this case they coincide.

The main goal of this paper is to investigate the following possibility for generalizing this relationship to higher dimensions. Consider the projection $\pi : G/B \to G/P$ for a parabolic $P \supseteq B$. Whenever a variety X is naturally a vector bundle, we denote by $D^b(\operatorname{Coh}_0 X)$ the derived category of the category of coherent sheaves on X set-theoretically supported on the zero section. We have the following diagram, where ψ, s are the inclusions as the zero section.

$$\begin{tikzcd} \tilde{\mathfrak{g}} & \omega_{\mathbb{P}^2} \\ G/B \arrow[u, "\psi_G"] \arrow[r, "\pi"] & G/P \arrow[u, "s"'] \end{tikzcd}$$

This gives us a corresponding diagram of derived categories.

$$\begin{tikzcd} D^b(\mathrm{Coh}_0\,\tilde{\mathfrak{g}}) & D^b(\mathrm{Coh}_0\,\omega_{\mathbb{P}^2}) \\ D^b(\mathrm{Coh}\,G/B) \arrow[u, "\psi_{G*}"] \arrow[r, "\pi_*"] & D^b(\mathrm{Coh}\,G/P) \arrow[u, "s_*"'] \end{tikzcd}$$

We will ask whether the actions on $D^b_0(\tilde{\mathfrak{g}})$ and $D^b_0(\omega_{\mathbb{P}^2})$ are related by way of this pushforward π_*.

We investigate this possibility by noting that we have equalities of Grothendieck groups

$$K(D^b(\mathrm{Coh}\,G/B)) = K(D^b(\mathrm{Coh}_0\,\tilde{\mathfrak{g}}))$$

and

$$K(D^b(\mathrm{Coh}\,G/P)) = K(D^b(\mathrm{Coh}_0\,\omega_{\mathbb{P}^2})),$$

so that $\pi_* : D^b(\mathrm{Coh}\,G/B) \to D^b(\mathrm{Coh}\,G/P)$ also gives a homomorphism of Grothendieck groups

$$\pi_* : K(D^b(\mathrm{Coh}_0\,\tilde{\mathfrak{g}})) \to K(D^b(\mathrm{Coh}_0\,\omega_{\mathbb{P}^2})).$$

Since the affine braid group action on $D^b(\mathrm{Coh}_0\,\tilde{\mathfrak{g}})$ gives an action on the Grothendieck group $K(D^b(\mathrm{Coh}_0\,\tilde{\mathfrak{g}}))$, and similarly the action by spherical twists gives an action on the Grothendieck group $K(D^b(\mathrm{Coh}_0\,\omega_{\mathbb{P}^2}))$, we can ask whether there is any shared subgroup of the braid groups involved such that this map π_* is equivariant with respect to the action of this shared subgroup.

In this paper we provide evidence suggesting a negative answer, by computing the actions on both Grothendieck groups, and giving empirical evidence, by exhaustive search, that the only possible subgroup making this map equivariant is the group $\mathbb{Z}$ generated by the Serre twist on G/P.

1.1. **Contents.** The paper will proceed in roughly four parts.

In Sections 2 and 3, we briefly cover background on the braid groups involved and the constructions of their actions on the categories.

In Sections 4 and 5, we prove that, when $G = SL_2$ on the affine braid group side, and $Z = \mathbb{P}^1$ on the spherical twist side (and the varieties are thus exactly the same), the two actions correspond in a precise way.

In Sections 6, 7, and 8, we use the results from Section 3 to derive the actions on the Grothendieck groups in the case where $G = SL_3$ and $Z = \mathbb{P}^2$.

Finally, in Section 9, we give results from an exhaustive search for subgroups of the braid groups that correspond, and conclude that there are likely no interesting ones.

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2. Preliminaries: the action by spherical twists

2.1. **The annular braid group.** The group that acts by spherical twists is a quotient of the *annular braid group*. The annular braid group can be imagined topologically as the group of n-stranded braids on a cylinder, or as the configuration space of n points in the punctured plane. It has the group presentation

$$C_n = \langle r, \sigma_0, \sigma_1, \dots \sigma_n \rangle$$

subject to the relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } i - j \neq 1, -1 \pmod{n}$$
$$r\sigma_i = \sigma_{i+1} r.$$

It so happens that in the action by spherical twists, r^n acts as 0, so we may more tightly write that the action is by B_n where

$$B_n = C_n/(r^n).$$

2.2. **Spherical twists.** Let n be a natural number (representing dimension), and let C be any category enriched over k-vector spaces for an algebraically closed field k. We call an object $E \in C$ *spherical* if the cohomology of its Ext with itself is the homology of the sphere; that is,

$$\dim \operatorname{Ext}^i(E, E) = \begin{cases} 1 & i \in 0, n \\ 0 & \text{otherwise} \end{cases}.$$

Given any spherical object E, we can form the *spherical twist functor* $\Phi_E : D^b(C) \to D^b(C)$ by mapping F to the cone of the evaluation map:

$$\Phi_E : F \mapsto \operatorname{cone}(\operatorname{Hom}(E, F) \otimes E \to F).$$

It was established in [3] that Φ_E is always an exact autoequivalence.

The braid group action by spherical twists acts on the set of length n *spherical collections*, i.e. ordered tuples of spherical objects. The action is defined for each generator σ_i and r as follows.

$$\sigma_i(S_0, S_1, \dots S_n)$$
$$= (S_0, S_1, \dots S_i[-1], \Phi_{S_i}(S_{i-1}), S_{i+1} \dots S_n).$$

$$r(S_0, S_1, \dots S_n)$$
$$= (S_n, S_0, \dots S_{n-1}).$$

It is established in [1] that this action satisfies the braid relations, so as to be an action by B_n.

In this paper we will be concerned with the case where C is the category of coherent sheaves on $\omega_{\mathbb{P}^n}$ set-theoretically supported on the zero section, which we will denote $D^b(\operatorname{Coh}_0 \omega_{\mathbb{P}^n})$. This category is the subcategory of $D^b(\omega_{\mathbb{P}^n})$ generated by the pushforwards of objects from $D^b(\mathbb{P}^n)$ along the inclusion $s : \mathbb{P}^n \to \omega_{\mathbb{P}^n}$ as the zero section.

Noting that s_*, since it is pushforward along a closed embedding, is exact, and recalling that $\mathrm{Coh}\,\mathbb{P}^n$ is generated by line bundles $\mathcal{O}(j)$ for $0 \leq j \leq n$, we may conclude that $D^b(\mathrm{Coh}_0\,\omega_{\mathbb{P}^n})$ is generated by $i_*\mathcal{O}(j)$ for $0 \leq j \leq n$.

It is established in [1] that the following collection (or any twist of it) is spherical.

$$(s_*\mathcal{O}, s_*\mathcal{O}(1), \dots s_*\mathcal{O}(n))$$

Note that, by acting by the braid group element $(\sigma_i r)^{n-1}$, we map

$$(\sigma_i r)^n(s_*\mathcal{O}, s_*\mathcal{O}(1), \dots s_*\mathcal{O}(n))$$
$$= (\Phi_{s_*\mathcal{O}(i)} s_*\mathcal{O}, \Phi_{s_*\mathcal{O}(i)} s_*\mathcal{O}(1), \dots \Phi_{s_*\mathcal{O}(i)} i_*\mathcal{O}(n)).$$

Moreover the following is established in [3].

Lemma 1 (Seidel, Thomas). $\Phi_{\Phi_{s_*\mathcal{O}(i)}(\mathcal{O}(j))}\Phi_{s_*\mathcal{O}(i)} = \Phi_{s_*\mathcal{O}(i)}\Phi_{s_*\mathcal{O}(j)}$

We therefore get the following action on the category by autoequivalences.

Lemma 2. *Let H be the subgroup of B_n^{op} generated by the following generators.*

$$\langle (\sigma_0 r)^{n-1}, (\sigma_1 r)^{n-1}, \dots (\sigma_n r)^{n-1}\rangle^{op} \subseteq B_n^{op}$$

Then the map sending $(\sigma_i r)^{n-1} \mapsto \Phi_{s_*\mathcal{O}(i)}$ generates a homomorphism of groups $H \to \mathrm{Aut}(D_\omega)$.

Proof. It suffices to show that if any word $\prod_{l=0}^m (\sigma_{i_l} r)^{n-1} = 1$ then $\prod_{l=0}^m \Phi_{s_*\mathcal{O}(i_{m-l})} = 1$. But if we have the former, then each $\mathcal{O}(j)$ in the spherical collection, that $(\prod_{l=0}^m (\sigma_{i_l} r)^{n-1})(\mathcal{O}(j)) = \mathcal{O}(j)$, and hence by Lemma 1 we will have $(\prod_{l=0}^m \Phi_{s_*\mathcal{O}(i_{m-l})})(\mathcal{O}(j)) = \mathcal{O}(j)$, and since the $\mathcal{O}(j)$ generate the category this means $\prod_{l=0}^m \Phi_{s_*\mathcal{O}(i_{m-l})} = 1$ as well. $\square$

3. Preliminaries: the action by the affine braid group

3.1. **The affine braid group.** The second action, from [2] is by *extended affine braid group* of an algebraic group. This group is derived from the algebraic group's Weyl group and its character lattice. Let G be a semisimple algebraic group, B be a Borel in it, and T be a maximal torus inside that Borel.

Recall the Weyl group has a presentation generated by the *simple reflections* r_α, one for each simple root α. The Weyl group can be represented as a group of linear transformations of the root lattice, where r_α acts by reflecting across a hyperplane orthogonal to the simple root α.

The *affine Weyl group* is formed by adjoining to the Weyl group the group of translations by roots; it is also a subgroup of the group of affine transformations of the root lattice. The *extended affine Weyl group* is formed by adjoining to the affine Weyl group the group of translations by any weights; this is a subgroup of the group of affine transformations of the weight lattice.

The *extended affine braid group*, then, is defined by taking a standard presentation for the extended affine Weyl group and removing the relations that say the reflections have square zero. More precisely, the extended affine braid group is generated by generators $\{s_\alpha\}$ for each root α, together with generators θ_λ for each weight λ, subject to the following relations.

$$(s_\alpha s_\beta)^n = (s_\beta s_\alpha)^n \text{ where } n \text{ is the order of } r_\alpha r_\beta \text{ in the Weyl group}$$

$$\theta_x\theta_y = \theta_{x+y}$$
$$s_\alpha\theta_x = \theta_x s_\alpha \quad \text{when } r_\alpha(x) = x$$
$$\theta_x = s_\alpha\theta_{x-\alpha}s_\alpha \quad \text{when } r_\alpha(x) = x - \alpha$$

The *affine braid group* is defined analogously, where the θ_λ are only included when λ is a root.

In the case of SL_2, the extended affine braid group is generated by a single simple reflection s_1, together with translation t by the single generating weight. The affine braid group is the subgroup of the extended affine braid group generated by s_1 and t^2; this can alternatively be seen as the subgroup generated by s_1 and $s_0 = ts_1t^{-1}$, since we can recover $t^2 = s_0s_1 = ts_1t^{-1}s_1 = t^2$ by the last relation above.

In the case of SL_3, the extended affine braid group is generated by two simple reflections s_1, s_2 together with the translation ω_1.

3.2. **The Grothendieck and Springer resolutions.** The action of the affine braid group acts on the derived category of the *Grothendieck simultaneous resolution* $\tilde{\mathfrak{g}}$, defined as the variety of pairs of a Borel together with an element of its Lie algebra

$$\{(gB, b) \mid b \in \operatorname{Lie} gBg^{-1}\} \subseteq G/B \times \operatorname{Lie} G.$$

There is also a Grothendieck resolution of *partial* flag varieties, i.e. G/P for a parabolic subgroup P. They are, analogously, defined as

$$\{(gP, p) \mid p \in \operatorname{Lie} gPg^{-1}\} \subseteq G/P \times \operatorname{Lie} G.$$

Both of these varieties are still a vector bundle over G/B; indeed, we have the following lemma.

Lemma 3. *We have the isomorphism of varieties $\tilde{\mathfrak{g}} \simeq (G \times \operatorname{Lie} B)/B$, where B acts on G by right inverse multiplication, and on $\operatorname{Lie} B$ by the adjoint action. Similarly $\tilde{\mathfrak{g}}^S \simeq (G \times \operatorname{Lie} P)/P$.*

Proof. We construct an explicit isomorphism, mapping

$$\tilde{\mathfrak{g}} \to (G \times \operatorname{Lie} B)/B$$
$$(gB, b) \mapsto (g, g^{-1}bg).$$

This map is well-defined, since any equivalent (gb_0B, b) on the left will be mapped to $(gb_0, b_0^{-1}g^{-1}bgb_0)$, which is equal to $(g, g^{-1}bg)$ under the quotient.

It is an isomorphism, since it has an inverse mapping

$$(g, b) \mapsto (gB, gbg^{-1}).$$

This inverse is also well-defined, since any equivalent $(gb_0^{-1}, b_0bb_0^{-1})$ will map to $(gb_0^{-1}B, gb_0^{-1}b_0gb_0^{-1}g) = (gB, gbg^{-1})$.

The explicit isomorphism between the partial Grothendieck resolutions is analogous. $\square$

The *Springer* resolution is the subvariety of the Grothendieck resolution consisting of pairs (g, b) where $b \in \operatorname{Lie} gBg^{-1}$ is also in the nilpotent radical of $\operatorname{Lie} gBg^{-1}$. The Springer resolution will be important to us because the Springer resolution of G is the cotangent space of the flag variety T^*G/B. In computations, however, we will generally simply do computations on the Grothendieck resolution and treat the Springer resolution as a subvariety.

3.3. **The affine braid group action.** The action by the extended affine braid group is defined on the derived category of the category of coherent sheaves on $\tilde{\mathfrak{g}}$ which are set-theoretically supported at the zero section (this is the subcategory of $D^b(\operatorname{Coh}\tilde{\mathfrak{g}})$ that is generated by the pushforwards of sheaves on G/B along the zero section). We define the action by giving the action of each generator of the extended affine braid group. The extended affine braid group is generated by simple reflections, and by translation by weight lattice elements.

Translation by a weight lattice element λ is defined to act by the twist functor $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{O}(\lambda)$.

For each root α, there is an associated subgroup U_α in G such that the action by $t \in T$ on U_α by conjugation is exactly multiplication by $\alpha(t)$. Then there is a parabolic $P_\alpha \supseteq B$ associated to s_α, generated by B together with U_α.

Let $\tilde{\mathfrak{g}}$ be the Grothendieck resolution of G/B and $\tilde{\mathfrak{g}}^S$ be the Grothendieck resolution of the partial flag variety G/P_α. Then there is a natural map:

$$\pi : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^S$$

formed by sending each pair (gB, b) to the pair (gP, b), noting that $b \in \operatorname{Lie} gBg^{-1} \subseteq \operatorname{Lie} gPg^{-1}$. The action of the extended affine braid group is then defined by taking letting each s_α^{-1} act by sending a sheaf $\mathcal{F}$ to $\operatorname{cone}(\pi^*\pi_*\mathcal{F} \to \mathcal{F})[-1]$ where the map $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is the unit map.

This action can also apply to the derived categories of sheaves supported at zero on the Springer resolutions. We will generally not bother to explicitly compute this because it is so similar to the action on the Grothendieck resolution. It is proven in [2] that the affine braid group can also act on the derived category of coherent sheaves set-theoretically supported on the zero section of the Springer resolution S, in such a way that the following diagram commutes for any braid group element g.

$$\begin{tikzcd} D_0^b(S) \arrow[r, "i_*"] \arrow[d, "g\cdot -"] & D_0^b(\tilde{\mathfrak{g}}) \arrow[d, "g\cdot -"] \\ D_0^b(S) \arrow[r, "i_*"] & D_0^b(\tilde{\mathfrak{g}}) \end{tikzcd}$$

In computations of this paper, every object we consider in $D^b(\operatorname{Coh}_0\tilde{\mathfrak{g}})$ will end up being the pushforward of a sheaf on G/B along the inclusion as the zero section $G/B \to \tilde{\mathfrak{g}}$. Since this inclusion as the zero section factors through the inclusion $S \to \tilde{\mathfrak{g}}$, one can deduce the action on $D^b(\operatorname{Coh}_0 S)$ easily by interpreting each of these pushforwards as being into S instead of into $\tilde{\mathfrak{g}}$; this will give the action on $D^b(\operatorname{Coh}_0 S)$.

4. Spherical twists in the SL_2 case

To motivate the idea that the two actions are related, and also to develop the tools required to deal with the SL_3 case, we first show that the two braid group actions in the SL_2 case coincide in cohomology. Here, the two braid group actions are in fact on the derived category of the same variety, namely $\omega_{\mathbb{P}^1} = T^*SL_2/B$. In this section and the following section, we will prove the following theorem.

Theorem 1. *Let $s : \mathbb{P}^1 \to \omega_{\mathbb{P}^1}$ be the inclusion as the zero section. Let $S_{SL_2} = \omega_{\mathbb{P}^1}$ be the Springer resolution of SL_2. Let s_α be the unique simple reflection generating* *the Weyl group of SL_2, and t be the translation by its generating positive weight. Then we have the following equalities for any sheaf $\mathcal{F}$.*

$$H^\bullet(\Phi_{s_*\mathcal{O}(-1)}(\mathcal{F})) = H^\bullet(s_\alpha(\mathcal{F}))$$
$$H^\bullet(\Phi_{s_*\mathcal{O}}(\mathcal{F})) = H^\bullet(ts_\alpha t^{-1}(\mathcal{F}))$$

We will compute the actions on the category by evaluating them on two generating objects of the category, in this case $s_*\mathcal{O}(-1)$ and $s_*\mathcal{O}$. We will only compute our results up to their cohomology.

To compute the spherical twist action we must compute the spherical twist functors $\Phi_{s_*\mathcal{O}(-1)}$ and $\Phi_{s_*\mathcal{O}}$. As mentioned before, it will suffice to compute these functors on the generating objects $s_*\mathcal{O}(-1)$ and $s_*\mathcal{O}$; this is to say we need to compute four values: $\Phi_{s_*\mathcal{O}(-1)}(\mathcal{O}(-1))$,$\Phi_{s_*\mathcal{O}(-1)}(\mathcal{O})$, $\Phi_{s_*\mathcal{O}}(\mathcal{O}(-1))$, and $\Phi_{s_*\mathcal{O}}(\mathcal{O})$.

4.1. **Computing $\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O}(-1))$ and $\Phi_{s_*\mathcal{O}}(s_*\mathcal{O})$.** First, we compute $\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O}(-1))$. Recall this is the cone of the map:

$$R\operatorname{Hom}(s_*\mathcal{O}(-1), s_*\mathcal{O}(-1)) \otimes s_*\mathcal{O}(-1) \to s_*\mathcal{O}(-1)$$

Recalling that $s_*\mathcal{O}(-1)$ is a spherical object, we have $R\operatorname{Hom}^i(s_*\mathcal{O}(-1), s_*\mathcal{O}(-1)) = \begin{cases} k & i \in \{0,2\} \\ 0 & \text{otherwise} \end{cases}$. This means we have the following long exact sequence in cohomology.

$$\begin{tikzcd}
s_*\mathcal{O}(-1) \arrow[r] & 0 \arrow[r] & H^2(\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O}(-1))) \\
0 \arrow[r] & 0 \arrow[r] & H^1(\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O}(-1))) \arrow[ull] \\
s_*\mathcal{O}(-1) \arrow[r] & s_*\mathcal{O}(-1) \arrow[r] & H^0(\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O}(-1))) \arrow[ull]
\end{tikzcd}$$

Here the $s_*\mathcal{O}(-1) \to s_*\mathcal{O}(-1)$ at the bottom is the evaluation map, i.e. the identity. Thus we get:

$$H^0(\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O}(-1))) = 0$$
$$H^1(\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O}(-1))) = s_*\mathcal{O}(-1)$$
$$H^2(\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O}(-1))) = 0.$$

Hence, on the level of cohomology, we have $\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O}(-1)) = s_*\mathcal{O}(-1)[-1]$. In a completely analogous fashion, we can obtain $\Phi_{s_*\mathcal{O}}(s_*\mathcal{O}) = s_*\mathcal{O}[-1]$.

4.2. **Computing $\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O})$.** The other two are slightly more difficult. We will compute $\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O})$ first. This is the cone of the map

$$R\operatorname{Hom}(s_*\mathcal{O}(-1), s_*\mathcal{O}) \otimes s_*\mathcal{O}(-1) \to s_*\mathcal{O}.$$

In order to compute this cone we must first compute what $R\operatorname{Hom}(s_*\mathcal{O}(-1), s_*\mathcal{O})$ is. To compute this, we use a resolution of $s_*\mathcal{O}(-1)$ (the sheaf supported at 0) by line bundles (supported on the whole variety). Let $\pi : \omega_{\mathbb{P}^1} \to \mathbb{P}^1$ be the natural projection. Then note there is an exact sequence

$$\pi^*\omega_{\mathbb{P}^1}^\vee \longrightarrow \pi^*\mathcal{O} \longrightarrow s_*\mathcal{O} \ .$$

Here by $\pi^*\omega_{\mathbb{P}^1}^\vee$ is included in $\pi^*\mathcal{O} = \mathcal{O}_{\omega_{\mathbb{P}^1}}$ as the ideal sheaf of functions vanishing on the zero section (locally, the total space of $\omega_{\mathbb{P}^1}$ looks like $\operatorname{Spec} k[\mu, \frac{x}{y}]$; locally $\pi^*\omega_{\mathbb{P}^1}^\vee$ is included as the ideal sheaf (μ)). Twisting, there is an exact triangle

$$\pi^*\omega_{\mathbb{P}^1}^\vee(1) \longrightarrow \pi^*\mathcal{O}(1) \longrightarrow s_*\mathcal{O}(1) \ .$$

We can now use the exactness of derived Hom to see there must be an exact triangle:

$$R\operatorname{Hom}(\pi^*\omega_{\mathbb{P}^1}^\vee(-1), s_*\mathcal{O}) \longrightarrow R\operatorname{Hom}(\pi^*\mathcal{O}(-1), s_*\mathcal{O}) \longrightarrow R\operatorname{Hom}(s_*\mathcal{O}(-1), s_*\mathcal{O}) \ .$$

By pushforward-pullback adjunction we may see this as an exact triangle

$$R\operatorname{Hom}(\omega_{\mathbb{P}^1}^\vee(-1), \pi_*s_*\mathcal{O}) \longrightarrow R\operatorname{Hom}(\mathcal{O}(-1), \pi_*s_*\mathcal{O}) \longrightarrow R\operatorname{Hom}(s_*\mathcal{O}(-1), s_*\mathcal{O}) \ .$$

Now, $\pi_*s_* = (\pi s)_*$ and πs is the identity morphism. Recalling also that $\omega_{\mathbb{P}^1} = \mathcal{O}(-2)$, so that $\omega_{\mathbb{P}^1}^\vee = \mathcal{O}(2)$, this gives us an exact triangle

$$R\operatorname{Hom}(\mathcal{O}(1), \mathcal{O}) \longrightarrow R\operatorname{Hom}(\mathcal{O}(-1), \mathcal{O}) \longrightarrow R\operatorname{Hom}(s_*\mathcal{O}(-1), s_*\mathcal{O}) \ .$$

By twisting, this is

$$R\operatorname{Hom}(\mathcal{O}, \mathcal{O}(-1)) \longrightarrow R\operatorname{Hom}(\mathcal{O}, \mathcal{O}(1)) \longrightarrow R\operatorname{Hom}(s_*\mathcal{O}(-1), s_*\mathcal{O})$$

.

Hence we have an exact triangle

$$R\Gamma(\mathcal{O}(-1)) \longrightarrow R\Gamma(\mathcal{O}(1)) \longrightarrow R\operatorname{Hom}(s_*\mathcal{O}(-1), s_*\mathcal{O})$$

and thus a long exact sequence in cohomology as follows.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \operatorname{Hom}^1(s_*\mathcal{O}(-1), s_*\mathcal{O}) \\
0 \arrow[r] & k^2 \arrow[r] & \operatorname{Hom}^0(s_*\mathcal{O}(-1), s_*\mathcal{O}) \arrow[uull]
\end{tikzcd}$$

Hence we get $R\operatorname{Hom}(s_*\mathcal{O}(-1), s_*\mathcal{O})$ has cohomology k^2 in degree 0 and zero elsewhere.

We are now equipped to compute $\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O})$, since it is the cone of the morphism

$$R\operatorname{Hom}(s_*\mathcal{O}(-1), s_*\mathcal{O}(-1)) \otimes s_*\mathcal{O}(-1) \to s_*\mathcal{O}(-1)$$

and hence has a long exact sequence in cohomology as follows.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & H^1(\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O})) \arrow[dll] \\
k^2 \otimes \mathcal{O}(-1) \arrow[r] & \mathcal{O} \arrow[r] & H^0(\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O})) \arrow[dll] \\
0 \arrow[r] & 0 \arrow[r] & H^{-1}(\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O}))
\end{tikzcd}$$

The map in degree zero here is the evaluation map, i.e. the twist of the inclusion as global sections $\mathcal{O}^{\oplus 2} \to \mathcal{O}(1)$. It thus has kernel $\mathcal{O}(-2)$. Thus we get that

$$H^{-1}(\Phi_{s_*\mathcal{O}(-1)}(s_*\mathcal{O})) = \mathcal{O}(-2)$$

and that this object has no cohomology elsewhere.

4.3. **Computing $\Phi_{s_*\mathcal{O}}(s_*\mathcal{O}(-1))$.** Finally, we can compute $\Phi_{s_*\mathcal{O}}(s_*\mathcal{O}(-1))$. This is the cone of the map

$$R\operatorname{Hom}(s_*\mathcal{O}, s_*\mathcal{O}(-1)) \otimes s_*\mathcal{O} \to s_*\mathcal{O}(-1).$$

Again, this means we must first compute $R\operatorname{Hom}(s_*\mathcal{O}, s_*\mathcal{O}(-1))$. Again, it is useful to use the exact sequence

$$\pi^*\omega_{\mathbb{P}^1}^{\vee} \longrightarrow \pi^*\mathcal{O} \longrightarrow s_*\mathcal{O}\ .$$

Following the same chain of reasoning as above, using pushforward-pullback adjunction and exactness of RHom, we obtain an exact triangle

$$R\operatorname{Hom}(\omega_{\mathbb{P}^1}^{\vee}, \mathcal{O}(-1)) \longrightarrow R\operatorname{Hom}(\mathcal{O}, \mathcal{O}(-1)) \longrightarrow R\operatorname{Hom}(s_*\mathcal{O}, s_*\mathcal{O}(-1))\ .$$

That is, we have an exact triangle

$$R\operatorname{Hom}(\mathcal{O}(2), \mathcal{O}(-1)) \longrightarrow R\operatorname{Hom}(\mathcal{O}, \mathcal{O}(-1)) \longrightarrow R\operatorname{Hom}(s_*\mathcal{O}, s_*\mathcal{O}(-1))$$

or equivalently

$$R\Gamma(\mathcal{O}(-3)) \longrightarrow R\Gamma(\mathcal{O}(-1)) \longrightarrow R\operatorname{Hom}(s_*\mathcal{O}, s_*\mathcal{O}(-1))\ .$$

Hence we get a long exact sequence in cohomology as follows.

$$\begin{tikzcd}
k^2 \arrow[r] & 0 \arrow[r] & \operatorname{Hom}^2(s_*\mathcal{O}, s_*\mathcal{O}(-1)) \\
0 \arrow[r] & 0 \arrow[r] & \operatorname{Hom}^1(s_*\mathcal{O}, s_*\mathcal{O}(-1)) \arrow[ull] \\
0 \arrow[r] & 0 \arrow[r] & \operatorname{Hom}^0(s_*\mathcal{O}, s_*\mathcal{O}(-1)) \arrow[ull]
\end{tikzcd}$$

This gives us that $R\operatorname{Hom}(s_*, s_*\mathcal{O}(-1))$ has cohomology k^2 in degree 1 and zero elsewhere. Thus when we can compute the cohomology of the desired cone

$$\operatorname{cone}(R\operatorname{Hom}(s_*\mathcal{O}, s_*\mathcal{O}(-1)) \otimes s_*\mathcal{O} \to s_*\mathcal{O}(-1))$$

by taking the following long exact sequence in cohomology.

$$\begin{tikzcd}
k^2 \otimes s_*\mathcal{O} \arrow[r] & 0 \arrow[r] & H^2(\Phi_{s_*\mathcal{O}}(s_*\mathcal{O}(-1))) \\
0 \arrow[r] & 0 \arrow[r] & H^1(\Phi_{s_*\mathcal{O}}(s_*\mathcal{O}(-1))) \arrow[ull] \\
0 \arrow[r] & s_*\mathcal{O}(-1) \arrow[r] & H^0(\Phi_{s_*\mathcal{O}}(s_*\mathcal{O}(-1))) \arrow[ull]
\end{tikzcd}$$

Thus we get

$$H^0(\Phi_{s_*\mathcal{O}}(s_*\mathcal{O}(-1))) = s_*\mathcal{O}(-1)$$
$$H^1(\Phi_{s_*\mathcal{O}}(s_*\mathcal{O}(-1))) = k^2 \otimes s_*\mathcal{O}$$

and that this object has vanishing cohomology elsewhere.

4.4. **Putting everything together for spherical twists.** Altogether, we get the following table for the cohomology of the resulting objects.

	$\Phi_{s_*\mathcal{O}(-1)} s_*\mathcal{O}(-1)$	$\Phi_{s_*\mathcal{O}(-1)} s_*\mathcal{O}$	$\Phi_{s_*\mathcal{O}} s_*\mathcal{O}(-1)$	$\Phi_{s_*\mathcal{O}} s_*\mathcal{O}$
H_{-1}	0	$s_*\mathcal{O}(-2)$	0	0
H_0	0	0	$s_*\mathcal{O}(-1)$	0
H_1	$s_*\mathcal{O}(-1)$	0	$k^2 \otimes s_*\mathcal{O}$	$s_*\mathcal{O}$

5. The affine braid group action in the SL_2 case

For the affine braid group action, we note that the affine braid group will be generated by a simple reflection s_α and its conjugation by the Serre twist, $ts_\alpha t^{-1}$. The simple reflection α has the entire group SL_2 as its associated parabolic group. Hence the action of s_α is induced by the morphism

$$\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$$

which sends $(gB, b) \mapsto b$. Then reflection by α will correspond to taking

$$\mathrm{cone}(\pi^*\pi_*X \to X)[-1].$$

This morphism is the unit of the pushforward-pullback adjunction $\pi^* \dashv \pi_*$. The main task here is to compute $\pi^*\pi_*X$.

We will explicitly compute π locally; to do so, we will first explicitly describe $\tilde{\mathfrak{g}}$.

5.1. **$\tilde{\mathfrak{g}}$ explicitly.** We can consider the variety $\tilde{\mathfrak{g}}$ as the set $(a : b, M)$ where $a : b$ is a point on the projective line and M is an element of $\mathfrak{sl}_2$ that fixes that line. We can cover $\tilde{\mathfrak{g}}$ by two affine opens, one where $a \neq 0$ and the other where $b \neq 0$.

If $a \neq 0$ then we can parameterize the possible M by two variables (x, y) by taking $\begin{bmatrix} x & y \\ 0 & -x \end{bmatrix}$, which fixes the line $(1, 0)$, and conjugating it so it fixes the line $(1, \frac{b}{a})$, as follows.

$$\begin{bmatrix} 1 & \\ \frac{b}{a} & 1 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & -x \end{bmatrix} \begin{bmatrix} 1 & \\ -\frac{b}{a} & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \\ \frac{b}{a} & 1 \end{bmatrix} \begin{bmatrix} x - \frac{b}{a}y & y \\ \frac{b}{a}x & -x \end{bmatrix}$$

$$= \begin{bmatrix} x - \frac{b}{a}y & y \\ 2\frac{b}{a}x - (\frac{b}{a})^2 y & \frac{b}{a}y - x \end{bmatrix}$$

The other affine open, symmetrically, allows us to parameterize M by two variables $\hat{x}, \hat{y}$ by setting M to be

$$\begin{bmatrix} \frac{a}{b} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} \hat{x} & \hat{y} \\ 0 & -\hat{x} \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & -\frac{a}{b} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{a}{b} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} \hat{y} & \hat{x} - \frac{a}{b}\hat{y} \\ -\hat{x} & \frac{a}{b}\hat{x} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{a}{b}\hat{y} - \hat{x} & 2\frac{a}{b}\hat{x} - (\frac{a}{b})^2\hat{y} \\ \hat{y} & \hat{x} - \frac{a}{b}\hat{y} \end{bmatrix}.$$

These should be glued together so that the point $(1 : \frac{b}{a}, M)$ is the same as the point $(\frac{a}{b} : 1, M)$. This gives us $\tilde{\mathfrak{g}}$ as two copies of $\mathbb{A}^3$ glued together along the following isomorphism.

$$\begin{tikzcd} k[\frac{b}{a}, x, y] \arrow[d] & k[\frac{a}{b}, \hat{x}, \hat{y}] \arrow[d] \\ k[\frac{b}{a}, x, y, \frac{a}{b}] \arrow[r, equal] & k[\frac{a}{b}, \hat{x}, \hat{y}, \frac{b}{a}] \end{tikzcd}$$

The maps here associate

$$y = M_{12} = 2\frac{a}{b}\hat{x} - (\frac{a}{b})^2\hat{y}$$

and

$$x = M_{11} + \frac{b}{a}M_{12} = (\frac{a}{b}\hat{y} - \hat{x}) + \frac{b}{a}(2\frac{a}{b}\hat{x} - (\frac{a}{b})^2\hat{y})$$
$$= \frac{a}{b}\hat{y} - \hat{x} + 2\hat{x} - \frac{a}{b}\hat{y} = \hat{x}.$$

Now, $\tilde{\mathfrak{g}}$ is a two-dimensional vector bundle over $\mathbb{P}^1$, so should split as a direct sum of line bundles. This gluing map allows us to see this split as follows.

$$x - \frac{1}{2}(\frac{b}{a})y \mapsto \hat{x} - \frac{1}{2}\frac{b}{a}(2\frac{a}{b}\hat{x} - (\frac{a}{b})^2 y)$$
$$= \hat{x} - \hat{x} + \frac{1}{2}\frac{a}{b}\hat{y} = \frac{1}{2}\frac{a}{b}\hat{y}$$
$$\frac{1}{2}y \mapsto \frac{a}{b}\hat{x} - \frac{1}{2}(\frac{a}{b})^2\hat{y} = \frac{a}{b}(\hat{x} - \frac{1}{2}(\frac{a}{b})\hat{y})$$

Using these bases, we get that, as a vector bundle, this is $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

5.2. **The map $\tilde{\mathfrak{g}} \to \mathfrak{g}$ explicitly.** The map π sends $(a : b, M) \in \tilde{\mathfrak{g}}$ to $M \in \mathbb{A}^3$. Locally (at, in this example, $a \neq 0$), this map looks like sends

$$\begin{bmatrix} x - \frac{b}{a}y & y \\ 2\frac{b}{a}x - (\frac{b}{a})^2 y & \frac{b}{a}y - x \end{bmatrix} \mapsto \begin{bmatrix} h & e \\ f & -h \end{bmatrix}.$$

Hence we see this map is, locally,

$$k[h, e, f] \to k[\frac{b}{a}, x, y]$$

$$h \mapsto x - \frac{b}{a}y$$
$$e \mapsto y$$
$$f \mapsto 2\frac{b}{a}x - (\frac{b}{a})^2 y.$$

On the other affine open $b \neq 0$ we get that the map is

$$\begin{bmatrix} \frac{a}{b}\hat{y} - \hat{x} & 2\frac{a}{b}\hat{x} - (\frac{a}{b})^2\hat{y} \\ \hat{y} & \hat{x} - \frac{a}{b}\hat{y} \end{bmatrix} \mapsto \begin{bmatrix} h & e \\ f & -h \end{bmatrix}$$

whence similarly

$$k[h,e,f] \to k[\frac{a}{b}, x, y]$$
$$h \mapsto \frac{a}{b}\hat{y} - \hat{x}$$
$$e \mapsto 2\frac{a}{b}\hat{x} - (\frac{a}{b})^2\hat{y}$$
$$f \mapsto \hat{y}.$$

5.3. **Computation of the functor $\pi_*\pi^*$.** We are mostly concerned with sheaves that are pushforwards of sheaves from $SL_2/B = \mathbb{P}^1$. We have the following commutative diagram.

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\pi} & \mathfrak{g} = \mathbb{A}^3 \\ {\scriptstyle\psi_G}\uparrow & & {\scriptstyle\psi_P}\uparrow \\ G/B & \xrightarrow{\pi'} & pt \end{array}$$

By commutativity we have $\pi_*\psi_G* = \psi_P^*\pi'_*$. Now π'_*, as a functor between categories of coherent sheaves, is just the global sections functor; so its derived functor is just the derived global sections. Then ψ_P is a closed embedding, hence exact. Hence to compute $\psi_P^*\pi'_*$ we can take derived global sections, and then take each element of the resulting chain and replace it with the corresponding skyscraper sheaf on $\mathbb{A}^3$ at 0.

Hence for any sheaf $\mathcal{F}$ that is the pushforward of a sheaf from G/B, $\pi_*(\mathcal{F})$ will have cohomology that is just some vector spaces supported at 0. Now, π^* is just the inverse image functor composed with tensor product. To take the derived functor of π^* we can note that $\mathbb{A}^3$ is affine, so we can use projective resolutions; in particular we can use the Koszul resolution of vector spaces that are supported at 0. These will be n direct-sum copies of the Koszul resolution of k (supported at 0), so our main task is to compute the Koszul resolution of k.

In the case of $\mathbb{A}^3$ the Koszul resolution looks like the following, where the rightmost $k[h,e,f]\langle 1\rangle$ is in degree 0 and degree decreases as you go left.

$$0 \to k[h,e,f]\langle\omega\rangle \to k[h,e,f]\langle\delta,\epsilon,\theta\rangle \to k[h,e,f]\langle\alpha,\beta,\gamma\rangle \to k[h,e,f]\langle 1\rangle \to 0$$

Here the the last map sends

$$\alpha \mapsto h1$$
$$\beta \mapsto e1$$
$$\gamma \mapsto f1.$$

The second-to-last map sends

$$\delta \mapsto f\beta - e\gamma$$
$$\epsilon \mapsto h\gamma - f\alpha$$
$$\theta \mapsto e\alpha - h\beta.$$

Finally, the first map sends

$$\omega \mapsto h\delta + e\epsilon + f\theta.$$

We can use this projective resolution to evaluate (up to cohomology) of the left derived functor of π^* on k. To do so, we pull back each of these modules (i.e. trivial vector bundles) by tensoring with $\mathcal{O}_{\tilde{\mathfrak{g}}}$, then compute the cohomology of the resulting chain.

We will compute this by computing it locally on $a \neq 0$ and then computing the gluing map. Locally on $a \neq 0$ we get the following chain.

$$0 \to k[\frac{b}{a}, x, y]\langle\omega\rangle \to k[\frac{b}{a}, x, y]\langle\delta, \epsilon, \theta\rangle \to k[\frac{b}{a}, x, y]\langle\alpha, \beta, \gamma\rangle \to k[\frac{b}{a}, x, y]\langle 1\rangle \to 0$$

Here the last map sends

$$\alpha \mapsto (x - \frac{b}{a}y)1$$
$$\beta \mapsto y1$$
$$\gamma \mapsto (2\frac{b}{a}x - (\frac{b}{a})^2 y)1.$$

The second-to-last map sends

$$\delta \mapsto (2\frac{b}{a}x - (\frac{b}{a})^2 y)\beta - y\gamma$$
$$\epsilon \mapsto (x - \frac{b}{a}y)\gamma - (2\frac{b}{a}x - (\frac{b}{a})^2 y)\alpha$$
$$\theta \mapsto y\alpha - (x - \frac{b}{a}y)\beta.$$

Finally, the last map sends

$$\omega \mapsto (x - \frac{b}{a}y)\delta + y\epsilon + (2\frac{b}{a}x - (\frac{b}{a})^2 y)\theta.$$

5.4. **The sheaf H^0.** We will first compute the cohomology sheaf H^0. Since this is the first element in the chain, its kernel is the whole sheaf, so we simply wish to take the quotient by the image from $k[\frac{b}{a}, x, y]\langle\alpha, \beta, \gamma\rangle$. We thus get, locally at $a \neq 0$,

$$H^0(\{a \neq 0\}) = k[\frac{b}{a}, x, y]/(y, x - \frac{b}{a}y, 2\frac{b}{a}x - (\frac{b}{a})^2 y).$$

Noting that these three elements $(y, x - \frac{b}{a}y, 2\frac{b}{a}x - (\frac{b^2}{a^2})y)$ generate x and y and are generated thereby, this is just the ideal (x, y) so we get

$$H_0(\{a \neq 0\}) = k[\frac{b}{a}, x, y]/(x, y).$$

Similarly, $H^0(\{b \neq 0\}) = k[\frac{a}{b}, x, y]/(x, y)$. The gluing map here sends the generators to each other, so this is the pushforward $\psi_{B*}\mathcal{O}_{\mathbb{P}^1}$.

5.5. **The module H^{-1} on $a \neq 0$.** We can now compute the cohomology module H^{-1}. First, we need to compute the kernel of this map into the $k[\frac{b}{a}, x, y]\langle 1\rangle$.

The kernel includes, at least, the following element

$$2\frac{b}{a}(\alpha + \frac{b}{a}\beta) - (\frac{b}{a})^2\beta - \gamma$$
$$= 2\frac{b}{a}\alpha + (\frac{b}{a})^2\beta - \gamma$$

since its image is $2\frac{b}{a}(x - \frac{b}{a}y + \frac{b}{a}y) - (\frac{b}{a})^2 y - (2\frac{b}{a}x - (\frac{b}{a})^2 y) = 0$. Hence, modulo the kernel, γ is generated by α and β. This means that we can factorize the map $k[\frac{b}{a}, x, y]\langle \alpha, \beta, \gamma\rangle \to k[\frac{b}{a}, x, y]\langle 1\rangle$ as follows.

$$k[\frac{b}{a}, x, y]\langle \alpha, \beta, \gamma\rangle \to k[\frac{b}{a}, x, y]\langle \alpha, \beta, \gamma\rangle/(2\frac{b}{a}\alpha + (\frac{b}{a})^2\beta - \gamma) \simeq k[\frac{b}{a}, x, y]\langle \alpha, \beta\rangle \to k[\frac{b}{a}, x, y]\langle 1\rangle$$

To compute the kernel of this factorized morphism, one can take the kernel of the map $k[\frac{b}{a}, x, y]\langle \alpha, \beta\rangle \to k[\frac{b}{a}, x, y]\langle 1\rangle$ and take its preimage under the quotient by $(2\frac{b}{a}\alpha + (\frac{b}{a})^2\beta - \gamma)$. Now, the kernel of the map $\alpha \mapsto (x - \frac{b}{a}y)1$ and $\beta \mapsto y1$ is generated by $y\alpha - (x - \frac{b}{a}y)\beta$ since the ideals y and $(x - \frac{b}{a}y)$ are coprime, so their intersection is their product. Then the preimage under the quotient, and hence the kernel of the map $k[\frac{b}{a}, x, y]\langle \alpha, \beta, \gamma\rangle \to k[\frac{b}{a}, x, y]\langle 1\rangle$, will be generated by the two elements

$$\langle y\alpha - (x - \frac{b}{a}y)\beta, 2\frac{b}{a}\alpha + (\frac{b}{a})^2\beta - \gamma\rangle.$$

To compute the cohomology module H^{-1} we want to take the quotient of this module by the image from $k[\frac{b}{a}, x, y]\langle \delta, \epsilon, \theta\rangle$, which is

$$\langle (2\frac{b}{a}x - (\frac{b}{a})^2 y)\beta - y\gamma, (x - \frac{b}{a}y)\gamma - (2\frac{b}{a}x - (\frac{b}{a})^2 y)\alpha, y\alpha - (x - \frac{b}{a}y)\beta\rangle.$$

We note that one of the generators of the kernel, $y\alpha - (x - \frac{b}{a}y)\beta$, lies in this image. Thus this cohomology group will be generated by the equivalence class of $2\frac{b}{a}\alpha + (\frac{b}{a})^2\beta - \gamma$.

Denote for convenience $z = 2\frac{b}{a}\alpha + (\frac{b}{a})^2\beta - \gamma$. Note that xz and yz are both in the image. We can generate xz in the following fashion.

$$\epsilon - \frac{b}{a}\delta - (\frac{b}{a})^2\theta =$$
$$(x - \frac{b}{a}y)\gamma - (2\frac{b}{a}x - (\frac{b}{a})^2 y)\alpha - \frac{b}{a}((2\frac{b}{a}x - (\frac{b}{a})^2 y)\beta - y\gamma) - (\frac{b}{a})^2(y\alpha - (x - \frac{b}{a}y)\beta)$$
$$= x\gamma - (2\frac{b}{a}x - (\frac{b}{a})^2 y)\alpha - (2(\frac{b}{a})^2 x - (\frac{b}{a})^3 y)\beta - (\frac{b}{a})^2(y\alpha - (x - \frac{b}{a}y)\beta)$$
$$= x\gamma - 2\frac{b}{a}x\alpha + (\frac{b}{a})^2 y\alpha - 2(\frac{b}{a})^2 x\beta + (\frac{b}{a})^3 y\beta - (\frac{b}{a})^2 y\alpha + (\frac{b}{a})^2 x\beta - (\frac{b}{a})^3 y\beta)$$
$$= x\gamma - 2\frac{b}{a}x\alpha - (\frac{b}{a})^2 x\beta$$
$$= -xz$$

We can also generate yz in the following fashion.

$$\begin{aligned}
&\delta + 2\frac{b}{a}\theta \\
&= (2\frac{b}{a}x - (\frac{b}{a})^2 y)\beta - y\gamma + 2\frac{b}{a}(y\alpha - (x - \frac{b}{a}y)\beta) \\
&= 2\frac{b}{a}x\beta - (\frac{b}{a})^2 y\beta - y\gamma + 2\frac{b}{a}y\alpha - 2\frac{b}{a}(x - \frac{b}{a}y)\beta \\
&= 2\frac{b}{a}x\beta - (\frac{b}{a})^2 y\beta - y\gamma + 2\frac{b}{a}y\alpha - 2\frac{b}{a}x\beta + 2(\frac{b}{a})^2\beta \\
&= 2\frac{b}{a}y\alpha + 2(\frac{b}{a})^2\beta - y\gamma \\
&= yz
\end{aligned}$$

Hence the submodule $(x, y)z$ is in the image, and conversely everything in the image is in $(x, y)z$ since every term in every coefficient in the image includes a factor of either x or y. Hence we get the submodule $k[\frac{b}{a}, x, y]\langle 2\frac{b}{a}\alpha + (\frac{b}{a})^2\beta - \gamma\rangle/(x, y)$.

We thus see that we have some pushforward of some line bundle on $\mathbb{P}^1$. To see what the twist of that line bundle is, we can compute the module on the other affine open and the relevant gluing map.

5.6. **The module H^{-1} on $b \neq 0$.** On the other affine open, the relevant map whose kernel we are trying to compute is as follows.

$$\alpha \mapsto (\frac{a}{b}\hat{x} - \hat{y})1$$

$$\beta \mapsto (2\frac{a}{b}\hat{x} - (\frac{a}{b})^2\hat{y})1$$

$$\gamma \mapsto \hat{y}1$$

Meanwhile the map whose image we are trying to compute is as follows.

$$\delta \mapsto \hat{y}\beta - (2\frac{a}{b}\hat{x} - (\frac{a}{b})^2\hat{y})\gamma$$

$$\epsilon \mapsto (\frac{a}{b}\hat{x} - \hat{y})\gamma - \hat{y}\alpha$$

$$\theta \mapsto (2\frac{a}{b}\hat{x} - (\frac{a}{b})^2\hat{y})\alpha - (\frac{a}{b}\hat{x} - \hat{y})\beta$$

We observe that we can obtain the previous case by doing the replacements $\alpha \mapsto -\alpha, \beta \mapsto \gamma, \gamma \mapsto \beta, \frac{a}{b} \mapsto \frac{b}{a}, \delta \mapsto -\delta, \epsilon \mapsto -\theta, \theta \mapsto -\epsilon$. Hence the kernel should be

$$k[\frac{a}{b}, \hat{x}, \hat{y}]\langle -2\frac{a}{b}\alpha + (\frac{a}{b})^2\gamma - \beta\rangle/(\hat{x}, \hat{y}).$$

5.7. **The sheaf H^{-1}.** We can write these two modules as $k[\frac{b}{a}, x, y]\langle z_1\rangle/(x, y)$ and $k[\frac{a}{b}, x, y]\langle z_2\rangle/(x, y)$ where $z_2 = (\frac{a}{b})^2 z_1$. This gives this module as having a twist of 2. Thus the first cohomology module is $\pi_*\mathcal{O}(-2)$.

5.8. **The sheaf H^{-2}.** Locally, we want to determine the kernel of the following map.

$$\delta \mapsto (2\frac{b}{a}x - (\frac{b}{a})^2 y)\beta - y\gamma$$
$$\epsilon \mapsto (x - \frac{b}{a}y)\gamma - (2\frac{b}{a}x - (\frac{b}{a})^2 y)\alpha$$
$$\theta \mapsto y\alpha - (x - \frac{b}{a}y)\beta$$

This kernel will be the submodule of linear combinations

$$c_\delta \delta + c_\epsilon \epsilon + c_\theta \theta$$

that are sent to zero. In order for such a linear combination to be sent to zero, the coefficient of γ in the image must be zero. This means $c_\epsilon(x - \frac{b}{a}y) - c_\delta y = 0$. Again by coprimality of $(x - \frac{b}{a}y)$ and (y), we have that $c_\epsilon = yc_0$ and $c_\delta = (x - \frac{b}{a}y)c_0$ for some c_0.

Now noting that the coefficient of α in the image must also be zero, we get

$$yc_\theta - (2\frac{b}{a}x - (\frac{b}{a})^2 y)c_\epsilon = 0$$
$$yc_\theta = (2\frac{b}{a}x - (\frac{b}{a})^2 y)yc_0.$$

Now since y is not a zero divisor this means:

$$c_\theta = (2\frac{b}{a}x - (\frac{b}{a})^2 y)c_0.$$

But then this gives that everything in the kernel is also in the image of the map $\omega \mapsto (x - \frac{b}{a}y)\delta + y\epsilon + (2\frac{b}{a}x - (\frac{b}{a})^2 y)\theta$. Thus the second cohomology module is zero (since this calculation works symmetrically on the other affine open).

5.9. **The sheaf H^{-3}.** The third cohomology module will (locally) be the kernel of the map $\omega \mapsto (x - \frac{b}{a}y)\delta + y\epsilon + (2\frac{b}{a}x - (\frac{b}{a})^2 y)\theta$. But this kernel is zero, since y is not zero divisor and anything in the kernel would have to have an image with vanishing coefficient of ϵ.

5.10. **Putting everything together for the affine braid group.** Hence π^* applied to k gives a chain with cohomology $H^0 = \psi_{G*}\mathcal{O}_{\mathbb{P}^1}$, $H^{-1} = \psi_{G*}\mathcal{O}_{\mathbb{P}^1}(-2)$, and all other cohomologies zero.

We will give this functor by applying it to the pushforwards of the line bundles $\mathcal{O}_{\mathbb{P}^1}$, $\mathcal{O}_{\mathbb{P}^1}(-1)$. Recall that the derived global sections of $\mathcal{O}(i)$ is k^{i+1} as the zeroth cohomology and k^{-i-1} is the first cohomology. We can tensor these together to see that $\pi^*\pi_*\psi_{G*}\mathcal{O}(i)$ gives something with cohomology

$$H^1(\pi^*\pi_*\psi_{G*}\mathcal{O}(i)) = k^{-i-1} \otimes \psi_{G*}\mathcal{O}$$
$$H^0(\pi^*\pi_*\psi_{G*}\mathcal{O}(i)) = k^{-i-1} \otimes \mathcal{O}(-2) \oplus k^{i+1} \otimes \psi_{G*}\mathcal{O}$$
$$H^{-1}(\pi^*\pi_*\psi_{G*}\mathcal{O}(i)) = k^{i+1} \otimes \mathcal{O}(-2).$$

Recall that s_α takes $\mathrm{cone}(\pi^*\pi_*\psi_{G*}\mathcal{O}(i) \to \mathcal{O}(i))[-1]$. We thus get the following long exact sequence of cohomology.

$$
\begin{tikzcd}
H^1(s_\alpha(\psi_{G*}\mathcal{O}(i))) \arrow[r] & k^{-i-1} \otimes \psi_{G*}\mathcal{O} \arrow[r] & 0 \\
H^0(s_\alpha(\psi_{G*}\mathcal{O}(i))) \arrow[r] & k^{-i-1} \otimes \psi_{G*}\mathcal{O}(-2) \oplus k^{i+1} \otimes \psi_{G*}\mathcal{O} \arrow[r] & \psi_{G*}\mathcal{O}(i) \arrow[ull] \\
H^{-1}(s_\alpha(\psi_{G*}\mathcal{O}(i))) \arrow[r] & k^{i+1} \otimes \psi_{G*}\mathcal{O}(-2) \arrow[r] & 0 \arrow[ull]
\end{tikzcd}
$$

Applying this in particular to $i = 0$ yields the following long exact sequence.

$$
\begin{tikzcd}
H^1(s_\alpha(\psi_{G*}\mathcal{O})) \arrow[r] & 0 \arrow[r] & 0 \\
H^0(s_\alpha(\psi_{G*}\mathcal{O})) \arrow[r] & \psi_{G*}\mathcal{O} \arrow[r] & \psi_{G*}\mathcal{O} \arrow[ull] \\
H^{-1}(s_\alpha(\psi_{G*}\mathcal{O})) \arrow[r] & \psi_{G*}\mathcal{O}(-2) \arrow[r] & 0 \arrow[ull]
\end{tikzcd}
$$

Here the map in degree zero is the identity map. Hence we obtain a module that has cohomology $\psi_{G*}\mathcal{O}(-2)$ in degree -1 and zero elsewhere.

Applying this in particular to $i = -1$ gives the following long exact sequence in cohomology.

$$
\begin{tikzcd}
H^1(s_\alpha(\psi_{G*}\mathcal{O}(-1))) \arrow[r] & 0 \arrow[r] & 0 \\
H^0(s_\alpha(\psi_{G*}\mathcal{O}(-1))) \arrow[r] & 0 \arrow[r] & \psi_{G*}\mathcal{O}(-1) \arrow[ull] \\
H^{-1}(s_\alpha(\psi_{G*}\mathcal{O}(-1))) \arrow[r] & 0 \arrow[r] & 0 \arrow[ull]
\end{tikzcd}
$$

Hence we get cohomology of $\psi_{G*}\mathcal{O}(-1)$ in degree 1 and zero elsewhere.

Finally, to compute $ts_\alpha t^{-1}$, we would also like to apply this to $\psi_{G*}\mathcal{O}(-2)$. Here we get the following long exact sequence in cohomology.

$$
\begin{tikzcd}
H^1(s_\alpha(\psi_{G*}\mathcal{O}(-2)) \arrow[r] & \psi_{G*}\mathcal{O} \arrow[r] & 0 \\
H^0(s_\alpha(\psi_{G*}\mathcal{O}(-2)) \arrow[r] & \psi_{G*}\mathcal{O}(-2) \arrow[r] & \psi_{G*}\mathcal{O}(-2) \arrow[ull] \\
H^{-1}(s_\alpha(\psi_{G*}\mathcal{O}(-2))) \arrow[r] & 0 \arrow[r] & 0 \arrow[ull]
\end{tikzcd}
$$

There are multiple possibilities for what the unknown cohomology groups here could be. However, we also know that this functor is exact, hence, from the exact sequence $\psi_{G*}\mathcal{O}(-2) \to \psi_{G*}\mathcal{O}(-1)^{\oplus 2} \to \psi_{G*}\mathcal{O}$, we should have an exact triangle $s_\alpha(\psi_{G*}\mathcal{O}(-2)) \to s_\alpha(\psi_{G*}\mathcal{O}(-1)^{\oplus 2}) \to s_\alpha(\psi_{G*}\mathcal{O})$, and thus the following long exact sequence of cohomology.

$$\begin{tikzcd}
H^1(s_\alpha(\psi_{G*}\mathcal{O}(-2)) \arrow[r] & \psi_{G*}\mathcal{O}(-1)^{\oplus 2} \arrow[r] & 0 \\
H^0(s_\alpha(\psi_{G*}\mathcal{O}(-2)) \arrow[r] & 0 \arrow[r] & 0 \arrow[ull] \\
H^{-1}(s_\alpha(\psi_{G*}\mathcal{O}(-2)) \arrow[r] & 0 \arrow[r] & \psi_{G*}\mathcal{O}(-2) \arrow[ull]
\end{tikzcd}$$

From this we get $H^0 = \psi_{G*}\mathcal{O}(-2)$ and $H^1 = \psi_{G*}\mathcal{O}(-1)^{\oplus 2}$, which also fits in to the long exact sequence from the pullback-pushforward adjunction above.

We thus get the following table for the action of the affine braid group.

	$s_\alpha(\psi_{G*}\mathcal{O}(-1))$	$s_\alpha(\psi_{G*}\mathcal{O})$	$ts_\alpha t^{-1}(\psi_{G*}\mathcal{O}(-1))$	$ts_\alpha t^{-1}(\psi_{G*}\mathcal{O})$
H^{-1}	0	$\psi_{G*}\mathcal{O}(-2)$	0	0
H^0	0	0	$\psi_{G*}\mathcal{O}(-1)$	0
H^1	$\psi_{G*}\mathcal{O}(-1)$	0	$k^2 \otimes \psi_{G*}\mathcal{O}$	$\psi_{G*}\mathcal{O}$

5.11. **The correspondence in the SL_2 case.** We are now ready to conclude our proof of Theorem 1.

Proof of Theorem 1. Recall that the affine braid group action on the $D^b(\mathrm{Coh}_0\, S_{SL_2})$ commutes with the pushforward functor $D^b(\mathrm{Coh}_0\, S_{SL_2}) \to D^b(\mathrm{Coh}_0\, \tilde{\mathfrak{g}})$; hence the table of Section 5.10 also applies to the action on $D^b(\mathrm{Coh}_0\, S_{SL_2})$ by replacing ψ_G with the inclusion $\iota : \mathbb{P}^1 \to S_{SL_2}$.

Then the tables from Sections 4.4 and 5.10, recalling that these functors are exact and that $\mathcal{O}$, $\mathcal{O}(-1)$ generate the category, we can see that the two actions correspond in cohomology in the SL_2 case, with

$$H^\bullet(\Phi_{s_*\mathcal{O}(-1)}(\mathcal{F})) = H^\bullet(s_\alpha(\mathcal{F}))$$
$$H^\bullet(\Phi_{s_*\mathcal{O}}(\mathcal{F})) = H^\bullet(ts_\alpha t^{-1}(\mathcal{F}))$$

This proves the theorem. □

6. Spherical twists in the SL_3 case

We are now ready to proceed to the SL_3 case. The action by spherical twists in the SL_3 case can be computed in a manner analogous to the SL_2 case. Instead of computing this here, we refer the reader to [4], where this computation is done.

The result from [4] uses as the basis for its Grothendieck group the three bundles $[S_0, S_1, S_2]$ where $S_0 = s_*\mathcal{O}$, $S_1 = s_*\Omega^1(1)[1]$, and $S_2 = s_*\mathcal{O}(-1)$. Using this basis, the resulting actions on the Grothendieck group are as follows.

$$\Phi_{S_0} = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Phi_{S_1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Phi_{S_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -3 & 1 \end{bmatrix}$$

To compare this action to the affine braid group action action, we will choose instead the basis $[\mathcal{O}, \mathcal{O}(1), \mathcal{O}(-1)]$. Performing the basis change to this basis instead (using the exact sequence $0 \to \Omega^1 \to \mathcal{O}(-1)^{\oplus 3} \to \mathcal{O} \to 0$), we obtain the following.

$$\Phi_{S_0} = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Phi_{S_1} = \begin{bmatrix} 10 & 27 & -9 \\ -3 & -8 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Phi_{S_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 6 & 1 \end{bmatrix}$$

7. The affine braid group action in the SL_3 Case

7.1. Lifting to equivariant sheaves on $G \times X$. Our strategy for computing the affine braid group action will be to notice that $G/B \to \mathbb{P}^2$ is a fibration with fiber $\mathbb{P}^1$, and that we can therefore lift the morphisms in such a way that the three-dimensional and two-dimensional cases become intimately related.

Specifically, as we did in the two-dimensional case, note that we have the following commutative diagram.

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, "\pi"] & \tilde{\mathfrak{g}}^S \\ G/B \arrow[u, "\psi_G"] \arrow[r, "\pi'"] & \mathbb{P}^2 \arrow[u, "\psi_P"'] \end{tikzcd}$$

To compute $\pi^*\pi_*\psi_{G*}\mathcal{F}$ for some sheaf $\mathcal{F}$ on G/B, we can note that $\pi_*\psi_{G*} = \psi_{P*}\pi'_*$, and so compute $\pi^*\psi_{P*}\pi'_*$. Now we claim the following.

Lemma 4. *For any P-space X, consider the variety $G \times X$ as a P-space by letting $p(g, x) = (gp^{-1}, px)$. Then there exist actions of P on $(P \times \mathrm{Lie}\, B)/B$, $\mathrm{Lie}\, P$, (P/B), and pt, and morphisms $\overline{\pi}, \overline{\pi}', \overline{\psi_G}, \overline{\psi_P}$, making the following diagram commute.*

$$\begin{tikzcd} G \times (P \times \mathrm{Lie}\, B)/B \arrow[r, "1\times\overline{\pi}"] \arrow[ddrr, bend right, "/P"] & G \times \mathrm{Lie}\, P \arrow[ddrr, bend left, "/P"] & & \\ G \times (P/B) \arrow[u, "1\times\overline{\psi_G}"] \arrow[r, "1\times\overline{\pi}'"] \arrow[ddrr, bend right, "/P"] & G \times pt \arrow[u, "1\times\overline{\psi_P}"'] \arrow[ddrr, bend left, "/P"] & & \\ & & \tilde{\mathfrak{g}} \arrow[r, "\pi"] & \tilde{\mathfrak{g}}^S \\ & & G/B \arrow[u, "\psi_G"] \arrow[r, "\pi'"] & \mathbb{P}^2 \arrow[u, "\psi_P"'] \end{tikzcd}$$

Proof. We choose the following actions.

- Choose the action of P on P/B to be by left multiplication.

- Choose the action on pt to be trivial.
- The action on $\operatorname{Lie} P$ to be the adjoint action.
- The action on $(P \times \operatorname{Lie} B)/B$ to be by left-multiplication on the copy of P. In $(P \times \operatorname{Lie} B)/B$, the action of B is by right inverse multiplication on P, and the adjoint action on $\operatorname{Lie} B$.

We first show that these actions indeed give the appropriate quotients. The fact that $(G \times P/B)/P = G/B$ and $(G \times pt)/P = G/P = \mathbb{P}^2$ is clear. We have $G \times \operatorname{Lie} P = \tilde{\mathfrak{g}}^S$ from Lemma 3. To see that $\tilde{\mathfrak{g}} = (G \times (P \times \operatorname{Lie} B)/B)/P$, we recall from Lemma 3 that $\tilde{\mathfrak{g}} = (G \times \operatorname{Lie} B)/B$, and note that

$$\begin{aligned}(G \times \operatorname{Lie} B)/B &= ((G \times P)/P \times \operatorname{Lie} B)/B = (G \times P \times \operatorname{Lie} B)/(P \times B) \\ &= (G \times (P \times \operatorname{Lie} B)/B)/P.\end{aligned}$$

We now choose the four maps $\overline{\pi}, \overline{\pi}', \overline{\psi_G}, \overline{\psi_P}$.

- The map $P/B \to pt$ is the only possible one.
- The map $pt \to \operatorname{Lie} P$ is the inclusion as the zero point.
- The map $(P/B) \to (P \times \operatorname{Lie} B)/B$ sends $pB \mapsto (p, 0)$.
- The map $(P \times \operatorname{Lie} B)/B \to \operatorname{Lie} P$ maps $(g, p, b) \mapsto (g, pbp^{-1})$.

To finish the proof we merely need to see that each of these causes the square attached to them to commute. To see that the square

$$\begin{array}{ccc} G \times (P/B) & \xrightarrow{1 \times \overline{\pi}'} & G \times pt \\ \big\downarrow{\scriptstyle /P} & & \big\downarrow{\scriptstyle /P} \\ G/B & \xrightarrow{\quad \pi \quad} & \mathbb{P}^2 \end{array}$$

commutes, we check the equation

$$\overline{(1 \times \overline{\pi}')(g, pB)} = \pi(\overline{(g, pB)}).$$

Evaluating both sides yields

$$\begin{gathered}\overline{g} = \pi(gpB) \\ gP = gpP\end{gathered}$$

and this is true. Similarly, to check the square for ψ_P we need to check the equation

$$\begin{gathered}\overline{(1 \times \overline{\psi_P})(g)} = \psi_P(\overline{g}) \\ \overline{(g, 0)} = \psi_P(gP) \\ (gP, 0) = (gP, 0)\end{gathered}$$

which is true. To check the square for ψ_G we check a similar equation

$$\begin{gathered}\overline{(1 \times \overline{\psi_G})(g, p)} = \psi_P(\overline{(g, p)}) \\ \overline{(g, p, 0)} = \psi_G(gpB) \\ (gpB, 0) = (gpB, 0)\end{gathered}$$

which is also true. Finally, to check the square for π, we check

$$\overline{(1 \times \overline{\pi P})(g, p, b)} = \pi(\overline{(g, p, b)})$$

$$\overline{(g, p, pbp^{1})} = \pi(gB, pbp^{-1})$$

$$(gpP, (gp)b(gp)^{-1}) = (gP, (gp)b(gp)^{-1})$$

which is also true. This finishes the proof. ∎

7.2. **Descending to equivariant sheaves on fibres.** The fact that each map takes the form $1 \times f$ means that all such maps are equivariant with respect to the action of G that acts by left-multiplication on the copy of G. This is useful, because all of the bundles $\mathcal{O}(\lambda)$ we begin with on G/B are equivariant with respect to the left multiplication action by G. This means that, after lifting, we will begin with a line bundle on $G \times (P/B)$ that is G-equivariant, and, since all the morphisms G-equivariant, all of the sheaves will stay equivariant after applying the functors.

$$\begin{tikzcd}
G \times (P \times \operatorname{Lie} B)/B \arrow[r, "1\times\overline{\pi}"] \arrow[ddrr, bend right, "/P"] \arrow[dddd, bend right, "/G"] & G \times \operatorname{Lie} P \arrow[ddrr, bend left, "/P"] \arrow[dddd, bend right, "/G"] & & \\
G \times (P/B) \arrow[u, "1\times\overline{\psi_G}"] \arrow[r, "1\times\overline{\pi}'"] \arrow[ddrr, bend right, "/P"] \arrow[dddd, bend right, "/G"] & G \times pt \arrow[u, "1\times\overline{\psi_P}"] \arrow[ddrr, bend left, "/P"] \arrow[dddd, bend left, "/G"] & & \\
& & \tilde{\mathfrak{g}} \arrow[r, "\pi"] & \tilde{\mathfrak{g}}^S \\
& & G/B \arrow[u, "\psi_G"] \arrow[r, "\pi'"] & \mathbb{P}^2 \arrow[u, "\psi_P"] \\
(P \times \operatorname{Lie} B)/B \arrow[r, "\overline{\pi}"] & \operatorname{Lie} P & & \\
P/B \arrow[u, "\overline{\psi_G}"] \arrow[r, "\overline{\pi}'"] & pt \arrow[u, "\overline{\psi_P}"] & &
\end{tikzcd}$$

This gives us the following lemma, which we will use as our main tool to compute the affine braid group action in the SL_3 case.

Lemma 5. *For each variety X of $P/B, pt, (P \times \operatorname{Lie} B)/B, G \times \operatorname{Lie} P$, there is an equivalence of categories $\mathcal{E}_X$*

$$\mathcal{E}_X : D^b(\operatorname{Coh} X) \to D^b(\operatorname{Coh}(G \times X)/P)$$

and for each of the morphisms $f : X \to Y$ in $\pi, \pi', \psi_G, \psi_P$ we have the following commutativity equations

$$\overline{f_*}\mathcal{E}_X = \mathcal{E}_X f_*$$

$$\overline{f^*}\mathcal{E}_X = \mathcal{E}_X f^*$$

Proof. We recall that since the actions of P and G on each $G \times X$ is free, we have two equivalences of categories $\mathcal{F} : D^b(\operatorname{Coh}(G \times X)/P) \to D^b(G \times X)$ and $\mathcal{G} : D^b(\operatorname{Coh}(G \times X)/P) \to D^b(X)$. We choose $\mathcal{E}_X = \mathcal{G} \circ \mathcal{F}$, which has the desired commutativity properties since $\mathcal{G}$ and $\mathcal{F}$ do. ∎

7.3. **Finding the SL_2 case in diagram of fibres.** The P-equivariant diagram (bottom square) is very similar to the diagram for the SL_2 case.

First, we observe that $P/B = \mathbb{P}^1 = SL_2/B_2$, where by B_2 we mean the Borel in SL_2. It will be useful later to also compute how P acts on SL_2 in this quotient.

Lemma 6. *We have an isomorphism of P-spaces $P/B \simeq SL_2/B_2$, where $p \in P$ acts on SL_2 by*

$$p(\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}) = \begin{bmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} p_{11} & \\ & 1 \end{bmatrix}.$$

Proof. Note that every element of P has a factorization of the following form.

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ & p_{22} & p_{23} \\ & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} 1 & & \\ & p_{11}p_{22} & p_{23} \\ & p_{11}p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ & p_{11}^{-1} & 0 \\ & 0 & 1 \end{bmatrix}$$

Since the latter matrix is in B, this means that every point $(p, b) \in P/B$ is equal to some $s \in P$ of the form $\begin{bmatrix} 1 & & \\ & * & * \\ & * & * \end{bmatrix}$. Let L subgroup of P consisting of such s, noting that $L \simeq SL_2$. We can thus consider this variety to be a quotient $L/(B \cap L)$ instead of $(P \times \operatorname{Lie} B)/B$.

To compute the action of p on SL_2/B_2, we pass to its action on P/B as follows.

$$\begin{aligned}
p(\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}) &\in SL_2/B_2 \\
&= \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ & p_{22} & p_{23} \\ & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & s_{11} & s_{12} \\ & s_{21} & s_{22} \end{bmatrix} \in P/B \\
&= \begin{bmatrix} p_{11} & p_{12}s_{11} + p_{13}s_{21} & p_{12}s_{12} + p_{13}s_{22} \\ & p_{22}s_{11} + p_{23}s_{21} & p_{22}s_{12} + p_{23}s_{22} \\ & p_{32}s_{11} + p_{33}s_{21} & p_{32}s_{12} + p_{33}s_{22} \end{bmatrix} \in P/B \\
&= (\begin{bmatrix} 1 & & \\ & p_{22}s_{11} + p_{23}s_{21} & p_{22}s_{12} + p_{23}s_{22} \\ & p_{32}s_{11} + p_{33}s_{21} & p_{32}s_{12} + p_{33}s_{22} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12}s_{11} + p_{13}s_{21} & p_{12}s_{12} + p_{13}s_{22} \\ & 1 & \\ & & 1 \end{bmatrix} \in P/B \\
&= \begin{bmatrix} 1 & & \\ & p_{22} & p_{23} \\ & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & s_{11} & s_{12} \\ & s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} 1 & & \\ & p_{11} & \\ & & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12}s_{11} + p_{13}s_{21} & p_{12}s_{12} + p_{13}s_{22} \\ & p_{11}^{-1} & \\ & & 1 \end{bmatrix} \in P/B \\
&= \begin{bmatrix} 1 & & \\ & p_{22} & p_{23} \\ & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & s_{11} & s_{12} \\ & s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} 1 & & \\ & p_{11} & \\ & & 1 \end{bmatrix} \in P/B \\
&\sim \begin{bmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} p_{11} & \\ & 1 \end{bmatrix}. \in SL_2/B_2
\end{aligned}$$

Noting that this means the action descends from an action on SL_2 given by $p(\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}) = \begin{bmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} p_{11} & \\ & 1 \end{bmatrix}$, we are done.

□

Lemma 7. *There is an isomorphism of varieties* $(P\times\operatorname{Lie} B)/B = (SL_2\times\operatorname{Lie} B)/B_2$*, where* B_2 *acts on* $\operatorname{Lie} B$ *by adjoint action on the lower-right-hand block.*

Proof. As in the proof of Lemma 6, recall every element $p \in P$ has a factorization $p = lb$ for $l \in L, b \in B$. Thus every point $(p, b) \in (P \times \operatorname{Lie} B)/B$ is equal to some (s, b') where $s \in L$. We can thus consider this variety to be a quotient $(L \times \operatorname{Lie} B)/(B \cap L)$ instead of $(P \times \operatorname{Lie} B)/B$. $\square$

We can now also note that $\operatorname{Lie} B$ can be decomposed into two B_2-invariant subspaces, namely

$$\operatorname{Lie} B = \{\begin{bmatrix} v_1 & v_2 & v_3 \\ & -\frac{v_1}{2} & 0 \\ & & -\frac{v_1}{2} \end{bmatrix}\} \oplus \{\begin{bmatrix} 0 & 0 & 0 \\ & u_1 & u_2 \\ & & -u_1 \end{bmatrix}\}.$$

The action of B_2 on the first subspace is trivial, and the second space is precisely $\operatorname{Lie} B_2$ with the adjoint action of B_2. We can thus regard this as

$$\begin{aligned}(P \times \operatorname{Lie} B)/B &\simeq \mathbb{A}^3 \times (SL_2 \times \operatorname{Lie} B_2)/B_2 \\ &\simeq \mathbb{A}^3 \times \tilde{\mathfrak{g}}_{SL_2}.\end{aligned}$$

Moreoever, since the adjoint action of SL_2 is also trivial on the copy of $\mathbb{A}^3$, and acts on the second copy exact as it does in the SL_2 case, the map $(p, b) \mapsto p^{-1}bp \in \operatorname{Lie} P$ can be written as

$$(1 \times \pi) : \mathbb{A}^3 \times \tilde{\mathfrak{g}}_{SL_2} \to \mathbb{A}^3 \times \mathfrak{g}_{SL_2}.$$

Here by π we mean the morphism $\tilde{\mathfrak{g}} \to \mathfrak{g}$ from the SL_2 case. We can thus see our entire P-equivariant diagram as follows.

$$\begin{tikzcd}
\mathbb{A}^3 \times \tilde{\mathfrak{g}}_{SL_2} \arrow[r, "1\times\pi_{SL_2}"] & \mathbb{A}^3 \times \mathfrak{g}_{SL_2} \\
SL_2/B_2 \arrow[u, "i_0\times\psi_{G,SL_2}"] \arrow[r, "\pi'_{SL_2}"] & SL_2/SL_2 \arrow[u, "i_0\times\psi_{P,SL_2}"']
\end{tikzcd}$$

Here by i_0 we mean inclusion of the point as the zero point in $\mathbb{A}^3$ (where we are viewing each variety SL_2/B_2 and SL_2/SL_2 as $pt \times SL_2/B_2$ and $pt \times SL_2/SL_2$, respectively), and by each of $\psi_{G,SL_2}, \psi_{P,SL_2}, \pi'_{SL_2}, \pi_{SL_2}$, we mean those maps from the SL_2 case. This gives us

$$\overline{\pi^*_{SL_2}\psi_{P*}\pi'_*} = (1 \times \pi_{SL_2})^*(i_0 \times \psi_{P,SL_2})_*\pi'_{SL_2*}.$$

By factorizing the morphism $i_0 \times \psi_{P,SL_2}$, we can also extend this diagram to a commutative diagram as follows.

$$\begin{tikzcd}
\mathbb{A}^3 \times \tilde{\mathfrak{g}}_{SL_2} \arrow[r, "1\times\pi_{SL_2}"] & \mathbb{A}^3 \times \mathfrak{g}_{SL_2} \\
\tilde{\mathfrak{g}}_{SL_2} \arrow[u, "i_0\times 1"] \arrow[r, "\pi_{SL_2}"] & \mathfrak{g}_{SL_2} \arrow[u, "i_0\times 1"'] \\
SL_2/B_2 \arrow[u, "\psi_{G,SL_2}"] \arrow[r, "\pi'_{SL_2}"] & SL_2/SL_2 \arrow[u, "\psi_{P,SL_2}"']
\end{tikzcd}$$

This let us rewrite

$$\overline{\pi^*_{SL_2}\psi_{P*}\pi'_*} = (1 \times \pi_{SL_2})^*(i_0 \times 1)_*\psi_{P,SL_2*}\pi'_{SL_2*}.$$

Then by flat base change on the uppermost square we have

$$\overline{\pi^*_{SL_2}\psi_{P*}\pi'_*} = (i_0 \times 1)_*\pi^*_{SL_2}\psi_{P,SL_2*}\pi'_{SL_2*}.$$

Thus it suffices to $\pi^*_{SL_2}\psi_{P,SL_2*}\pi'_{SL_2}$ equivariantly, and our desired sheaves will simply be these sheaves pushed forward along $i_0 \times 1$ into $\mathbb{A}^3 \times \tilde{\mathfrak{g}}_{SL_2}$. We will compute this for $\mathcal{O}, \mathcal{O}(-1)$, and $\mathcal{O}(1)$.

7.4. **The correspondence between G/B-bundles and equivariant P/B-bundles explicity.** Before we begin computing this, we need to lift a particular line bundle $\mathcal{O}(\lambda)$ on G/B to a line bundle on P/B through this process. Define $\mathcal{O}(\lambda)$ to be the vector bundle given by the projection $(G \times k)/B \to G$ where $b(g, x) = (gb^{-1}, \lambda(b)^{-1}x)$. We choose this sign so that when λ is dominant then $\mathcal{O}(\lambda)$ has global sections.

Lemma 8. *Let ω_1, ω_2 be the fundamental weights, i.e. the characters of B where $\omega_1(b) = b_{33}^{-1}$ and $\omega_2(b) = b_{11}$. Let ω_0 be the character of B_2 such that $\omega_0(b') = b'_{11} = b'^{-1}_{22}$. Then $\mathcal{E}_{\mathbb{P}^1}(\mathcal{O}(a\omega_1 + b\omega_2)) = \mathcal{O}(a\omega_0)$, equipped with the P-equivariant structure where $p \in P$ acts on $\mathcal{O}(a\omega_0) = (SL_2 \times k)/B$ by sending $p(s, x) =$*
$$(p(s), \lambda(\begin{bmatrix} p_{11} & & \\ & p_{11}^{-1} & \\ & & 1 \end{bmatrix})^{-1}x).$$

Proof. Recall that $\mathcal{O}(\lambda)$ is defined as the line bundle $(G \times k)/B$ where B acts by right inverse multiplication on G and by the character $-\lambda$ on k. Using our isomorphism $G/B = (G \times P/B)/P$ gives us a vector bundle defined by the map

$$(G \times k)/B = (G \times (P \times k)/B)/P \longrightarrow (G \times P/B)/P\ .$$

We can lift this map to a P-equivariant map

$$G \times (P \times k)/B \longrightarrow G \times P/B\ .$$

This gives us the desired vector bundle on $G \times P/B$. Quotienting this map out by G-equivariance will give us a map

$$(P \times k)/B \longrightarrow P/B\ .$$

Here the map is the quotient by the action of B by right inverse multiplication on P, and by the chosen character on k. Using an analogous argument to that used in 7.3, we can rewrite $P/B = SL_2/B_2$ and $(P \times k)/B = (SL_2 \times k)/B_2$, where the action of B_2 is the action of B restricted to the subgroup of matrices of the form $\begin{bmatrix} 1 & & \\ & * & * \\ & & * \end{bmatrix}$. This gives us a vector bundle

$$(SL_2 \times k)/B_2 \longrightarrow SL_2/B_2\ .$$

This is a recognizable vector bundle on $\mathbb{P}^2$.

The P-action on $(SL_2 \times k)/B_2$ is defined as begin inherited from its action on $(P \times k)/B$ follows.

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ & p_{22} & p_{23} \\ & p_{32} & p_{33} \end{bmatrix} (\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}, v)$$
$$= \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ & p_{22} & p_{23} \\ & p_{32} & p_{33} \end{bmatrix} (\begin{bmatrix} 1 & & \\ & s_{11} & s_{12} \\ & s_{21} & s_{22} \end{bmatrix}, v)$$
$$= (\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ & p_{22} & p_{23} \\ & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & s_{11} & s_{12} \\ & s_{21} & s_{22} \end{bmatrix}, v)$$
$$= (\begin{bmatrix} p_{11} & p_{12}s_{11} + p_{13}s_{21} & p_{12}s_{12} + p_{13}s_{22} \\ & p_{22}s_{11} + p_{23}s_{21} & p_{22}s_{12} + p_{23}s_{22} \\ & p_{32}s_{11} + p_{33}s_{21} & p_{32}s_{12} + p_{33}s_{22} \end{bmatrix}, v)$$
$$= (\begin{bmatrix} 1 & & \\ & p_{22}s_{11} + p_{23}s_{21} & p_{22}s_{12} + p_{23}s_{22} \\ & p_{32}s_{11} + p_{33}s_{21} & p_{32}s_{12} + p_{33}s_{22} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12}s_{11} + p_{13}s_{21} & p_{12}s_{12} + p_{13}s_{22} \\ & 1 & \\ & & 1 \end{bmatrix}, v)$$
$$= (\begin{bmatrix} 1 & & \\ & p_{22} & p_{23} \\ & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & s_{11} & s_{12} \\ & s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} 1 & & \\ & p_{11} & \\ & & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12}s_{11} + p_{13}s_{21} & p_{12}s_{12} + p_{13}s_{22} \\ & p_{11}^{-1} & \\ & & 1 \end{bmatrix}, v)$$
$$= (\begin{bmatrix} 1 & & \\ & p_{22} & p_{23} \\ & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & s_{11} & s_{12} \\ & s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} 1 & & \\ & p_{11} & \\ & & 1 \end{bmatrix}, \lambda(\begin{bmatrix} p_{11} & p_{12}s_{11} + p_{13}s_{21} & p_{12}s_{12} + p_{13}s_{22} \\ & p_{11}^{-1} & \\ & & 1 \end{bmatrix})^{-1}v)$$
$$= (\begin{bmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} p_{11} & \\ & 1 \end{bmatrix}, \lambda(\begin{bmatrix} p_{11} & p_{12}s_{11} + p_{13}s_{21} & p_{12}s_{12} + p_{13}s_{22} \\ & p_{11}^{-1} & \\ & & 1 \end{bmatrix})^{-1}v)$$

Recalling that the characters λ are uniquely defined by their action on the torus, we may equivalently write

$$= (\begin{bmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} p_{11} & \\ & 1 \end{bmatrix}, \lambda(\begin{bmatrix} p_{11} & & \\ & p_{11}^{-1} & \\ & & 1 \end{bmatrix})^{-1}v).$$
$$= (p(s), \lambda(\begin{bmatrix} p_{11} & & \\ & p_{11}^{-1} & \\ & & 1 \end{bmatrix})^{-1}v)$$

This gives the action of P on our bundle in terms of λ. Hence we may concretely describe the correspondence between line bundles as sending $\mathcal{O}(\lambda)$ to the sheaf $(SL_2 \times k)/B_2$ where $b \in B_2$ acts as $b(s, v) = (sb^{-1}, \lambda|_{L\cap B}(x)^{-1}v)$ (i.e. the sheaf $\mathcal{O}(a\omega_0)$, and P acts on this sheaf by $p(s, v) = (p(s), \lambda(\begin{bmatrix} p_{11} & & \\ & p_{11}^{-1} & \\ & & 1 \end{bmatrix})^{-1}s)$.

Now suppose we begin with a line bundle $\mathcal{O}(\lambda = a\omega_1 + b\omega_2)$ on G/B. Then the restriction of λ to the subgroup $L \cap B$ is $a\omega_0$, and $\lambda(\begin{bmatrix} p_{11} & & \\ & p_{11}^{-1} & \\ & & 1 \end{bmatrix})^{-1} = p_{11}^{-b}$. Thus $\mathcal{O}(\lambda)$ corresponds to $\mathcal{O}(a\omega_0)$, equipped with the P-equivariant structure where $p \in P$ acts on $\mathcal{O}(a\omega_0) = (SL_2 \times k)/B$ by sending $p(s,x) = (p(s), \lambda(\begin{bmatrix} p_{11} & & \\ & p_{11}^{-1} & \\ & & 1 \end{bmatrix})^{-1}x)$. This finishes the proof.

□

7.5. **Global sections equivariantly.** We now compute π_{SL_2*} equivariantly, recalling that this functor is simply the derived global sections functor. We begin with the line bundles

$$\mathcal{O}(1), \mathcal{O}, \mathcal{O}(-1).$$

These can be constructed, respectively, as

$$(SL_2 \times k_{\omega_0})/B_2, (SL_2 \times k_0)/B_2, (SL_2 \times k_{-\omega_0})/B_2.$$

Here P acts trivially on the k component of each, and in the standard way (described in the previous section) on the SL_2 component of each.

Recall that, ignoring equivariant structure, taking the derived global sections of $\mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1)$ will give the following cohomologies in degree zero, with no cohomology in any other degree.

$$R\Gamma(\mathcal{O}(-1)) = 0$$
$$R\Gamma(\mathcal{O}) = k$$
$$R\Gamma(\mathcal{O}(1)) = k^2$$

The P-equivariant structure on the first two of these are clear, since the first is zero (so can only have the trivial structure) and the second is the space of constant functions on $\mathbb{P}^1$, which has the trivial equivariant structure. The only difficulty is with $\Gamma(\mathcal{O}(1))$.

Lemma 9. *The space of global sections $\Gamma(\mathcal{O}(1))$ comes equipped with equivariant structure where $p \in P$ acts by sending*

$$p(\begin{bmatrix} u \\ v \end{bmatrix}) = p_{11}^{-1} \begin{bmatrix} p_{22} & p_{32} \\ p_{23} & p_{33} \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Proof. We can deduce the P-action on the space of global sections by noting that there should be a natural P-equivariant map

$$\mathcal{O} \otimes \Gamma(\mathcal{O}(1)) \to \mathcal{O}(1).$$

These are $\mathbb{P}^1$-bundles, so we can write them as

$$(SL_2 \times k^2)/B_2 \to (SL_2 \times k)/B_2.$$

This map is the map which descends from the B_2-equivariant map

$$SL_2 \times k^2 \to SL_2 \times k$$

$$(\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, u, v) \mapsto (\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, m_{11}u + m_{21}v).$$

We wish to equip k^2 with a P-action such that this map becomes P-equivariant. Recall from Lemma 6 that the action of P on SL_2 is

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \mapsto \begin{bmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} p_{11} & \\ & 1 \end{bmatrix} = \begin{bmatrix} p_{11}(p_{22}m_{11} + p_{23}m_{21}) & p_{22}m_{12} + p_{23}m_{22} \\ p_{11}(p_{32}m_{11} + p_{33}m_{12}) & p_{32}m_{12} + p_{33}m_{22} \end{bmatrix}.$$

Thus we would like to solve for an action making the following diagram commute.

$$\begin{tikzcd}
(\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, u, v) \arrow[r] \arrow[d] & (\begin{bmatrix} p_{11}(p_{22}m_{11} + p_{23}m_{21}) & p_{22}m_{12} + p_{23}m_{22} \\ p_{11}(p_{32}m_{11} + p_{33}m_{12}) & p_{32}m_{12} + p_{33}m_{22} \end{bmatrix}, u', v') \arrow[d] \\
(\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, m_{11}u + m_{21}v) \arrow[r] & (\begin{bmatrix} p_{11}(p_{22}m_{11} + p_{23}m_{21}) & p_{22}m_{12} + p_{23}m_{22} \\ p_{11}(p_{32}m_{11} + p_{33}m_{12}) & p_{32}m_{12} + p_{33}m_{22} \end{bmatrix}, m_{11}u + m_{21}v)
\end{tikzcd}$$

This is to say we would like to solve the following equation.

$$\begin{aligned}
m_{11}u + m_{21}v &= p_{11}(p_{22}m_{11} + p_{23}m_{21})u' + p_{11}(p_{32}m_{11} + p_{33}m_{21})v' \\
&= p_{11}((p_{22}u' + p_{32}v')m_{11} + (p_{32}u' + p_{33}v')m_{21} \\
\begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} \end{bmatrix} &= p_{11} \begin{bmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} \end{bmatrix}
\end{aligned}$$

Since this should be true for arbitrary m_{11}, m_{21}, we obtain

$$\begin{bmatrix} u \\ v \end{bmatrix} = p_{11} \begin{bmatrix} p_{22} & p_{32} \\ p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}$$

$$p_{11}^{-1} \begin{bmatrix} p_{22} & p_{32} \\ p_{23} & p_{33} \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u' \\ v' \end{bmatrix}.$$

as desired.

□

7.6. **Descending the global sections of $\mathcal{O}(1)$.** Because we will need it later, it is worth computing how $\mathcal{O} \otimes \Gamma(\mathcal{O}(1))$ as a P-equivariant sheaf on P/B descends to a sheaf on G/B. Since we are eventually going to pass to the Grothendieck group anyway, we will only compute its class in the Grothendieck group. We will do this by embedding in an exact sequence on $\mathbb{P}^1$ whose other elements are easier to descend.

Lemma 10. *The exact sequence of $\mathbb{P}^1$ sheaves*

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \otimes \Gamma(\mathcal{O}(1)) \to \mathcal{O}(1) \to 0$$

can be made P-equivariant, where P acts on $\mathcal{O}(-1)$ through ζ. Therefore, in the Grothendieck group,

$$[\mathcal{E}_{\mathbb{P}^1}^{-1}(\mathcal{O} \otimes \Gamma(\mathcal{O}(1)))] = [\mathcal{O}(-\omega_1 + \omega_2)] + [\mathcal{O}(\omega_1)]$$

Proof. In this exact sequence, the map $\mathcal{O}(-1) \to \mathcal{O} \otimes \Gamma(\mathcal{O}(1))$ descends from the B_2-equivariant map $\iota : SL_2 \times k \to SL_2 \times k^2$ mapping

$$\iota : (\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, v) \mapsto (\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, m_{12}v, -m_{11}v).$$

As before, we can deduce the P-action on $\mathcal{O}(-1)$ by solving for the commutativity of the following diagram.

$$\begin{array}{ccc} (\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, v) & \xrightarrow{p\cdot -} & (\begin{bmatrix} p_{11}(p_{22}m_{11} + p_{23}m_{21}) & p_{22}m_{12} + p_{23}m_{22} \\ p_{11}(p_{32}m_{11} + p_{33}m_{12}) & p_{32}m_{12} + p_{33}m_{22} \end{bmatrix}, v') \\ \downarrow \iota & & \downarrow \iota \\ (\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, -m_{21}v, m_{11}v) & \xrightarrow{p\cdot -} & (\begin{bmatrix} p_{11}(p_{22}m_{11} + p_{23}m_{21}) & p_{22}m_{12} + p_{23}m_{22} \\ p_{11}(p_{32}m_{11} + p_{33}m_{12}) & p_{32}m_{12} + p_{33}m_{22} \end{bmatrix}, A, B) \end{array}$$

Here

$$\begin{bmatrix} A \\ B \end{bmatrix} = p_{11}^{-1} \begin{bmatrix} p_{22} & p_{32} \\ p_{23} & p_{33} \end{bmatrix}^{-1} \begin{bmatrix} -m_{21}v \\ m_{11}v \end{bmatrix}.$$

This means we need to solve the following equation.

$$\begin{bmatrix} -p_{11}(p_{32}m_{11} + p_{33}m_{21})v' \\ p_{11}(p_{22}m_{11} + p_{23}m_{21})v' \end{bmatrix} = p_{11}^{-1} \begin{bmatrix} p_{22} & p_{32} \\ p_{23} & p_{33} \end{bmatrix}^{-1} \begin{bmatrix} -m_{21}v \\ m_{11}v \end{bmatrix}$$

$$p_{11} \begin{bmatrix} p_{33} & -p_{32} \\ -p_{23} & p_{22} \end{bmatrix} \begin{bmatrix} -m_{21}v' \\ m_{11}v \end{bmatrix} = p_{11}^{-1} \begin{bmatrix} p_{22} & p_{32} \\ p_{23} & p_{33} \end{bmatrix}^{-1} \begin{bmatrix} -m_{21}v \\ m_{11}v \end{bmatrix}$$

$$p_{11} \begin{bmatrix} p_{22} & p_{32} \\ p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} p_{33} & -p_{32} \\ -p_{23} & p_{22} \end{bmatrix} \begin{bmatrix} -m_{21}v' \\ m_{11}v \end{bmatrix} = p_{11}^{-1} \begin{bmatrix} -m_{21}v \\ m_{11}v \end{bmatrix}$$

$$p_{11}^2 \begin{bmatrix} p_{22}p_{33} - p_{32}p_{23} & \\ & p_{22}p_{33} - p_{23}p_{32} \end{bmatrix} \begin{bmatrix} -m_{21}v' \\ m_{11}v \end{bmatrix} = \begin{bmatrix} -m_{21}v \\ m_{11}v \end{bmatrix}$$

Recalling that $p_{22}p_{33} - p_{32}p_{23} = p_{11}^{-1}$, this gives us

$$v' = p_{11}^{-1}v.$$

Thus the equivariant structure can be viewed as having $p(s, x) = (p(s), \lambda(\begin{bmatrix} p_{11} & & \\ & p_{11}^{-1} & \\ & & 1 \end{bmatrix})^{-1}x)$ where $\lambda(\begin{bmatrix} p_{11} & & \\ & p_{11}^{-1} & \\ & & 1 \end{bmatrix})^{-1} = p_{11}^{-1}$, so $\lambda(\begin{bmatrix} p_{11} & & \\ & p_{11}^{-1} & \\ & & 1 \end{bmatrix}) = p_{11}$. By Lemma 8 we can thus descend $\mathcal{O} \otimes \Gamma(\mathcal{O}(1))$ to something with Grothendieck group

$$[\mathcal{O}(-\omega_1 + \omega_2)] + [\mathcal{O}(\omega_1)]$$

as desired. $\square$

7.7. **Derived pullback equivariantly.** The pushforward $pt \to \operatorname{Lie} P$ simply gives the skyscraper sheaves on $\operatorname{Lie} P$ with the same P-actions.

To finish computing $\pi^*_{SL_2}\psi_{P,SL_2*}\pi'_{SL_2*}$ equivariantly, then, the main task is to compute the derived pullback $\pi^*_{SL_2}$ of a point equivariantly.

To do so, we can compute the P-action on the Koszul resolution of the point on $\operatorname{Lie} P$, then tensor each element of this resolution with $\mathcal{O}_{\mathbb{A}^3 \times \tilde{\mathfrak{g}}_{SL-2}}$ to pull it back, and find the induced P-action on the cohomology of the resulting chain.

We can simplify this computation somewhat by recalling from Section 5 that the cohomology will end up consisting only of (one-dimensional) line bundles. This means the P-action on each of these line bundles will be determined by a single character of P. But, by Jordan decomposition, a character of P is determined by the action of the torus inside P. Hence it suffices to look at the action of the torus $T \subseteq P$. Write $t = \begin{bmatrix} l & & \\ & z & \\ & & w \end{bmatrix}$ for an arbitrary element of T.

Moreoever, we can recall that the Koszul resolution only ended up having cohomology in grades zero and one. Thus we only need to look at the action on the first and second elements in the Koszul resolution.

Lemma 11. *Let $\mathcal{F}$ be the one-dimensional sksycraper sheaf on* $\operatorname{Lie} P$*. Then $H^0(\pi^*_{SL_2}\mathcal{F}) = \overline{\psi_{G*}}\mathcal{O}$, corresponding to $\psi_{G_*}\mathcal{O}$ on $\tilde{\mathfrak{g}}$, and $H^{-1}(\pi^*\mathcal{F}) = \overline{\psi_{G_*}}\mathcal{O}(-2)$, corresponding to $\psi_{G_*}\mathcal{O}(-2\omega_1 + \omega_2)$ on $\tilde{\mathfrak{g}}$.*

Proof. We will prove this in the following two sections.

7.8. **The H^0 sheaf equivariantly.** Write the coordinate ring of $\mathfrak{g}$ as $k[h,e,f]$, representing $\begin{bmatrix} h & e \\ f & -h \end{bmatrix}$. Now, the Koszul resolution begins with the structure sheaf $k[h,e,f]$, with the trivial equivariant structure.

This pulls back to the sheaf $\mathcal{O}$ on $\tilde{\mathfrak{g}}$, again with the trivial equivariant structure. Finally, the zeroth cohomology is a quotient of this module, so has the trivial equivariant structure as well.

7.9. **The H^{-1} sheaf equivariantly.** As a P-equivariant sheaf, the second element of Koszul resolution is $\mathcal{O}_{\mathfrak{g}} \otimes \mathfrak{g}$, where $\mathcal{O}_{\mathfrak{g}}$ has the trivial equivariant structure, and $\mathfrak{g}$ is the P-representation given by the space of linear sections of $\mathcal{O}_{\mathfrak{g}}$.

Recalling $\mathcal{O}_{\mathfrak{g}}$ is just the module $k[\mathfrak{g}] = k[h,e,f]$, and writing $\mathfrak{g}$ as $\langle \alpha, \beta, \gamma \rangle$ we wish for the following map ι to be P-equivariant.

$$\iota : k[\mathfrak{g}] \otimes \mathfrak{g} \to k[\mathfrak{g}]$$

$$g \otimes \alpha \mapsto hg$$

$$g \otimes \beta \mapsto eg$$

$$g \otimes \gamma \mapsto fg$$

where the maps are defined for any polynomial $g \in k[h,e,f]$.

This map will be equivariant if its pullback is compatible with the equivariance isomorphisms; that is to say, if the following diagram commutes:

$$\begin{tikzcd} k[\mathfrak{g}] \otimes_{k[\mathfrak{g}]} (k[\mathfrak{g}] \otimes \mathfrak{g}) \arrow[r, "\iota \otimes_{k[\mathfrak{g}]} k[\mathfrak{g}]"] \arrow[d] & k[\mathfrak{g}] \otimes_{k[\mathfrak{g}]} k[\mathfrak{g}] \arrow[d] \\ k[\mathfrak{g}] \otimes \mathfrak{g} \arrow[r, "\iota"] & k[\mathfrak{g}] \end{tikzcd}$$

Now, the action of t sends

$$\begin{bmatrix} h & e \\ f & -h \end{bmatrix} \mapsto \begin{bmatrix} h & \frac{z}{w}e \\ \frac{w}{z}f & -h \end{bmatrix}.$$

Hence, upstairs, we have $(\alpha, \beta, \gamma) \mapsto (h, \frac{z}{w}e, \frac{w}{z}f)$. The equivariant structure on the right is trivial, so on the left we must have an equivariianct structure on the left as $t(\alpha, \beta, \gamma) = (\alpha, \frac{z}{w}\beta, \frac{w}{z}\gamma)$.

Pulling this back into $\tilde{\mathfrak{g}}$ gives us the equivariant sheaf $\mathcal{O}_{\tilde{\mathfrak{g}}} \otimes \mathfrak{g}$. Here $\mathfrak{g}$ is equipped with the same action as before, and $\mathcal{O}_{\tilde{\mathfrak{g}}}$ is equipped with the trivial equivariant structure.

The H^{-1} cohomology sheaf then inherits its P-action as a subsheaf of this sheaf. Noting that the first cohomology sheaf is supported at the zero section, we may first consider the restriction of this to the zero section. The zero section is $\mathbb{P}^1$, so we can view its structure sheaf as the graded module $k[a, b]$. We thus get the H^{-1} cohomology sheaf as a subsehaf of

$$k[a, b] \otimes \mathfrak{g}.$$

Then, we recall that the first cohomology sheaf is generated by the element $2ab\alpha + b^2\beta - a^2\gamma$. That is to say, we have an inclusion ι

$$\iota : k[a, b][-2] \to k[a, b]\langle \alpha, \beta, \gamma \rangle$$

mapping $1 \mapsto 2ab\alpha + b^2\beta - a^2\gamma$. To obtain the P-equivariant structure on the H^{-1} cohomology sheaf, we simply need to solve so that this is P-equivariant.

We first wish to put these bundles into the form $(G \times (P \times V)/B)/P$, so that we can understand how the resulting P-action descends using Lemma 8. To do so, we would like to turn this map of sheaves into a map between the total spaces of these bundles.

We first get dual map between the dual space, which will correspond to the scheme map between total spaces. The dual space of $k[a, b] \otimes \mathfrak{g}$ is generated by $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$, where each of these represents the map that sends the respective generator to 1 and the rest to zero (e.g. $\overline{\alpha}(\alpha) = 1$ but $\overline{\alpha}(\beta) = 0$). Then the dual map sends

$$\overline{\alpha} \mapsto 2ab$$
$$\overline{\beta} \mapsto b^2$$
$$\overline{\gamma} \mapsto -a^2.$$

We can then take this and deduce the map of points between the schemes

$$(SL_2 \times k)/B_2 \to (SL_2 \times k^3)/B_2$$

as descending from the B_2-equivariant map

$$SL_2 \times k \to SL_2 \times k^3$$

which sends

$$(\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, v) \mapsto (\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, m_{11}m_{21}v, m_{11}^2 v, -m_{21}^2 v).$$

We would now like to solve for the T-action on $SL_2 \times k$ making this P-equivariant, i.e. such making the following diagram commute.

$$\begin{array}{ccc} (\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, v) & \longrightarrow & (\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, m_{11}m_{21}v, m_{11}^2 v, -m_{21}^2 v) \\ \downarrow & & \downarrow \\ (\begin{bmatrix} lzm_{11} & zm_{12} \\ lwm_{21} & wm_{21} \end{bmatrix}, v') & \longrightarrow & (\begin{bmatrix} lzm_{11} & zm_{12} \\ lwm_{21} & wm_{22} \end{bmatrix}, m_{11}m_{21}v, m_{11}^2 \frac{z}{w} v, -m_{21}^2 \frac{w}{z} v) \end{array}$$

Hence we get the equations

$$(l^2zw)m_{11}m_{21}v' = m_{11}m_{21}v$$
$$(l^2z^2)m_{11}^2 v' = m_{11}^2 \frac{z}{w} v$$
$$(l^2w^2)m_{21}^2 v' = m_{31}^2 \frac{w}{z} v.$$

All three of these simplify to the same $v' = (l^2zw)^{-1}v = l^{-1}v$. Hence the action of T on this sheaf sends $t(s,v) = (t(s), \lambda(\begin{bmatrix} l & & \\ & l^{-1} & \\ & & 1 \end{bmatrix})^{-1}v)$ where $\lambda(\begin{bmatrix} l & & \\ & l^{-1} & \\ & & 1 \end{bmatrix})^{-1} = l^{-1}$. Thus by Lemma 8 this sheaf corresponds to $\mathcal{O}(-2\omega_1 + \omega_2)$. Thus the sheaf on $\tilde{\mathfrak{g}}$ must be $\psi_G \mathcal{O}(-2\omega_1 + \omega_2)$. This completes the proof.

□

7.10. **Combining the derived pullback with derived global sections.** Using the previous two sections, we can compute the P-equivariant structure on $\pi_{SL_2}^* \psi_{P,SL_2*} \pi_{SL_2*} \mathcal{F}$ (i.e. $\pi_{SL_2}^* \pi_{SL_2*} \psi_{G,SL_2*} \mathcal{F}$) for each of our three line bundles $\mathcal{O}(-1)$, $\mathcal{O}$, and $\mathcal{O}(1)$. By applying the equivalences $\mathcal{E}_X$ we can then compute all the necessary sheaves $\pi^*\pi_*\psi_G\mathcal{F}$ in the SL_3 case.

First, $\pi_{SL_2}^* \psi_{P,SL_2*} \pi_{SL_2*} \mathcal{O}(-1) = 0$, since $\psi_{P,SL_2*} \pi_{SL_2*} \mathcal{O}(-1)$ is already zero.

Second, $\pi_{SL_2}^* \psi_{P,SL_2*} \pi_{SL_2*} \mathcal{O}$ has

$$H^{-1} = \overline{\psi_{G*}} \mathcal{O}(-2)(\omega_2)$$
$$H^0 = \overline{\psi_{G*}} \mathcal{O}(0)(0).$$

Here the second parenthetical indicates the P-action on the factor of k in each line bundle $(SL_2 \times k)/B_2$. Finally, $\pi_{SL_2}^* \psi_{P,SL_2*} \pi_{SL_2*} \mathcal{O}$ has

$$H^{-1} = (\overline{\psi_{G*}} \mathcal{O}(-2)) \otimes \Gamma(\mathcal{O}(1))$$
$$H^0 = (\overline{\psi_{G*}} \mathcal{O}) \otimes \Gamma(\mathcal{O}(1)).$$

Here $\Gamma(\mathcal{O}(1))$ comes equipped with the action described in Lemma 9.

We can then descend these, at least in the Grothendieck group. We have clearly

$$[\pi^*\pi_*\psi_{G*}\mathcal{O}(-\omega_1)] = 0$$
$$[\pi^*\pi_*\psi_{G*}\mathcal{O}] = [\psi_{G*}\mathcal{O}] - [\psi_{G*}\mathcal{O}(-2\omega_1+\omega_2)].$$

Using the descent of $\mathcal{O}\otimes\Gamma(\mathcal{O}(1))$ from Lemma 10, we also have

$$[\pi^*\pi_*\psi_{G*}\mathcal{O}(\omega_1)] = [\psi_{G*}\mathcal{O}(-\omega_1+\omega_2)]+[\psi_{G*}\mathcal{O}(\omega_1)]-[\psi_{G*}\mathcal{O}(-\omega_1+2\omega_2)]-[\psi_{G*}\mathcal{O}(-3\omega_1+\omega_2)].$$

These three, together with the projection formula, allow us to compute the Grothendieck groups of the functor $\pi^*\pi_*$ for a full basis for $D^b_0(\tilde{\mathfrak{g}})$ in the SL_3 case.

From the appendix of [5] we have that the sheaves

$$\psi_{G*}\mathcal{O}, \psi_{G*}\mathcal{O}(-\omega_1), \psi_{G*}\mathcal{O}(-\omega_2), \psi_{G*}\pi_1^*\Omega^1(1), \psi_{G*}\pi_2^*\Omega^1(1), \psi_{G*}\mathcal{O}(-\omega_1-\omega_2)$$

generate the category $D^b(\mathrm{Coh}_0\,\tilde{\mathfrak{g}})$ (here π_1,π_2 are the two projections $G/B\to G/P$). We moreoever have that $\mathcal{O}(\omega_1),\mathcal{O}$ together generate $\pi_1^*\Omega^1(1)$, and similarly $\mathcal{O}(\omega_2),\mathcal{O}$ together generate $\pi_2^*\Omega^1(1)$. We thus get that the following basis generates

$$D^b(\mathrm{Coh}_0\,\tilde{\mathfrak{g}})$$

.

$$\psi_{G*}\mathcal{O}, \psi_{G*}\mathcal{O}(-\omega_1), \psi_{G*}\mathcal{O}(-\omega_2), \psi_{G*}\mathcal{O}(\omega_1), \psi_{G*}\mathcal{O}(\omega_2), \psi_{G*}\mathcal{O}(-\omega_1-\omega_2)$$

Hence to compute the action on the Grothendieck group it suffices to give the Grothendieck group classes of the functor applied to each of these. Recalling that the functor was defined by sending $\mathcal{F}$ to $\mathrm{cone}(\pi^*\pi_*\mathcal{F}\to\mathcal{F})[-1])$, we may compute this as $s_1(\mathcal{F}) = [\pi^*\pi_*\mathcal{F}]-[\mathcal{F}]$. Hence we may write the following table, where the last three rows are computed by using the projection formula to see that the functor commutes with twists by ω_2.

$\mathcal{F}$	$[s_1\mathcal{F}]$
$\psi_{G*}\mathcal{O}(-\omega_1)$	$-[\psi_{G*}\mathcal{O}(-\omega_1)]$
$\psi_{G*}\mathcal{O}$	$-[\psi_{G*}\mathcal{O}(-2\omega_1+\omega_2)]$
$\psi_{G*}\mathcal{O}(\omega_1)$	$[\psi_{G*}\mathcal{O}(-\omega_1+\omega_2)]-[\psi_{G*}\mathcal{O}(-\omega_1+2\omega_2)]-[\mathcal{O}(-3\omega_1+\omega_2)]$
$\psi_{G*}\mathcal{O}(-\omega_2-\omega_1)$	$-[\psi_{G*}\mathcal{O}(-\omega_2-\omega_1)]$
$\psi_{G*}\mathcal{O}(-\omega_2)$	$-[\psi_{G*}\mathcal{O}(-2\omega_1)]$
$\psi_{G*}\mathcal{O}(\omega_2)$	$[\psi_{G*}\mathcal{O}(-2\omega_1+2\omega_2)]$

8. The affine braid group action on the Grothendieck group

The above table is not quite enough for us to examine the action on the Grothendieck group, because we do not know how to express the Grothendieck group classes of the resulting vector bundles in terms of the original basis. In this section we will leverage the well-known basis of *Schubert subvarieties* for the Grothendieck group to express the resulting bundles in terms of the Schubert basis and then base-changing back to the original line bundle basis.

Recall that there is an equivalence of categories between G/B and bigraded modules over the bigraded algebra $R = k[x,y,z,\overline{x},\overline{y},\overline{z}]/(x\overline{x},y\overline{y},z\overline{z})$, where (x,y,z) are in degree $(1,0)$ and $(\overline{x},\overline{y},\overline{z})$ are in degree $(0,1)$. The Schubert cells correspond to the six modules

$$R/(y,z,\overline{x},\overline{y}), R/(z,\overline{x},\overline{y}), R/(y,z,\overline{x}), R/(z), R/(\overline{x}), R/(0).$$

For notation's sake we will label these, respectively,

$$Z, P_1, P_2, X_1, X_2, R.$$

Here Z is a point; the P_i are copies of $\mathbb{P}^1$; the X_i are codimension-1; and R is the existing algebra. In this section we will prove the following.

Lemma 12. *For each line bundle $\mathcal{O}(a\omega_1 + b\omega_2)$ we have the following equality in the Grothendieck group.*

$$[\mathcal{O}(a\omega_1 + b\omega_2)] =$$

$$[R] + a[X_1] + b[X_2] + \frac{a(a+2b+1)}{2}[P_1] + \frac{b(2a+b+1)}{2}[P_2] + \frac{ab(a+b)}{2}[Z].$$

Proof. We will prove this in the following four sections.

8.1. **Line bundles on the whole variety.** We would like to express arbitrary line bundles in terms of these Schubert cells. To do this, note that any line bundle takes, as an R-module, the form

$$R[a, b]$$

for some a, b. Note also that there are exact sequences

$$R[a, b-1] \to R[a, b] \to R[a, b]/(\overline{x})$$

$$R[a-1, b] \to R[a, b] \to R[a, b]/(z).$$

This allows us to write the recurrence

$$[R[a, b]] = [R[a, b-1]] + [X_2[a, b]].$$

Hence

$$[R[a, b]] = [R[a, 0]] + ([X[a, 1]] + [X[a, 2]] + \dots [X[a, b]])$$

$$= [R[a, 0]] + (\sum_{i=1}^{b} [X[a, i]]).$$

We similarly may write the recurrence

$$[R[a, 0]] = [R[0, 0]] + \sum_{i=1}^{a} X_1[i, 0].$$

Hence, in general,

$$[R[a, b]] = [R] + [\sum_{i=b}^{k} X_2[a, i])] + (\sum_{i=1}^{a} X_1[i, 0]).$$

8.2. **Line bundles on the codimension-1 Schubert subvarieties.** We will consider the case of $R/(\overline{x})$; the case of $R/(z)$ is analogous. We will make use of three exact sequences. The first is

$$R[0,-1]/(\overline{x}) \xrightarrow{\cdot\overline{y}} R/(\overline{x}) \longrightarrow R/(\overline{x},\overline{y}) \simeq R/(\overline{x},\overline{y},z).$$

That the sequence is exact is clear; to see the last isomorphism $R/(\overline{x},\overline{y}) \simeq R/(\overline{x},\overline{y},z)$ (which is an isomorphism of sheaves, even though it is not one of graded modules), we note that $R/(\overline{x},\overline{y})$ is the structure sheaf of the subvariety

$$k[x,y,z,\overline{z}]/(z\overline{z}).$$

This subvariety can be covered by three affine opens, corresponding to $(x,\overline{z}) \neq 0$, $(y,\overline{z}) \neq 0$, and $(z,\overline{z}) \neq 0$. Localizing to the first affine open gives

$$k[\frac{x}{x},\frac{y}{x},\frac{z}{x},\frac{\overline{z}}{\overline{z}}]/(\frac{z}{x}\frac{\overline{z}}{\overline{z}}) \simeq k[\frac{y}{x}] \simeq (R/(\overline{x},\overline{y},z))_{x,\overline{z}}.$$

Localizing at each other affine open similarly gives that this is locally isomorphic to $R/(\overline{x},\overline{y},\overline{z})$, and it is easy to check that the local isomorphisms commute with the gluing maps.

The second exact sequence we use is

$$R[0,-1]/(\overline{x}) \xrightarrow{\cdot z} R/(\overline{x}) \longrightarrow R/(\overline{x},z)\ .$$

The exactness of this sequence is clear. The third exact sequence is:

$$R/(\overline{x},z) \xrightarrow{/(y)\oplus/(\overline{y})} R/(\overline{x},y,z) \oplus R/(\overline{x},\overline{y},z) \xrightarrow{e_2/(y)-e_1/(\overline{y})} R/(\overline{x},\overline{y},y,z)\ .$$

To see that this is exact, we'll first show the first map is injective. To see this, we note that its kernel is the set of elements $r \in R/(\overline{x},z)$ that lie in both (y) and $(\overline{y})$. Since these are both prime ideals, their intersection is their product $(y\overline{y})$. Thus the kernel is the ideal $(y\overline{y})$; but since we have $x\overline{x} + y\overline{y} + z\overline{z} + 0$ we have $y\overline{y} = -x\overline{x} - z\overline{z} \in (\overline{x},z)$ and hence $(y\overline{y}) = 0$ within $R/(\overline{x},z)$. Thus this first map is injective.

To then see that the kernel of the second map is exactly the image of the first map, we first note that the image of the first map lies within the kernel of the second map, since for any $r \in R/(\overline{x},z)$ we have:

$$r \mapsto (r/(y), r/(\overline{y})) \mapsto r/(y,\overline{y}) - r/(y,\overline{y}) = 0.$$

We then note that the kernel of the second map lies within the image of the first map, since if e_1, e_2 have $e_1/(\overline{y}) = e_2/(y)$, then since (y) and $(\overline{y})$ are coprime in $R/(\overline{x},z)$ we have some element $r \in R/(\overline{x},z)$ with $r/(y) = e_1$ and $r/(\overline{y}) = e_2$ by the Chinese Remainder Theorem.

Put together, this gives us the following three equations in the Grothendieck group.

$$\begin{gathered}
[X_2] = [X_2[0,-1]] + [P_1] \\
[X_2] = [X_2[-1,0]] + [R/(x,\overline{z})] \\
[R/(x,\overline{z})] = [P_1] + [P_2] - [Z]
\end{gathered}$$

Hence

$$[X_2] = [X_2[-1,0]] + [P_1] + [P_2] - [Z].$$

By twisting the first equation, we get, more generally,

$$[X_2[a,b]] = [X_2[a,b-1]] + [P_1[a,b]]$$
$$[X_2[a,b]] = [X_2[a,0]] + [P_1[a,1]] + [P_1[a,2]] + \cdots + [P_1[a,b]]$$
$$[X_2[a,b]] = [X_2[a,0]] + \sum_{i=1}^{b} P_1[a,i].$$

By twisting the second equation, we get

$$[X_2[a,0]] = [X_2[a-1,0]] + [P_1[a,0]] + [P_2[a,0]] - [Z[a,0]].$$

We note immediately the $Z[a,0] \simeq Z$ as sheaves since Z is just the structure sheaf of a point, and that similarly $P_2[a,0] = P_2$. Hence

$$[X_2[a,0]] = [X_2[a-1,0]] + [P_1[a,0]] + [P_2] - [Z]$$

$$\begin{aligned} X_2[a,0] &= [X_2] + ([P_1[1,0]] + [P_1[2,0]] + \ldots [P_1[a,0]]) + a[P_2] - a[Z] \\ &= [X_2] + \sum_{i=1}^{a} P_1[i,0] + a([P_2] - [Z]). \end{aligned}$$

Hence, more generally,

$$[X_2[a,b]] = [X_2[a,0]] + \sum_{i=1}^{b} P_1[a,i]$$

$$[X_2[a,b]] = [X_2] + \sum_{i=1}^{a} P_1[i,0] + a([P_2] - [Z]) + \sum_{i=1}^{b} P_1[a,i].$$

Symmetrically, we have

$$[X_1[a,b]] = [X_1] + \sum_{i=1}^{b} P_2[0,i] + a([P_1] - [Z]) + \sum_{i=1}^{a} P_2[i,b].$$

8.3. **Line bundles on copies of $\mathbb{P}^1$.** We note that there is an exact sequence

$$R[0,-1]/(x,y,\overline{z}) \xrightarrow{\cdot\overline{y}} R/(x,y,\overline{z}) \longrightarrow R/(x,y,\overline{y},\overline{z}) \ .$$

Hence in the Grothendieck group we have

$$[P_2] = [P_2[0,-1]] + [Z].$$

Hence

$$[P_2[a,b]] = [P_2[0,b]] = [P_2] + b[Z].$$

Symmetrically,

$$[P_1[a,b]] = [P_1] + a[Z].$$

8.4. **Putting it all together.** We have the following equations in the Grothendieck group.

$$[R[a,b]] = [R] + (\sum_{i=1}^{b}[X_2[a,i]]) + (\sum_{i=1}^{a} X_1[i,0])$$

$$[X_2[a,b]] = [X_2] + \sum_{i=1}^{a} P_1[i,0] + a([P_2] - [Z]) + \sum_{i=1}^{b} P_1[a,i]$$

$$[X_1[a,b]] = [X_1] + \sum_{i=1}^{b} P_2[0,i] + b([P_1] - [Z]) + \sum_{i=1}^{a} P_2[i,b]$$

$$[P_2[a,b]] = [P_2] + b[Z]$$

$$[P_1[a,b]] = [P_1] + a[Z]$$

Substituting upwards, we get

$$[X_1[a,b]] = [X_1] + \sum_{i=1}^{b}([P_2] + i[Z]) + b([P_1] - [Z]) + \sum_{i=1}^{a}([P_2] + b[Z])$$

$$= [X_1] + (a+b)[P_2] + b[P_1] + \frac{b(b+1)}{2}[Z] + ab[Z] - b[Z]$$

$$= [X_1] + (a+b)[P_2] + b[P_1] + (\frac{b(b-1)}{2} + ab)[Z].$$

Symmetrically,

$$[X_2] = [X_2] + a[P_2] + (a+b)[P_1] + (\frac{a(a-1)}{2} + ab)[Z].$$

And finally

$$[R[a,b]] = [R] + (\sum_{i=1}^{b}[X_2] + a[P_2] + (a+i)[P_1] + (\frac{a(a-1)}{2} + ai)[Z]) + (\sum_{i=1}^{a}[X_1] + i[P_2])$$

$$= [R] + b[X_2] + ab[P_2] + \frac{b(b+1)}{2}[P_1] + ab[P_1] + \frac{ba(a-1)}{2}[Z] + \frac{ab(b+1)}{2}[Z]$$

$$+ a[X_1] + \frac{a(a+1)}{2}[P_2].$$

Collecting like terms:

$$= [R] + a[X_1] + b[X_2] + \frac{a(a+2b+1)}{2}[P_1] + \frac{b(2a+b+1)}{2}[P_2] + \frac{ab(a+b)}{2}[Z].$$

This completes the proof. $\square$

8.5. **Change-of-basis matrices.** This gives us a general way to express line bundles on G/B in terms of the Schubert cells. In particular, it gives us the following base-change matrix between the line bundle basis $[\mathcal{O}, \mathcal{O}(-1,0), \mathcal{O}(0,-1), \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(-1,-1)]$.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 1 & 1 & -2 & -2 & -5 \\ 0 & -1 & 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

8.6. **The action as Grothendieck group automorphisms.** Equipped with these change-of-basis matrices and the formula for putting line bundles into the Schubert basis, we can now compute the action of affine braid group action on the Grothendieck group of $D^b_0(\tilde{\mathfrak{g}})$. Beginning with the basis $[\mathcal{O}, \mathcal{O}(-1,0), \mathcal{O}(0,-1), \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(-1,-1)]$, we can compute the functor in terms of line bundles, then reexpress those line bundles as linear combinations of the Schubert basis. We obtain the following table.

	$[R]$	$[X_1]$	$[X_2]$	$[P_1]$	$[P_2]$	$[Z]$
$\mathcal{O}$	-1	2	-1	1	1	-1
$\mathcal{O}(-\omega_1)$	1	1	0	0	0	0
$\mathcal{O}(-\omega_2)$	-1	2	0	-1	0	0
$\mathcal{O}(\omega_1)$	-1	3	-2	3	3	-3
$\mathcal{O}(-\omega_2)$	-1	2	-2	3	1	0
$\mathcal{O}(-\omega_1-\omega_2)$	-1	1	1	-1	-1	1

We can then use the change-of-basis matrix again to convert these back into the line bundle basis, giving the following table.

	$\mathcal{O}$	$\mathcal{O}(-\omega_1$	$\mathcal{O}(-\omega_2)$	$\mathcal{O}(\omega_1)$	$\mathcal{O}(-\omega_2)$	$\mathcal{O}(-\omega_1-\omega_2)$
$\mathcal{O}$	1	-3	0	0	0	1
$\mathcal{O}(-\omega_1)$	0	-1	0	0	0	0
$\mathcal{O}(-\omega_2)$	3	-3	0	-1	0	0
$\mathcal{O}(\omega_1)$	3	-6	-1	0	0	3
$\mathcal{O}(-\omega_2)$	-9	1	3	3	1	0
$\mathcal{O}(-\omega_1-\omega_2)$	0	0	0	0	0	-1

This is to say that the action of s_1 on the Grothendieck group (using the our basis of six line bundles in the order listed in the table) can be written as the matrix

$$s_1 = \begin{bmatrix} 1 & 0 & 3 & 3 & -9 & 0 \\ -3 & -1 & -3 & -6 & 1 & 0 \\ 0 & 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 & 0 & -1 \end{bmatrix}.$$

By symmetry, we get that s_2 must be the matrix

$$s_2 = \begin{bmatrix} 1 & 3 & 0 & -9 & 3 & 0 \\ 0 & 0 & 0 & 3 & -1 & 0 \\ -3 & -3 & -1 & 1 & -6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 & -1 \end{bmatrix}.$$

9. Failure of the correspondence

9.1. **The kernel of the pushforward $G/B \to \mathbb{P}^2$.** As discussed in the introduction, our aim is to see whether the two actions can be related by the pushforward $G/B \to \mathbb{P}^2$ by noting that this pushforward induces a homomorphism of Grothendieck groups, and seeing if the two actions of the Grothendieck groups are compatible with this homomorphism. We will here concern ourselves with specifically the pushforward forgetting the line (and preserving the plane) of the flag; this is the same as the morphism $G/B \to \mathbb{P}^2$ which would be used in the computation of of s_1 in the affine braid group action. As we saw in Section 7 (invoking Lemma 5), this pushforward is the functor mapping $\mathcal{O}(a\omega_1 + b\omega_2) \mapsto \mathcal{O}(b) \otimes R\Gamma(\mathcal{O}_{\mathbb{P}^1}(a))$.

If any subgroup of the affine braid group will be compatible with this homomorphism, it must certainly perserve the kernel of this pushforward. We can thus search for elements of the extended affine braid group whose actions on the Grothendieck group $K(D^b_0(\tilde{\mathfrak{g}})) = K(G/B)$ preserves the kernel of the pushforward $G/B \to P^2$.

To do so, we can first compute said kernel.

Lemma 13. *The kernel of the pushforward π_* between Grothendieck groups is generated by the following.*

$$\langle 3[\mathcal{O}] - [\mathcal{O}(-\omega_2)] - [\mathcal{O}(\omega_1)], [\mathcal{O}(-\omega_1)], [\mathcal{O}(-\omega_1 - \omega_2)] \rangle$$

Proof. We know that the morphism of Grothendieck group is surjective, since $[\mathcal{O}], [\mathcal{O}(-1)], [\mathcal{O}(1)]$ generate $K(\mathbb{P}^2)$ and these are the pushforwards of $\mathcal{O}, \mathcal{O}(-\omega_2), \mathcal{O}(\omega_2)$, respectively. Hence, since this is a morphism $\mathbb{Z}^6 \to \mathbb{Z}^3$, we should have a rank 3 kernel. We know that each of $[\mathcal{O}(-\omega_2)]$, $[\mathcal{O}(-\omega_1 + \omega_2]$, and $[\mathcal{O}(-\omega_1 - \omega_2)]$ all do lie within the kernel. They are moreover linearly independent, so they must generate the kernel at least as a $\mathbb{Q}$-vector space.

The Grothendieck group class of $[\mathcal{O}(-\omega_1 + \omega_2)]$ in terms of our line bundle basis is $3[\mathcal{O}] - [\mathcal{O}(-\omega_2)] - [\mathcal{O}(\omega_1)]$ (this can be computed e.g. expressing this line bundle in the Schubert basis and then base-changing back to the lie bundle basis). Thus our kernel is generated by:

$$\langle 3[\mathcal{O}] - [\mathcal{O}(-\omega_2)] - [\mathcal{O}(\omega_1)], [\mathcal{O}(-\omega_1)], [\mathcal{O}(-\omega_1 - \omega_2)] \rangle$$

as desired. □

9.2. **Results from exhaustive search.** We can now recall that the extended affine braid group is generated by s_1, s_2, ω_1, where ω_1 is the twist functor by ω_1. The twist functor by ω_1 is easy to compute as an action on the Grothendieck group, using our formula for expressing bundles in terms of the Schubert basis and then base-changing back. The resulting action is:

$$\omega_1 = \begin{bmatrix} 0 & 1 & 3 & -3 & -10 & 0 \\ 0 & 0 & -1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

The affine braid group has a natural set of generators $s_1, s_2, r = \omega_1 s_1 s_2$, and $s_0 = r s_2 r^{-1}$. Using the above matrices, we can get matrices representing each of these generators. We ran a computer program to search for products of the three matrices s_0, s_1, s_2, r, and their inverses, which preserve the subspace

$$\langle \begin{bmatrix} 3 & 0 & -1 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rangle$$

We ran a computer program to find such elements, searching for products up to thirteen elements long. Every element found was generated by the following two words.

$$\{s_1, \omega_1\}$$

The twist ω_1 descends to the Serre twist on $\mathbb{P}^2$. Meanwhile we can ask what s_1 descends to by base-changing into a basis that isolates the kernel. We can shift to the basis $[\mathcal{O}, \mathcal{O}(-\omega_2), \mathcal{O}(-\omega_1), \mathcal{O}(-\omega_1 - \omega_2), \mathcal{O}(-\omega_1 + \omega_2)]$, wherein the last three basis elements span the kernel. The base-change matrix from the line bundle basis to this basis is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Conjugating by this basis change matrix gives s_1 as:

$$s_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & -3 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 & 0 & -1 \end{bmatrix}.$$

From this it is clear that s_1, when pushed forward acts trivially on the Grothendieck group of $\mathbb{P}^1$.

Thus we get that the only nontrivial element of the affine braid group (of length less than 13 when generated by s_1, s_2, ω_1) which preserves the kernel is ω_1 itself, which descends to the Serre twist on $\mathbb{P}^2$. This is a significantly poorer correspondence than in the SL_2 case, suggesting that the generalization to higher dimensions may require a different kind of relationship.

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