Mod 2 Prismatic Cohomology of $BSpin(n)$

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Abstract

We give an algorithm for computing the mod 2 prismatic cohomology $H^{\bullet}_{\Delta/2}(BSpin(n))$ of the classifying stack of the spin group $Spin(n)$ for any n via the spectral sequence associated with the short exact sequence $1 \to \mu_2 \to Spin(n) \to SO(n) \to 1$. We explicitly describe the result of the computation for $n \leq 13$ and give a precise conjecture telling what happens for general n. If true it gives an exact formula for the discrepancy between the dimensions of the de Rham and the singular $\mathbb{F}_2$-cohomology, extending the work [Tot17] of Totaro, where he was able to show that

$$\dim_{\mathbb{F}_2} H^{32}_{dR}(BSpin(11)/\mathbb{F}_2) > \dim_{\mathbb{F}_2} H^{32}_{sing}(BSpin(11)(\mathbb{C}), \mathbb{F}_2).$$

1 Introduction

Given a smooth algebraic variety X over $\mathbb{C}$, a classical result of Grothendieck identifies the singular cohomology $H^\bullet_{sing}(X(\mathbb{C}),\mathbb{C})$ of the topological space $X(\mathbb{C})$ with the algebraic de Rham cohomology $H^\bullet_{dR}(X/\mathbb{C})$ of X. This gives a description of $H^\bullet_{sing}(X(\mathbb{C}),\mathbb{C})$ in purely algebraic terms.

If one wants to replace coefficients in $\mathbb{C}$ with some torsion group $\mathbb{F}_p$, there is also an algebraic description using the étale cohomology of X. Namely, Artin's comparison theorem identifies $H^\bullet_{sing}(X(\mathbb{C}),\mathbb{F}_p)$ with the étale cohomology $H^\bullet_{et}(X,\mathbb{F}_p)$.

However, if X has a model over $\mathbb{Z}$ which is smooth at p, there is another de Rham style cohomology theory which takes values in $\mathbb{F}_p$-vector spaces. Namely, we can consider the reduction $X_{\mathbb{F}_p}$ and take its de Rham cohomology $H^\bullet_{dR}(X_{\mathbb{F}_p}/\mathbb{F}_p)$. The natural question then is how this compares with $H^\bullet_{sing}(X(\mathbb{C}),\mathbb{F}_p)$.

For $X/\mathbb{Z}_p$ proper this question is naturally put into the framework of the integral p-adic Hodge theory developed in great detail in [BMS16] and [BS19] by Bhatt-Morrow-Scholze and Bhatt-Scholze correspondingly. Namely there exists the mod-p prismatic cohomology theory $H^\bullet_{\Delta/p}(X)$ (or rather a complex $R\Gamma_{\Delta/p}(X)$) over the mod p Kisin ring $\mathfrak{S}/p \simeq \mathbb{F}_p[[T]]$ which exactly interpolates between the two cohomology theories above: the étale $\mathbb{F}_p$-cohomology of (the generic fiber of) X roughly appear as the generic fiber of $R\Gamma_{\Delta/p}(X)$ over $\operatorname{Spec}\mathbb{F}_p[[T]]$ and it specializes to the de Rham cohomology $R\Gamma_{dR}(X_{\mathbb{F}_p}/\mathbb{F}_p)$ after taking (the derived) quotient modulo T. In particular,

$$\dim_{\mathbb{F}_p} H^i_{dR}(X/\mathbb{F}_p) \geq \dim_{\mathbb{F}_p} H^i(X(\mathbb{C}),\mathbb{F}_p)$$

for all i, and for a given i, the dimensions of $H^i_{dR}(X_{\mathbb{F}_p}/\mathbb{F}_p)$ and $H^i_{sing}(X(\mathbb{C}),\mathbb{F}_p)$ coincide if and only if there is no T-torsion both in $H^i_{\Delta/p}(X)$ and $H^{i+1}_{\Delta/p}(X)$.

In [Tot17] Totaro studied the analogous question for the classifying stack BG of a split reductive group G. In particular, (Theorem 10.2 in [Tot17]) for a given G he showed that

$$\dim_{\mathbb{F}_p} H^i_{dR}(BG/\mathbb{F}_p) = \dim_{\mathbb{F}_p} H^i_{sing}(BG(\mathbb{C}),\mathbb{F}_p)$$

outside of some (in fact very small) set of primes that depends only on the root data of G. He also showed that the equality does not hold in general, for example

$$\dim_{\mathbb{F}_2} H^{32}_{dR}(BSpin(11)/\mathbb{F}_2) > \dim_{\mathbb{F}_2} H^{32}_{sing}(BSpin(11)(\mathbb{C}),\mathbb{F}_2).$$

The idea of this paper is to apply the integral p-adic Hodge theory to understand the reason behind this discrepancy and get an understanding of its size. However, at least from the first glance, the prismatic cohomology theory is badly suited to establish such inequality, since the stack BG is not proper. In particular, having an arbitrary Artin stack $\mathcal{Y}$ over $\mathbb{Z}_p$, a priory there is no clear reason why the natural (Kan) extension of the prismatic cohomology to $\mathcal{Y}$ should give a deformation between algebraic de Rham and algebraic étale cohomology, though it indeed gives one if you take the étale cohomology of the Raynaud generic fiber instead. One of the main results of [KP] is that for BG with G reductive these two étale cohomology theories in fact coincide. Thus prismatic cohomology can be used to study the discrepancy between $\dim_{\mathbb{F}_p} H^i_{dR}(BG/\mathbb{F}_p)$ and $\dim_{\mathbb{F}_p} H^i_{sing}(BG(\mathbb{C}),\mathbb{F}_p)$.

1.1 Results of the paper and the idea behind the proof

In this paper we describe an algorithm of computing the mod 2 prismatic cohomology $H^\bullet_{\Delta/2}(BSpin(n))$ of the classifying stack of the spin group $Spin(n)$. We explicitly describe the result of the computation for $n \leq 13$ and give a conjecture for what we expect to happen for a general n. Assuming the conjecture one also gets the answer for all dimensions of the de Rham cohomology of $BSpin(n)$. In fact this leaves only one option for the algebra structure on it as well but it is not immediate and we won't discuss this here.

Generally, the idea is to adapt the computation of $H^\bullet_{sing}(BSpin(n), \mathbb{F}_2)$ done by Quillen in [Qui71] to the case of the prismatic cohomology. Namely $Spin(n)$ fits into a short exact sequence

$$1 \to \mu_2 \to Spin(n) \to SO(n) \to 1$$

and there is the associated spectral sequence

$$E_2^{p,q} = H^\bullet_{\Delta/2}(B\mu_2) \otimes H^\bullet_{\Delta/2}(BSO(n)) \Rightarrow H^\bullet_{\Delta/2}(BSpin(n)).$$

Now the key observation is that the Totaro's computation of the de Rham cohomology of $BSO(n)$ over $\mathbb{F}_2$ (Theorem 12.1 of [Tot17]) implies that $H^\bullet_{\Delta/2}(BSO(n))$ is a polynomial ring over $\mathfrak{S}/2$ with generators in degree $2, 3, \dots, n$. Moreover, restricting to a maximal elementary 2-subgroup $\Gamma \subset SO(n)$ it is possible to describe $H^\bullet_{\Delta/2}(BSO(n))$ as a Breuil-Kisin module: namely it is a sum of Breuil-Kisin twists and one can explicitly say what the weights of the generators are. $H^\bullet_{\Delta/2}(B\mu_2)$ can also be computed explicitely and it turns out to be a sum of Breuil-Kisin twists as well. Differentials in the spectral sequence should be maps in the category of Breuil-Kisin modules. Luckily, between the twists there are not that many (Lemma 3.4):

$$\mathrm{Hom}(\mathfrak{S}/2\{-i\}, \mathfrak{S}/2\{-j\}) = \begin{cases} \mathbb{F}_2, & i \geq j \\ 0, & \text{otherwise.} \end{cases}$$

There is a unique non-zero map and moreover it is enough to check that whether it is equal to zero or not after inverting T. From this it roughly follows that all maps in the spectral sequence are uniquely defined by the analogous spectral sequence in the singular cohomology. This in turn was completely described by Quillen in [Qui71].

More precisely, Quillen has shown that, applying the spectral sequence, $H^\bullet_{sing}(BSpin(n), \mathbb{F}_2)$ is obtained from $H^\bullet_{sing}(BSO(n), \mathbb{F}_2)$ as follows. Let $f_1, f_2, \dots, f_h \in H^\bullet_{sing}(BSO(n), \mathbb{F}_2)$ be defined by $f_1 := w_2$ and $f_i = Sq^{2^{i-2}} f_{i-1}$ where w_2 is the second Stiefel-Whitney class and Sq^i denotes the i-th Steenrod square. Then Quillen proved that the sequence $f_1, f_2, \dots, f_h \in H^\bullet_{sing}(BSO(n), \mathbb{F}_2)$ is regular and

$$H^\bullet_{sing}(BSpin(n), \mathbb{F}_2) \simeq H^\bullet_{sing}(BSO(n), \mathbb{F}_2)/(f_1, \dots, f_h) \otimes \mathbb{F}_2[z_{2^h}]$$

where z_{2^h} is a certain element of degree 2^h and h is roughly $n/2$ but the precise value depends on n modulo 8. The values for h are listed in Table 1, taken from [Qui71].

We expect there to be a similar formula for the mod 2 prismatic cohomology. Namely, let $u_2 := w_2^\Delta \in H^2_{\Delta/2}(BSO(n))$ be the second prismatic Stiefel-Whitney class (see Definition 4.4). Let $\widetilde{f_1}, \widetilde{f_2}, \dots, \widetilde{f_h} \in H^\bullet_{\Delta/2}(BSO(n))$ be the sequence given by $\widetilde{f_1} = u_2$ and $\widetilde{f_i} = Sq^{2^{i-2}} \widetilde{f_{i-1}}$. Note that $\deg \widetilde{f_i} = 2^{i-1} + 1$. We reduce everything to the following

Conjecture 1.1. The sequence $\widetilde{f_1}, \widetilde{f_2}, \dots, \widetilde{f_{h-1}}, T \in H^\bullet_{\Delta/2}(BSO(n))$ is regular.

n	type	h
$8l+1$	$\mathbb{R}$	$4l+0$
$8l+2$	$\mathbb{C}$	$4l+1$
$8l+3$	$\mathbb{H}$	$4l+2$
$8l+4$	$\mathbb{H}$	$4l+2$
$8l+5$	$\mathbb{H}$	$4l+3$
$8l+6$	$\mathbb{C}$	$4l+3$
$8l+7$	$\mathbb{R}$	$4l+3$
$8l+8$	$\mathbb{R}$	$4l+3$

Table 1: Value of h for each residue of $n \pmod 8$

Theorem 1.2. *Assume Conjecture 1.1. Then $\widetilde{f_1}, \widetilde{f_2}, \ldots, \widetilde{f_h} \in H^\bullet_{\Delta/2}(BSO(n))$ is a regular sequence and*

$$H^\bullet_{\Delta/2}(BSpin(n)) \simeq H^\bullet_{\Delta/2}(BSO(n))/(\widetilde{f_1}, \ldots, \widetilde{f_h}) \otimes_{\mathfrak{S}/2} \mathfrak{S}/2[\widetilde{z_{2^h}}]$$

where $\deg(\widetilde{z_{2^h}}) = 2^h$ *and* $\mathrm{wt}(\widetilde{z_{2^h}}) = 2^{h-1}$

Moreover, we have a conjecture describing the T-torsion:

Conjecture 1.3. Assume Conjecture 1.1. Then

- If $n \neq 3, 4, 5 \mod 8$, $H^\bullet_{\Delta/2}(BSpin(n))$ is T-torsion free. In particular, in this case

  $$\dim H^i_{dR}(BSpin(n)/\mathbb{F}_2) = \dim H^i_{sing}(BSpin(n)(\mathbb{C}), \mathbb{F}_2)$$

  for all i.

- If $n = 3, 4, 5 \mod 8$, the T-torsion in $H^\bullet_{\Delta/2}(BSpin(n))$ (as a graded $\mathfrak{S}/2$-module) is isomorphic to $H^\bullet_{\Delta/2}(BSO(n))/(\widetilde{f_1}, \ldots, \widetilde{f_{h-1}}, T)[-2^{h-1}-1]$. In particular

  $$\dim H^i_{dR}(BSpin(n)/\mathbb{F}_2) = \dim H^i_{sing}(BSpin(n)(\mathbb{C}), \mathbb{F}_2)$$

  for $i < 2^{h-1}$ and for general i the dicrepancy is given explicitly by the value of the Hilbert function for $H^\bullet_{\Delta/2}(BSO(n))/(\widetilde{f_1}, \ldots, \widetilde{f_{h-1}}, T)$ (shifted by $-2^{h-1}-1$), which is a quasi-polynomial of degree $n-h$, see Section 10

Note that if true the T-torsion is quite huge, but it is elementary (every element killed by T^n for some n is in fact killed by T) and this we can prove in general.

We show that the Conjectures 1.1 and 1.3 are true for $n \leq 13$ by an explicit computation. We also have some promising ideas about how to prove them in general by considering the restriction to a maximal elementary 2-subgroup $\Gamma \subset SO(n)$, which we plan to turn into a rigorous proof very soon.

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2 Notation and conventions

Starting from now on we put the prime p to be equal to 2. $\mathfrak{S}/2 \cong \mathbb{F}_2[[T]]$ will denote the mod 2 Breuil-Kisin ring. Let $\mathfrak{S}/2\{n\}$ be the n-th Breuil-Kisin twist mod 2: the twisted Frobenius action is $\varphi = T^{-n}\,\mathrm{Fr}$ where Fr is the regular Frobenius on $\mathbb{F}_2[[T]]$. The mod 2 prismatic cohomology complex of an object X will be denoted by $R\Gamma_{\Delta/2}(X)$ with individual cohomology denoted by $H^i_{\Delta/2}(X)$. $(\mathfrak{S}/2) - \mathsf{Mod}$ will denote the ∞-category of (complexes of) modules over $\mathfrak{S}/2$.

3 Overview of the (mod 2) prismatic cohomology

Let $\mathbb{C}_2 := \widehat{\overline{\mathbb{Q}}}_2$ be the completion of the algebraic closure of $\mathbb{Q}_2$ and let $\mathcal{O}_{\mathbb{C}_2} \subset \mathbb{C}_2$ be the ring of integers. Following Fontaine we define the tilt

$$\mathcal{O}^\flat_{\mathbb{C}_2} := \lim_{\mathrm{Fr}} \mathcal{O}_{\mathbb{C}_2}/2$$

This is naturally a topological ring via the discrete topology on $\mathcal{O}_{\mathbb{C}_2}/2$. Note that it is a ring of characteristic 2. As a multiplicative monoid $\mathcal{O}^\flat_{\mathbb{C}_2}$ can also be identified with the limit

$$\mathcal{O}^\flat_{\mathbb{C}_2} = \lim_{x\mapsto x^2} \mathcal{O}_{\mathbb{C}_2}$$

Let's fix a sequence $2, 2^{\frac{1}{2}}, 2^{\frac{1}{4}}, \ldots$ of 2^n-th roots of 2, then this gives an element $(2, 2^{\frac{1}{2}}, 2^{\frac{1}{4}}, \ldots)$ in $\lim_{x\mapsto x^2} \mathcal{O}_{\mathbb{C}_2} \simeq \mathcal{O}^\flat_{\mathbb{C}_2}$ which we denote by $2^\flat$. If we localise at $2^\flat$ then we get $\mathbb{C}_2^\flat := \mathcal{O}_{\mathbb{C}_2^\flat}[\frac{1}{2^\flat}]$, which is the fraction field of $\mathcal{O}^\flat_{\mathbb{C}_2}$.

We have a map $\mathfrak{S}/2 \simeq \mathbb{F}_2[[T]] \to \mathcal{O}^\flat_{\mathbb{C}_2}$ sending T to $(2^\flat)^2$. Under this map $\mathbb{C}_2^\flat$ is identified with $\widehat{\overline{\mathbb{F}_2((T))}}$, the completion of the algebraic closure of the Laurent series $\mathbb{F}_2((T))$.

Theorem 3.1 ([BS19], [KP]). *Let $\mathcal{Y}$ be a smooth Artin stack over $\mathbb{Z}_2$. Then there exists an E_∞-algebra $R\Gamma_{\Delta/2}(\mathcal{Y})$ in $(\mathfrak{S}/2) - \mathsf{Mod}$ and a Frobenius-linear endomorphism $\phi : R\Gamma_{\Delta/2}(\mathcal{Y}) \to R\Gamma_{\Delta/2}(\mathcal{Y})$ such that*

- $R\Gamma_{\Delta/2}(\mathcal{Y}) \otimes^{\mathbb{L}}_{\mathfrak{S}/2} \mathfrak{S}/(2,T) \simeq R\Gamma_{dR}(\mathcal{Y}/\mathbb{F}_2)$;
- $R\Gamma_{\Delta/2}(\mathcal{Y}) \otimes^{\mathbb{L}}_{\mathfrak{S}/2} \mathbb{C}_2^\flat \simeq R\Gamma_{et}(\widehat{\mathcal{Y}}_{\mathbb{C}_2}, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{C}_2^\flat$ *and* $R\Gamma_{et}(\widehat{\mathcal{Y}}_{\mathbb{C}_2}, \mathbb{F}_2) \simeq (R\Gamma_{\Delta/2}(\mathcal{Y}) \otimes^{\mathbb{L}}_{\mathfrak{S}/2} \mathbb{C}_2^\flat)^{\phi=1}$

Here $R\Gamma_{dR}(\mathcal{Y}/\mathbb{F}_2)$ denotes the de Rham cohomology of the reduction of $\mathcal{Y}$ modulo 2 and $R\Gamma_{et}(\widehat{\mathcal{Y}}_{\mathbb{C}_2}, \mathbb{F}_2)$ denotes the $\mathbb{F}_2$-étale cohomology of the Raynaud (geometric) generic fiber of $\mathcal{Y}$. A priory it is not clear whether $R\Gamma_{et}(\widehat{\mathcal{Y}}_{\mathbb{C}_2}, \mathbb{F}_2)$ gives anything reasonable. However, if $\mathcal{Y} = BG$ with G reductive this can be identified with the singular cohomology of its complex points:

Theorem 3.2 ([KP]). *Let G be a split reductive group and let BG be its classifying stack. Let $G(\mathbb{C})$ be the topological space of its complex points and let $BG(\mathbb{C})$ be its classifying space. Then*

$$R\Gamma_{et}(\widehat{BG}_{\mathbb{C}_2}, \mathbb{F}_2) \simeq R\Gamma_{sing}(BG(\mathbb{C}), \mathbb{F}_2)$$

We will be primarily interested in the case $G = Spin(n)$.

Under some cohomological properness assumption (satisfied by BG) each cohomology $H^i_{\Delta/2}(\mathcal{Y})$ carries a structure of a (2-torsion) Breuil-Kisin module: namely $H^i_{\Delta/2}(\mathcal{Y})$ are finitely generated over $\mathfrak{S}/2$ and there is a Frobenius-linear endomorphism $\phi : H^i_{\Delta/2}(\mathcal{Y}) \to H^i_{\Delta/2}(\mathcal{Y})$ which becomes an isomorphism after inverting T. All Breuil-Kisin modules which we will meet in this paper will be Breuil-Kisin twists.

Definition 3.3. The $(-n)$-th (mod 2) Breuil-Kisin twist $\mathfrak{S}/2\{-n\}$ has

- $\mathfrak{S}/2$ as an underlying $\mathfrak{S}/2$-module;
- $T^n\,\mathrm{Fr}$ as ϕ, where $\mathrm{Fr} : \mathfrak{S}/2 \to \mathfrak{S}/2$ is the standard Frobenius.

All Breuil-Kisin modules that we will meet will be Breuil-Kisin twists.

Lemma 3.4.

$$\mathrm{Hom}(\mathfrak{S}/2\{-i\}, \mathfrak{S}/2\{-j\}) = \begin{cases} \mathbb{F}_2, & i \geq j \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Recall that $\mathrm{Hom}(\mathfrak{S}/2\{-i\}, \mathfrak{S}/2\{-j\}) = \mathrm{Hom}(\mathfrak{S}/2, \mathfrak{S}/2\{i-j\})$ so it is enough to prove it for this special case. We will prove:

$$\mathrm{Hom}(\mathfrak{S}/2, \mathfrak{S}/2\{-k\}) = \begin{cases} \mathbb{F}_2, & k \leq 0 \\ 0, & \text{otherwise.} \end{cases}$$

To give $f\colon \mathfrak{S}/2 \to \mathfrak{S}/2\{-k\}$, it is enough to define $f(1)$ by Frobenius-linearity. Let $f(1) = P(T) = \sum_{i\geq 0} a_i T^i$. We must have

$$f(\varphi(1)) = f(1) = P(T) = \varphi(f(1)) = \sum_{i\geq 0} a_i T^{2i+k}. \tag{1}$$

As such, we have the relation $a_i = a_{(i-k)/2}$. Suppose $k > 0$, $P(T) \neq 0$. Let $d = \min_i a_i \neq 0$. Then the minimum degree on the left hand side of Equation (1) is d while the minimum on the right hand side is $2d + k > d$. Thus $P(T) = 0$.

Suppose $k \leq 0$. As above, we have the relation

$$a_i = a_{(i-k)/2} = a_{((i-k)/2-k)/2} = \dots$$

which gives $a_i = a_{(i+k)/2^n - k}$ for every n. If $i \neq -k$, then there exists n large so that $(i+k)/2^n - k \notin \mathbb{Z}$, which forces $a_i = 0$. It follows that only a_{-k} is allowed to be nonzero and if it is, it is necessarily equal to 1 (since $\mathbb{F}_2 = \{0, 1\}$). □

Remark 3.5. Let $i \geq j$. The unique non-zero map $f\colon \mathfrak{S}/2\{-i\} \to \mathfrak{S}/2\{-j\}$ is given by multiplication by T^{i-j}.

We will also need the following lemma:

Lemma 3.6. *Let v be the generator of $\mathfrak{S}/2\{-n\}$ such that $\phi(v) = T^n \cdot v$. Then ϕ-invariants in $\mathfrak{S}/2\{-n\} \otimes_{\mathfrak{S}/2} \mathbb{C}_2^\flat$ are given by $T^{-n} \cdot v$ and 0.*

Proof. Indeed, since $\mathfrak{S}/2\{-n\} \otimes_{\mathfrak{S}/2} \mathbb{C}_2^\flat \simeq \mathbb{C}_2^\flat$ as vector spaces over $\mathbb{C}_2^\flat$ and at the same moment $\mathfrak{S}/2\{-n\} \otimes_{\mathfrak{S}/2} \mathbb{C}_2^\flat \simeq (\mathfrak{S}/2\{-n\} \otimes_{\mathfrak{S}/2} \mathbb{C}_2^\flat)^{\phi=1} \otimes_{\mathbb{F}_2} \mathbb{C}_2^\flat$ by Theorem 3.1 we get that

$$\dim_{\mathbb{F}_2}(\mathfrak{S}/2\{-n\} \otimes_{\mathfrak{S}/2} \mathbb{C}_2^\flat)^{\phi=1} = 1.$$

On the other hand

$$\phi(T^{-n} \cdot v) = T^{-2n} \cdot \phi(v) = T^{-n} \cdot v,$$

so $T^{-n} \cdot v$ is a non-zero invariant vector and consequently

$$(\mathfrak{S}/2\{-n\} \otimes_{\mathfrak{S}/2} \mathbb{C}_2^\flat)^{\phi=1} = \mathbb{F}_2 \cdot (T^{-n} \cdot v)$$

. $\square$

Corollary 3.7. *Let $\mathcal{Y}$ be such that $H^\bullet_{\Delta/2}(\mathcal{Y})$ is a sum of Breuil-Kisin twists. Then $\mathbb{C}_2^\flat$ in Theorem 3.1 can be replaced by $\mathbb{F}_2((T))$. Namely $R\Gamma_{\Delta/2}(\mathcal{Y})[\frac{1}{T}] \simeq R\Gamma_{et}(\widehat{\mathcal{Y}}_{\mathbb{C}_2}, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2((T))$ and $R\Gamma_{et}(\widehat{\mathcal{Y}}_{\mathbb{C}_2}, \mathbb{F}_2) \simeq (R\Gamma_{\Delta/2}(\mathcal{Y})[\frac{1}{T}])^{\phi=1}$.*

Definition 3.8. If $\mathfrak{S}/2 \cdot v \subset H^i_{\Delta/2}(\mathcal{Y})$ forms a Breuil-Kisin submodule isomorphic to $\mathfrak{S}/2\{-n\}$ for some n we say that $v \in H^i_{\Delta/2}(\mathcal{Y})$ has weight n.

4 Prismatic cohomology of $BSO(n)$

Theorem 4.1. $H^\bullet_{\Delta/2}(BSO(n)) \cong (\mathfrak{S}/2)[u_2, \dots, u_n]$ *where u_i is in degree i.*

Proof. For any reductive group G, each cohomology group $H^i_{\Delta/2}(BG)$ is finitely generated over $\mathfrak{S}/2$ (see [KP]). From [Tot17], we know that $H^\bullet_{dR}(BSO(n))$ is a polynomial algebra with generators of degree $2, 3, \dots, n$, and the same is true for $H^\bullet_{sing}(BSO(n)(\mathbb{C}), \mathbb{F}_2)$.

By Theorem 3.1 of prismatic cohomology, we have a quasi-isomorphism

$$R\Gamma_{dR}(BSO(n)/\mathbb{F}_2) \cong R\Gamma_{\Delta/2}(BSO(n)) \otimes^{\mathbb{L}}_{\mathbb{F}_2[[T]]} (\mathbb{F}_2[[T]]/T).$$

This gives rise to a long exact sequence

$$\dots \to H^i_{dR}(BSO(n)/\mathbb{F}_2) \to H^{i+1}_{\Delta/2}(BSO(n)) \xrightarrow{\cdot T} H^{i+1}_{\Delta/2}(BSO(n)) \to H^{i+1}_{dR}(BSO(n)/\mathbb{F}_2) \to \cdots.$$

By the structure theorem for modules over a PID, if M is finitely generated over $\mathbb{F}_2[[T]]$, we have an isomorphism

$$M \cong (\mathbb{F}_2[[T]])^r \bigoplus_{i=1}^{k} \mathbb{F}_2[[T]]/T^{n_i}.$$

Define

$$\begin{aligned} r_i &\coloneqq \dim_{\mathbb{F}_2((T))} H^i_{\Delta/2}(BSO(n)) \otimes \mathbb{F}_2((T)) = \dim_{\mathbb{F}_2} H^i_{sing}(BSO(n)(\mathbb{C}), \mathbb{F}_2) \\ t_i &\coloneqq \dim_{\mathbb{F}_2} H^i_{\Delta/2}(BSO(n))[T-tors] \\ d_i &\coloneqq \dim_{\mathbb{F}_2} H^i_{\Delta/2}(BSO(n)/\ T \cdot H^i_{\Delta/2}(BSO(n)). \end{aligned}$$

Also let k_i be k in the formula above for $M = H^i_{\Delta/2}(BSO(n))$. Then $t_i = k_i$ and $d_i = r_i + k_i$ in the decomposition above. From the long exact sequence, we see that

$$\dim_{\mathbb{F}_2} H^i_{\Delta/2}(BSO(n)) \otimes^{\mathbb{L}}_{\mathbb{F}_2[[T]]} (\mathbb{F}_2[[T]]/T) = t_{i+1} + d_i.$$

Also, $BSO(n)$ is connected, so $H^0_{dR}(BSO(n)/\mathbb{F}_2) = H^0_{sing}(BSO(n)(\mathbb{C}), \mathbb{F}_2) = \mathbb{F}_2$. Then we have

$$0 = \dim_{\mathbb{F}_2} H^{-1}_{dR}(BSO(n)/\mathbb{F}_2) = t_0 + d_{-1} = t_0$$

and therefore $H^0_{\Delta/2}(BSO(n))$ is a free module over $\mathbb{F}_2[[T]]$.

Note also that if $H^i_{\Delta/2}(BSO(n))$ is free over $\mathbb{F}_2[[T]]$ if and only if $t_i = 0$ if and only if $d_i = r_i = \dim_{\mathbb{F}_2} H^i_{sing}(BSO(n)(\mathbb{C}), \mathbb{F}_2)$. We then proceed by induction. From above we know that $d_0 = \dim_{\mathbb{F}_2} H^0_{sing}(BSO(n)(\mathbb{C}), \mathbb{F}_2)$. Assume the same is true for d_i. Then we have

$$d_i + t_{i+1} = \dim H^i_{sing}(BSO(n)(\mathbb{C}), \mathbb{F}_2) + t_{i+1} = \dim H^i_{dR}(BSO(n))$$

and since the dimension of de Rham and singular cohomology are the same, we have $t_{i+1} = 0$, so $H^{i+1}_{\Delta/2}(BSO(n))$ is a free $\mathbb{F}_2[[T]]$-module as well.

To show that $H^\bullet_{\Delta/2}(BSO(n)) \cong \mathfrak{S}/2[u_2, \ldots, u_n]$, tautologically, it is enough to find elements $\widetilde{u_2}, \ldots, \widetilde{u_n} \in H^\bullet_\Delta(BSO(n))$ such that the induced map

$$\mathbb{F}_2[[T]][\widetilde{u_2}, \ldots, \widetilde{u_n}] \xrightarrow{\widetilde{u_i} \mapsto u_i} H^\bullet_\Delta(BSO(n))$$

is an isomorphism of graded rings, where the grading on the left is given by $\deg \widetilde{u_i} = i$. To check that such a map is an isomorphism, it is enough to check it modulo T. This is because both each graded components of both source and target are free finitely generated modules of the same rank, and so a map is an isomorphism if and only if its determinant is invertible, which happens if and only if it is nonzero mod T.

Thus we reduced to finding $\widetilde{u_2}, \ldots, \widetilde{u_n}$, such that

$$(\mathbb{F}_2[[T]][\widetilde{u_2}, \ldots, \widetilde{u_n}])_{(k)} \to H^k_{\Delta/2}(BSO(n)) \tag{2}$$

is an isomorphism modulo T for all $k \geq 0$. But for this we can take $\widetilde{u_i}$ to be a lift of the generators u_i in degree i of $H^\bullet_{dR}(BSO(n)/\mathbb{F}_2)$ (see [Tot17] Theorem 12.1). □

Proposition 4.2 ([KP]). *Moreover,* $\mathrm{wt}(\widetilde{u_i}) = \lfloor \frac{i}{2} \rfloor$. *In particular* $H^\bullet_{\Delta/2}(BSO(n))$ *is a direct sum of Breuil-Kisin twists.*

Remark 4.3. Proposition 4.2 provides is a nice choice of generators for $H^\bullet_{\Delta/2}(BSO(n))$. Since there is no T-torsion in $H^\bullet_{\Delta/2}(BSO(n))$, there is an embedding

$$H^\bullet_{\Delta/2}(BSO(n)) \hookrightarrow H^\bullet_{\Delta/2}(BSO(n))[T^{-1}] \simeq H^\bullet_{sing}(BSO(n), \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2((T)).$$

If we denote by w_i the elements in $H^\bullet_{\Delta/2}(BSO(n))[T^{-1}]$ corresponding to the Stiefel-Whitney classes $w_i \otimes 1 \in H^\bullet_{sing}(BSO(n), \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2((T))$, by Lemma 3.6 the elements $T^{\lfloor \frac{i}{2} \rfloor} w_i$ will give generators for $H^\bullet_{\Delta/2}(BSO(n))$.

Definition 4.4. We will call the elements $w_i^\Delta = T^{\lfloor i/2 \rfloor} w_i \in H^\bullet_{\Delta/2}(BSO(n))$ the *prismatic Stiefel-Whitney classes.*

As was noted above, we have $H^\bullet_{\Delta/2}(BSO(n)) \simeq \mathfrak{S}/2[w_2^\Delta, \ldots, w_n^\Delta]$. Also $\deg w_i^\Delta = i$ and $\mathrm{wt}\, w_i^\Delta = \lfloor i/2 \rfloor$. For convenience we also put $w_1^\Delta = 0$.

These generators are very nice because for them we can easily deduce the action of the Steenrod algebra

Proposition 4.5. *Let $0 < j \leq i$. Then*

$$\text{if } i+j \text{ is odd} \quad Sq^j(w_i^{\Delta}) = \sum_{l=0}^{j} \binom{i-l-1}{j-l} w_l^{\Delta} w_{i+j-l}^{\Delta} \tag{3}$$

$$\text{if } i+j \text{ is even} \quad Sq^j(w_i^{\Delta}) = T \cdot \sum_{l=0}^{j} \binom{i-l-1}{j-l} w_l^{\Delta} w_{i+j-l}^{\Delta} \tag{4}$$

Proof. This follows from the Wu's formula for the usual Stiefel-Whitney classes plus the Frobenius-linearity of Steenrod squares over $\mathbb{F}_2[[T]]$. □

The appearence of a T-factor in the action by Steenrod squares in the even case is exactly what (for some n) produces a non-trivial T-torsion in the prismatic cohomology of $BSpin(n)$.

5 Mod 2 prismatic cohomology of $B\mu_2$

Proposition 5.1 ([KP])**.**

$$H^{\bullet}_{\Delta/2}(B\mu_2) \cong \mathfrak{S}/2[v, c]/(v^2 = T \cdot c)$$

where $\deg v = 1$, $\deg c = 2$ *and* $\mathrm{wt}\, c = \mathrm{wt}\, v = 1$.

6 Mod 2 prismatic cohomology of $BSpin(n)$

There is a short exact sequence

$$1 \to B\mu_2 \to BSpin(n) \to BSO(n) \to 1.$$

For a given n there is associated a number h, see Table 1 in the introduction. The way we are going to compute the cohomology is via a spectral sequence. Namely,

Proposition 6.1 ([KP])**.** *There is a spectral sequence of graded algebras*

$$E_2^{p,q} = H^p_{\Delta/2}(BSO(n)) \otimes H^q_{\Delta/2}(B\mu_2) \Rightarrow H^{p+q}_{\Delta/2}(BSpin(n)).$$

Let $v, c \in H^{\bullet}_{\Delta/2}(B\mu_2)$ be the generators of the first column. Firstly, $d_2(v) \in H^{\bullet}_{\Delta/2}(BSO(n))$ should have degree 2 and weight 1 and should be non-zero since it is non-zero after localizing T (see [Qui71]), so there is only one option for it, namely $d_2(v) = w_2^{\Delta}$. There is no T-torsion on the second page, so all computations can be done after localizing T. In particular

$$d_2(c) = d_2(T^{-1}v^2) = T^{-1}d_2(v^2) = 0$$

Obviously d_2 kills the first row and this (together with the discussion above) extends it uniquely to the whole 2nd page via the Leibnitz rule. This shows that

$$E_3^{\bullet,\bullet} = \mathfrak{S}/2[w_2^{\Delta}, \ldots, w_n^{\Delta}]/(w_2^{\Delta}) \otimes_{\mathfrak{S}/2} \mathfrak{S}/2[c]$$

Let $\widetilde{f_1}, \dots, \widetilde{f_h} \in H^\bullet_{\Delta/2}(BSO(n)) \simeq \mathfrak{S}/2[w_2^\Delta, \dots, w_n^\Delta]$ be a sequence of polynomials defined by $\widetilde{f_1} := w_2^\Delta$ and $\widetilde{f_i} := Sq^{2^{i-2}}\widetilde{f_{i-1}}$. Let also $\bar{f}_1, \dots \bar{f}_h \in \mathbb{F}_2[w_2^{dR}, \dots, w_n^{dR}] \simeq H^\bullet_{dR}(BSO(n)/\mathbb{F}_2)$ be their reduction mod T (with $w_i^{dR} = w_i^\Delta \mod T$ correspondingly).

We assume the following

Conjecture 6.2. The sequence $\bar{f}_1, \dots, \bar{f}_{h-1} \in \mathfrak{S}/2[w_2^{dR}, \dots, w_n^{dR}]$ is regular.

Note that this is equivalent to Conjecture 1.1. Indeed from 4.5 and the Cartan formula we see that $\widetilde{f_1}, \dots, \widetilde{f_h}$ in fact lie in $\mathbb{F}_2[T][w_2^\Delta, \dots, w_n^\Delta] \subset \mathbb{F}_2[[T]][w_2^\Delta, \dots, w_n^\Delta]$. If instead of grading by the cohomological degree we consider grading by the weight, then all $\widetilde{f_1}, \dots, \widetilde{f_h}, T$ will be homogeneous. In particular $\widetilde{f_1}, \dots, \widetilde{f_{h-1}}, T$ is regular if and only if $T, \widetilde{f_1}, \dots, \widetilde{f_{h-1}}$ is regular which exactly means that $\bar{f}_1, \dots, \bar{f}_{h-1} \in \mathfrak{S}/2[w_2^{dR}, \dots, w_n^{dR}]$ is regular.

Corollary 6.3. *The sequence $\widetilde{f_1}, \dots, \widetilde{f_h} \in \mathfrak{S}/2[w_2^\Delta, \dots, w_n^\Delta]$ is regular.*

Proof. We know that this is true after inverting T by [Qui71]. We then proceed by induction. Assume that $\widetilde{f_1}, \dots, \widetilde{f_i}$ are regular. Then we need to show that $\widetilde{f_{i+1}}$ is not a zero divisor in $\mathfrak{S}/2[w_2^\Delta, \dots, w_n^\Delta]/(\widetilde{f_1}, \dots, \widetilde{f_i})$. Let $K \subset \mathfrak{S}/2[w_2^\Delta, \dots, w_n^\Delta]/(\widetilde{f_1}, \dots, \widetilde{f_i})$ be the kernel of multiplication by $\widetilde{f_{i+1}}$. But then $K \otimes_{\mathfrak{S}/2} \mathbb{F}_2((T)) = 0$ and so K is T-torsion. $\bar{f}_1, \dots, \bar{f}_i \in \mathfrak{S}/2[w_2^{dR}, \dots, w_n^{dR}]$ is regular, so by the discussion above there is no T-torsion in $\mathfrak{S}/2[w_2^\Delta, \dots, w_n^\Delta]/(\widetilde{f_1}, \dots, \widetilde{f_i})$ and we are done. □

Proposition 6.4. *Assume Conjecture 6.2. Then for all $r \le h$ the 2^r+1-th sheet $E^{\bullet,\bullet}_{2^r+1}$ looks like*

$$E^{\bullet,\bullet}_{2^r+1} \simeq \mathfrak{S}/2[w_2^\Delta, \dots, w_n^\Delta]/(\widetilde{f_1}, \dots, \widetilde{f_r}) \otimes_{\mathfrak{S}/2} \mathfrak{S}/2[c^{2^{r-1}}].$$

Moreover if $r < h$, it does not have any T-torsion.

Proof. The argument is completely analogous to Quillen's. See [Qui71]. The absense of T-torsion follows from the regularity of the sequence $\bar{f}_1, \dots, \bar{f}_r$. □

Consider the map $i^*: H^\bullet_{\Delta/2}(BSpin(n)) \to H^\bullet_{\Delta/2}(B\mu_2)$ dual to the embedding $\mu_2 \hookrightarrow Spin(n)$.

Lemma 6.5. *$c^{2^{h-1}}$ lies in the image of i^*.*

Proof. Let's take the Spin representation $\theta : Spin(n) \to SO(2^h)$. Then the restriction to μ_2 is given by the standard 1-dimensional representation $\mu_2 \to \mathbb{G}_m$ with multiplicity 2^h. Then the pull-back of $i^*\theta^* w_{2^h}^\Delta$ of $w_{2^h}^\Delta \in H^\bullet_{\Delta/2}(BSO(2^h))$ is non-zero (since it is non-zero after inverting T) and should have degree 2^h and weight 2^{h-1}. Thus it should be equal to $c^{2^{h-1}}$ and consequently $c^{2^{h-1}} \in \mathrm{Im}(i^*)$. □

This means that $d_{2^h+1}(c^{2^{h-1}}) = 0$ and starting from this moment the spectral sequence stabilizes. Thus, fixing any homogeneous (with respect to the weight) lift z_{2^h} of $c^{2^{h-1}}$ to $H^\bullet_{\Delta/2}(BSpin(n))$ (e.g. $z_{2^h} = \theta^* w_{2^h}^\Delta$) we get

Theorem 6.6. *Assuming the Conjecture 6.2*

$$H^\bullet_{\Delta/2}(BSpin(n)) \simeq S/2[w_2^\Delta, \dots, w_n^\Delta]/(\widetilde{f_1}, \dots, \widetilde{f_h}) \otimes_{\mathfrak{S}/2} \mathfrak{S}/2[z_{2^h}].$$

7 Description of the T-torsion

To describe the T-torsion it remains to understand by what power of T we can divide $\widetilde{f_h}$ inside $S/2[w_2^\Delta,\dots,w_n^\Delta]/(\widetilde{f_1},\dots,\widetilde{f_{h-1}})$. We give the equivalent description in terms of monomial expressions for $\widetilde{f_h}$.

Definition 7.1. Let $w_I^\Delta = w_{i_1}^\Delta w_{i_2}^\Delta \cdots w_{i_n}^\Delta$ be a monomial (with repeats possibly among the i_k). We have $\deg(w_I^\Delta) = \sum i_k$ and $\mathrm{weight}(w_I^\Delta) = \sum \lfloor i_k/2 \rfloor$. We will call the weight of a homogeneous polynomial $f \in \mathfrak{S}/2[w_2^\Delta,\dots,w_n^\Delta]$ the maximum of the weights of the monomials in it. Let I be a homogenous ideal with respect to degree. The weight of a polynomial f modulo I is the minimum of the weights of all representatives of f modulo I.

The following lemma is crucial but trivial to prove.

Lemma 7.2. *Let f be a homogeneous polynomial in w_i^Δ. Let I be a homogenous ideal. Let d be the degree of f. Then*

- $0 \le \mathrm{weight}(f) \le \lfloor \deg(f)/2 \rfloor$
- $f = T^{\lfloor \deg(f)/2 \rfloor - \mathrm{weight}(f)} g$ *where* $g \notin (T)$.

We consider $I = (\widetilde{f_1},\dots,\widetilde{f_k})$ and $f = \widetilde{f_{k+1}}$. In particular if $\widetilde{f_{k+1}}$ is not divisible by T modulo $(\widetilde{f_1},\dots,\widetilde{f_k})$ (e.g. if $\bar{f}_1,\dots \bar{f}_{k+1}$ is a regular sequence) it means that any representative of $\widetilde{f_{k+1}}$ modulo I has a monomial with no more than 1 odd variable. After applying Sq^i for some even i by Proposition 4.5 and the Cartan formula we get that the number of odd indices is at most 3 and so by Lemma 7.2 $\widetilde{f_{k+2}}$ can't be be divisible by T^2 in $\mathfrak{S}/2[w_2^\Delta,\dots,w_n^\Delta]/(\widetilde{f_1},\dots,\widetilde{f_{k+1}})$. This shows that there is only elementary T-torsion in $H^\bullet(BSO(n))$. Unfortunately for now we can't say for sure when we have T-torsion there at all. However we conjecture the following:

Conjecture 7.3.

- If $n \neq 3,4,5 \mod 8$, $H^\bullet_{\Delta/2}(BSpin(n))$ is T-torsion free. In particular, in this case

$$\dim H^i_{dR}(BSpin(n)/\mathbb{F}_2) = \dim H^i_{sing}(BSpin(n)(\mathbb{C}),\mathbb{F}_2)$$

for all i.

- If $n = 3,4,5 \mod 8$, the T-torsion in $H^\bullet_{\Delta/2}(BSpin(n))$ (as a graded $\mathfrak{S}/2$-module) is isomorphic to $H^\bullet_{\Delta/2}(BSO(n))/(\widetilde{f_1},\dots,\widetilde{f_{h-1}},T)[-2^{h-1}-1]$. In particular

$$\dim H^i_{dR}(BSpin(n)/\mathbb{F}_2) = \dim H^i_{sing}(BSpin(n)(\mathbb{C}),\mathbb{F}_2)$$

for $i < 2^{h-1}$ and for general i the dicrepancy is given explicitly by the value of the Hilbert function for $H^\bullet_{\Delta/2}(BSO(n))/(\widetilde{f_1},\dots,\widetilde{f_{h-1}},T)$ (shifted by $-2^{h-1}-1$), which is a quasi-polynomial of degree $n-h$ see Section 10 for more detail.

8 Verification of Conjecture 7.3 assuming Conjecture 6.2 for $n \le 13$

Further we present some computations of the Steenrod polynomials $\widetilde{f_i}$ and prove the conjectures above for $n \le 13$. In what follows $w_i = w_i^\Delta$. If $i > n$, then we put $w_i = 0$. For each $\widetilde{f_i}$ we pick the nicest representative. We have:

$$\widetilde{f}_2 = w_3, \qquad \widetilde{f}_3 = w_5, \qquad \widetilde{f}_4 = w_9.$$

Then, if $n \geq 10$

n	$\widetilde{f}_5$	$\widetilde{f}_6$	$\widetilde{f}_7$
10	w_7w_{10}	-	-
11	$w_6w_{11} + w_7w_{10}$	$T \cdot (w_4w_7w_{11}^2 + w_7^2w_8w_{11} + w_{11}^3)$	-
12	$w_6w_{11} + w_7w_{10}$	$T \cdot (w_4w_7w_{11}^2 + w_7^2w_8w_{11} + w_7^3w_{12} + w_{11}^3)$	-

Note that $\widetilde{f}_6$ is exactly $\widetilde{f}_h$ for $n = 11, 12$ which confirms Conjecture 7.3 in these cases. Also if $n \leq 10$ there is no T-torsion at all.

For $n = 13$,

$$Sq_4w_2 = w_4w_{13} + w_6w_{11} + w_7w_{10}$$

$$Sq_5w_2 = w_4w_6w_{11}w_{12} + w_4w_7w_{10}w_{12} + Tw_4w_7w_{11}^2 + w_6w_7w_{10}^2 + Tw_6w_7^2w_{13} + w_6^2w_8w_{13} + w_6^2w_{10}w_{11} + Tw_7w_{13}^2 + Tw_7^2w_8w_{11} + Tw_7^3w_{12} + w_{10}^2w_{13} + Tw_{11}^3$$

$$Sq_6w_2 = T^2w_4w_6w_7^2w_8w_{11}^3 + Tw_4w_6w_7^2w_{10}^3w_{11} + Tw_4w_6w_7^3w_{10}w_{12}^2 + T^2w_4w_6w_7^3w_{11}^2w_{12} + Tw_4w_6^2w_7w_8w_{11}^2w_{12} + Tw_4w_6^2w_7w_{10}^2w_{11}^2 + T^2w_4w_7^3w_8w_{10}w_{11}^2 + Tw_4w_7^3w_8w_{10}^2w_{12} + T^2w_4w_7^4w_{10}w_{11}w_{12} + Tw_4^2w_7^2w_8w_{11}w_{12}^2 + Tw_4^2w_7^2w_{10}^2w_{11}w_{12} + Tw_4^2w_7^3w_{12}^3 + Tw_4^2w_{11}^3w_{12}^2 + Tw_4^3w_7w_{11}^2w_{12}^2 + T^2w_6w_7w_{13}^4 + Tw_6w_7^2w_{10}^2w_{12}w_{13} + T^2w_6w_7^2w_{11}^3w_{12} + Tw_6w_7^3w_8w_{10}^3 + T^2w_6w_7^3w_{12}w_{13}^2 + T^2w_6w_7^4w_8w_{10}w_{13} + T^2w_6w_7^4w_8w_{11}w_{12} + T^2w_6w_7^4w_{10}^2w_{11} + Tw_6^2w_7w_8w_{12}w_{13}^2 + Tw_6^2w_7w_{10}w_{11}w_{12}w_{13} + Tw_6^2w_7w_{10}^2w_{13}^2 + Tw_6^2w_7w_{11}^2w_{12}^2 + T^2w_6^2w_7w_{11}^3w_{13} + Tw_6^2w_7^2w_8^2w_{10}w_{13} + Tw_6^2w_7^2w_8^2w_{11}w_{12} + T^2w_6^2w_7^3w_8w_{11}w_{13} + Tw_6^2w_7^3w_8w_{12}^2 + Tw_6^2w_7^3w_{10}^2w_{12} + Tw_6^2w_8w_{10}w_{11}^2w_{13} + Tw_6^2w_8w_{11}^3w_{12} + Tw_6^2w_{10}^2w_{11}^3 + Tw_6^3w_7w_8w_{10}w_{11}^2 + Tw_6^3w_7^2w_{10}^2w_{13} + Tw_6^3w_7^2w_{11}^3 + Tw_6^4w_7w_8w_{13}^2 + Tw_6^4w_7w_{10}w_{11}w_{13} + Tw_6^4w_7w_{11}^2w_{12} + T^2w_7w_{10}w_{11}^2w_{13}^2 + Tw_7w_{10}^2w_{12}w_{13}^2 + Tw_7^2w_8w_{10}^2w_{11}w_{12} + Tw_7^2w_8w_{10}^3w_{13} + Tw_7^2w_{10}^4w_{11} + T^2w_7^3w_8w_{10}w_{13}^2 + T^2w_7^3w_{10}w_{11}^2w_{12} + T^2w_7^3w_{10}^2w_{11}w_{13} + T^2w_7^4w_8^3w_{13} + T^2w_7^4w_{12}^2w_{13} + T^2w_7^5w_8w_{10}w_{12} + T^2w_8w_{11}^4w_{13} + Tw_{10}^2w_{11}^3w_{12} + Tw_{10}^3w_{11}^2w_{13} + T^2w_{13}^5$$

Note that in this case $h = 7$ and we see that $\widetilde{f}_7$ is divisible by T. So Conjecture 7.3 is confirmed.

9 Verification of Conjecture 6.2 for $n \leq 13$

9.1 About Gröbner bases.

We first recall some material about Gröbner bases. Material from this section is from [Eis08], Chapter 15.

Let S be a polynomial ring over a field, $S = k[x_1, \ldots, x_n]$. Fix an order on the monomials of S. Let F be a finitely generated free module over S with basis $\{e_i\}$. Choose an order on monomial basis elements of F, elements of the form x^Ae_i where x^A is a monomial in S.

Definition 9.1. If $>$ is a monomial order, then for any $f \in F$ we define the initial term, $in_>(f)$ to be the smallest term of f with respect to the order $>$. If M is a submodule of F then $in_>(M)$ is the submodule generated by $in_>(f)$ for all $f \in M$.

Going forward, we'll just write $in(f)$ if the order is clear from context.

Definition 9.2. If $f, g_1, \dots, g_t \in F$, then a standard expression is one of the form

$$f = \sum f_i g_i + f'$$

with $f' \in F$, $f_i \in S$, where none of the monomials of f' are in $(in(g_1), \dots, in(g_t))$, and $in(f) \leq in(f_i g_i)$.

Proposition 9.3. *Standard expressions always exist.*

The proof consists of an algorithm where one repeatedly removes monomials from f' by subtracting a suitable multiple of g_j. Details can be found in [Eis08], Division Algorithm 15.7.

Definition 9.4. A Gröbner basis with respect to an order $>$ on F is a set of elements $g_1, \dots, g_t \in F$ such that if M is the submodule generated by $g_1, \dots, g_t$ then $in_>(g_1), \dots, in_>(g_t)$ generate $in_>(M)$.

Let $g_1, \dots, g_t$ be elements of F. Let $\oplus S\varepsilon_i$ be a free module with basis $\{\varepsilon_i\}$ corresponding to the elements $\{g_i\}$ of F and let

$$\varphi\colon \oplus S\varepsilon_i \to F; \qquad \varepsilon_i \mapsto g_i \tag{5}$$

be the corresponding map of modules. For each pair of indices i, j such that $in(g_i)$ and $in(g_j)$ involve the same basis element of F, we define

$$m_{ij} = in(g_i)/GCD(in(g_i), in(g_j)) \in S$$

and we set $\sigma_{ij} = m_{ij}\varepsilon_i - m_{ij}\varepsilon_j$. For each pair i, j, we choose a standard expression

$$m_{ji}g_i - m_{ij}g_j = \sum f_u^{(ij)} g_u + h_{ij}$$

with respect to $g_1, \dots, g_t$.

Theorem 9.5 (Buchberger)**.** *The elements $g_1, \dots, g_t$ form a Gröbner basis if and only if $h_{ij} = 0$ for all i, j.*

Theorem 9.6 (Schreyer)**.** *Using the set-up above, further assume that $g_1, \dots, g_t$ form a Gröbner basis. Define*

$$\tau_{ij} = m_{ji}\varepsilon_i - m_{ij}\varepsilon_j - \sum_u f_u^{(ij)} \varepsilon_u,$$

Then the τ_{ij} generate $\ker(\varphi)$ *(c.f. Equation* (5)*).*

Proofs can be found in [Eis08], Theorems 15.8 and 15.10.

9.2 Verification of Conjecture 6.2

In this section we verify Conjecture 6.2 for $n \leq 13$. Here we put $w_i = w_i^{dR}$ and consider $\bar{f}_i \in \mathbb{F}_2[w_2, \dots, w_n]$.

It is clear that

$$\bar{f}_1 = w_2, \qquad \bar{f}_2 = w_3, \qquad \bar{f}_3 = w_5, \qquad \bar{f}_4 = w_9, \qquad \bar{f}_5 = w_4w_{13} + w_6w_{11} + w_7w_{10}$$

is regular. It remains to prove that $\bar{f}_6$ is not a zero divisor modulo the previous elements.

Using Buchberger's Criterion ([Eis08], Theorem 15.8), one can verify that

$$w_4w_6w_{11}w_{12} + w_4w_7w_{10}w_{12} + w_6^2w_8w_{13} + w_6^2w_{10}w_{11} + w_6w_7w_{10}^2 + w_{10}^2w_{13}$$

$$w_4w_{13} + w_6w_{11} + w_7w_{10}$$

$$w_6^2w_8w_{13}^2 + w_6^2w_{10}w_{11}w_{13} + w_6^2w_{11}^2w_{12} + w_6w_7w_{10}^2w_{13} + w_7^2w_{10}w_{12} + w_{10}^2w_{13}^2$$

is a Gröbner basis for the ideal $J = (\bar{f}_5, \bar{f}_6)$. Using Schreyer's theorem ([Eis08], Theorem 15.10), one verifies that the module of syzygies, expressed as the kernel of the map

$$\mathbb{F}_2[w_4, w_6, w_7, w_8, w_{10}, w_{11}, w_{12}, w_{13}]\langle\varepsilon_1, \varepsilon_2\rangle \xrightarrow{\varepsilon_1 \mapsto \bar{f}_5, \varepsilon_2 \mapsto \bar{f}_6} \mathbb{F}_2[w_4, w_6, w_7, w_8, w_{10}, w_{11}, w_{12}, w_{13}]$$

is generated by $\bar{f}_6\varepsilon_1 - \bar{f}_5\varepsilon_2$. As such, it follows that the sequence $\bar{f}_1, \ldots, \bar{f}_6$ is regular.

10 Description of the Hilbert function of $H^\bullet_{dR}(BSO(n)/\mathbb{F}_2)/(\bar{f}_1, \ldots, \bar{f}_{h-1})$

We give a partial description of the Hilbert series for $H^\bullet_{\Delta/2}(BSO(n))/(\widetilde{f_1}, \ldots, \widetilde{f_{h-1}}, T)$. First, note that $\widetilde{f_i}, T$ are homogenous elements with respect to the weight, therefore regularity of the sequence does not depend on the order of elements. With this in mind, we first find the Hilbert series of $H^\bullet(BSO(n))/T \cong \mathbb{F}_2[u_2, \ldots, u_n]$. The Hilbert series of $\mathbb{F}_2[u_i]$ for $\deg(u_i) = i$ is $(1 - t^i)^{-1}$. Since Hilbert series of tensor products multiply, we have

$$f_M(t) \coloneqq \mathrm{Hilb}\left(H^\bullet(BSO(n))/T\right) = \prod_{i=2}^{n}(1 - t^i)^{-1}.$$

Write

$$M = H^\bullet(BSO(n))/T) \qquad N = H^\bullet_{\Delta/2}(BSO(n))/(\widetilde{f_1}, \ldots, \widetilde{f_{h-1}}, T)$$

Now by regularity of the sequence, there is a Koszul resolution

$$0 \to \bigwedge^{h-1} M^{h-1} \to \bigwedge^{h-2} M^{h-1} \to \cdots \to \bigwedge^{2} M^{h-1} \to M^{h-1} \to M \to N \to 0$$

which allows us to compute the Hilbert function of N. Another description of the Hilbert function can be obtained as follows.

Let

$$P(M, t) = \prod_{i=2}^{n}(1 - t^i)^{-1}$$

and define recursively

$$P_0 = P(M, t), \qquad P_i = P_{i-1}(M, t)/P_{i-1}(M, t - 2^{i-1} + 1).$$

Define $M_i = M/(\widetilde{f_0}, \ldots, \widetilde{f_{i-1}})$. From the exact sequences

$$0 \to f_iM_i \to M_i \to M_i/f_iM_i \to 0$$

we see by induction that the Poincaré series of N is given by $P(N, t) = P_{h-2}(M, t)$. If we write

$$P(N, t) = \sum \dim N_{(n)}t^n$$

then according to [Bav95], Theorem 2.2, $\dim N_{(n)}$ is a quasi-polynomial in n, i.e. there exist $n!$ polynomials $p_0, \ldots, p_{n!-1}$ such that $\dim N_{(n)} = p_k(n)$ for $n \equiv k \pmod{n!}$. Moreover, these polynomials have the same degree $n - h$ and the same leading coefficient.

11 References

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