Dual Pairs in Complex Reductive Groups

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Abstract. In Roger Howe's 1989 paper [5], Howe introduces the notion of a *dual pair of Lie subalgebras* – a pair $(\mathfrak{g}_1, \mathfrak{g}_2)$ of reductive Lie subalgebras of a Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are each other's centralizers in $\mathfrak{g}$. This notion has a natural analog for algebraic groups; namely, a *dual pair of subgroups* is a pair (G_1, G_2) of reductive subgroups of an algebraic group G such that G_1 and G_2 are each other's centralizers in G. This paper presents substantial progress towards classifying the dual pairs of the complex classical groups ($GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $Sp(2n,\mathbb{C})$, $O(n,\mathbb{C})$, and $SO(n,\mathbb{C})$) and their projective counterparts ($PGL(n,\mathbb{C})$, $PSp(2n,\mathbb{C})$, $PO(n,\mathbb{C})$, $PSO(n,\mathbb{C})$). The classification of dual pairs in $Sp(2n,\mathbb{C})$ already exists in the literature (see [6, Chapter 2]) and follows easily from Howe's analysis in [5]; the classifications of dual pairs in $GL(n,\mathbb{C})$ and $O(n,\mathbb{C})$ are also likely known, but an explicit treatment is lacking. In this paper, we provide a straightforward presentation of the classifications of dual pairs in $GL(n,\mathbb{C})$, $Sp(2n,\mathbb{C})$, and $O(n,\mathbb{C})$ using basic techniques from representation theory. Additionally, we classify the dual pairs in $SL(n,\mathbb{C})$ and $SO(n,\mathbb{C})$, and present partial progress towards classifying the dual pairs in $PGL(n,\mathbb{C})$ and $PSp(2n,\mathbb{C})$.

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra with reductive Lie subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. If $\mathfrak{g}_1$ is the centralizer of $\mathfrak{g}_2$ in $\mathfrak{g}$ and $\mathfrak{g}_2$ is the centralizer of $\mathfrak{g}_1$ in $\mathfrak{g}$, then $(\mathfrak{g}_1, \mathfrak{g}_2)$ is said to be a *dual pair of Lie subalgebras* in $\mathfrak{g}$. This notion of duality was introduced by Roger Howe in his seminal 1989 paper [5], and has a natural analog for algebraic groups:

Definition 1.1 ([5]). A pair (G_1, G_2) of reductive subgroups of an algebraic group G form a *dual pair of subgroups* (or simply a *dual pair*) in G if $C_G(G_1) = G_2$ and $C_G(G_2) = G_1$.

While the dual pairs in Lie algebras have been extensively studied (see, for instance, [5] or [8]), the dual pairs in algebraic groups have received significantly less attention. In fact, the only existing treatment of this latter problem in the literature appears to be the observation that Howe's analysis in [5] provides a classification of the dual pairs in $Sp(2n,\mathbb{C})$ (see [6, Chapter II]). In this instance, Howe's treatment of dual pairs in certain Lie algebras gives rise to a classification of the dual pairs in $Sp(2n,\mathbb{C})$. However, it is not, in general, the case that the dual pairs in an algebraic group G can be understood by looking at the dual pairs in the Lie algebra associated to G. Nonetheless, we expect that the classifications of dual pairs in $GL(n,\mathbb{C})$ and $O(n,\mathbb{C})$ are likely known to (or at least easily-derivable for) anyone who has studied dual pairs in Lie algebras. However, since explicit classifications of the dual pairs in these groups appear to be missing from the literature, we provide straightforward classifications of the dual pairs in $GL(n,\mathbb{C})$, $Sp(2n,\mathbb{C})$, and $O(n,\mathbb{C})$ in Sections 3, 4, and 5, respectively.

In addition to being a natural analog of the well-studied notion of dual pairs in Lie algebras, the topic of dual pairs in algebraic groups is made interesting by its potential to play an important role in the study of nilpotent orbits in complex semisimple Lie algebras. To see this, let H be a complex reductive algebraic group, and let $\varphi : H \to G$ be a homomorphism of algebraic groups. Then

$$G^{\varphi} := \{g \in G \ : \ g\varphi(x)g^{-1} = \varphi(x) \ \forall x \in H\}$$

is a reductive algebraic group and is usually disconnected. Since G^{φ} is usually disconnected, it cannot be completely understood using based root datum and the structure theory of connected reductive algebraic groups. However, the following fact shows that $(G^{\varphi}, C_G(G^{\varphi}))$ is a dual pair in G:

Fact 1.2. Let G be a group, and $S \subseteq G$ a subset. Then $C_G(C_G(C_G(S))) = C_G(S)$, where $C_G(S)$ denotes the centralizer of S in G.

Moreover, we note that all dual pairs in G arise in this way:

Remark 1.3. Let G be a complex reductive algebraic group. Then any dual pair (G_1, G_2) in G can be written in the form $(G^{\varphi}, C_G(G^{\varphi}))$. Indeed, take φ to be the inclusion $G_2 \hookrightarrow G$. We get $G_1 = C_G(\varphi(G_2)) =: G^{\varphi}$ and $G_2 = C_G(G_1) = C_G(G^{\varphi})$.

Consequently, a classification of the dual pairs in G would determine the possibilities for the pairs $(G^{\varphi}, C_G(G^{\varphi}))$. Since groups of the form G^{φ} are crucial for understanding the structure of nilpotent orbits in complex semisimple Lie algebras (see [1]), this speaks to the importance of classifying dual pairs. Although classifying dual pairs in an arbitrary complex reductive algebraic group appears to be a very difficult problem, this classification problem becomes much more manageable when G is taken to be a classical group ($GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $Sp(2n,\mathbb{C})$, $O(n,\mathbb{C})$, or $SO(n,\mathbb{C})$) or a complex projective classical group ($PGL(n,\mathbb{C})$, $PSp(2n,\mathbb{C})$, $PO(n,\mathbb{C})$, or $PSO(n,\mathbb{C})$).

We start in Section 2, where we discuss some embeddings of dual pairs, which will help the reader better understand the proofs and examples in the remainder of the paper. In Sections 3, 4, and 5, we present classifications of the dual pairs in $GL(n,\mathbb{C})$, $Sp(2n,\mathbb{C})$, and $O(n,\mathbb{C})$, respectively. In Section 6, we discuss how the dual pairs in a complex reductive algebraic group G relate to the dual pairs in certain subgroups of G; this sets us up for our classifications of the dual pairs in $SL(n,\mathbb{C})$ and $SO(n,\mathbb{C})$, which we complete in Sections 7 and 8, respectively. We proceed in Section 9 to discuss how the dual pairs in a complex reductive algebraic group G relate to the dual pairs in certain quotients of G; this discussion prepares us for Sections 10 and 11, in which we present progress towards classifying the dual pairs in $PGL(n,\mathbb{C})$ and $PSp(2n,\mathbb{C})$, respectively.

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2. Embeddings

Let U be a finite dimensional complex vector space. In the remainder of the paper, we use tools from representation theory to classify dual pairs in various complex classical groups and their projective counterparts. For instance, in Corollary 3.3, we prove that the dual pairs in $GL(U)$ are exactly the pairs of groups of the form

$$\left(\prod_{i=1}^{r} GL(V_i),\ \prod_{i=1}^{r} GL(W_i) \right),$$

where $U = \bigoplus_{i=1}^{r} V_i \otimes W_i$ is a vector space decomposition of U. However, the proof of Corollary 3.3 does not describe the embeddings of $\prod_{i=1}^{r} GL(V_i)$ and $\prod_{i=1}^{r} GL(W_i)$ into $GL(U)$. In fact, very few of the other proofs that follow make use of particular embeddings of members of dual pairs as subgroups of classical groups (or projective counterparts). Although these embeddings do not play a crucial role for our proofs, a good understanding of these embeddings is likely to help the reader better understand some of the proofs and examples that follow. For this reason, we describe these embeddings here in detail.

2.1. $GL(U)$**.** Let U, V, and W be finite dimensional complex vector spaces such that $U = V \otimes W$. Write $n := \dim V$ and $m := \dim W$, and let $A = (a_{ij}) \in GL(V)$ and $B = (b_{k\ell}) \in GL(W)$. We define the embedding $\iota : GL(V) \to GL(V \otimes W)$ by

$$A \overset{\iota}{\mapsto} \begin{pmatrix} a_{11} I_m & \cdots & a_{1n} I_m \\ \vdots & \ddots & \vdots \\ a_{n1} I_m & \cdots & a_{nn} I_m \end{pmatrix} \in GL(V \otimes W)$$

and the embedding $\kappa : GL(W) \to GL(V \otimes W)$ by

$$B \overset{\kappa}{\mapsto} \begin{pmatrix} B & & \\ & \ddots & \\ & & B \end{pmatrix} \in GL(V \otimes W),$$

where the image of B under κ has n copies of B along its diagonal. If instead $U = \bigoplus_{i=1}^{r} V_i \otimes W_i$ with $n_i := \dim V_i$ and $m_i := \dim W_i$, then ι (adjusted for the appropriate dimensions) can be applied to each factor of $(g_1, \ldots, g_r) \in \prod_{i=1}^{r} GL(V_i)$, and κ (adjusted for the appropriate dimensions) can be applied to each factor of $(g'_1, \ldots, g'_r) \in \prod_{i=1}^{r} GL(W_i)$; each tensor factor $V_i \otimes W_i$ corresponds to a $m_i \times n_i$-block of the resulting block diagonal matrix in $GL(U)$. This realizes $\prod_{i=1}^{r} GL(V_i)$ and $\prod_{i=1}^{r} GL(W_i)$ as subgroups of $GL(U)$.

2.2. $Sp(U)$**.** Let U, V, and W be finite dimensional complex vector spaces such that $U = V \otimes W$. Additionally, assume that U and W admit symplectic forms, and that V admits an orthogonal form. Assume, without loss of generality, that W has the standard symplectic form Ω_m, so that the matrix Ω giving the symplectic form on $U = V \otimes W$ can be written as

$$\Omega := \begin{pmatrix} \Omega_m & & \\ & \ddots & \\ & & \Omega_m \end{pmatrix}, \qquad \text{where } \Omega_m := \begin{pmatrix} & I_{m/2} \\ -I_{m/2} & \end{pmatrix}.$$

(Note that all symplectic forms are isomorphic, so we are free to make these assumptions regarding the symplectic forms of W and U.)

Any matrix in $M \in Sp(V \otimes W)$ satisfies $M \Omega M^T = \Omega$. Restricting the map ι defined above to $O(V)$ gives an embedding $O(V) \hookrightarrow Sp(V \otimes W)$. This simply follows from checking that $\iota(A)\Omega\iota(A)^T = \Omega$ for $A \in O(V)$ (which follows easily from the fact that $AA^T = I_n$ for all $A \in O(V)$). Similarly, restricting the map κ to $Sp(W)$ gives an embedding $Sp(W) \hookrightarrow$

$Sp(V \otimes W)$. This follows from checking that $\kappa(B)\Omega\kappa(B)^T = \Omega$ for $B \in Sp(W)$ (which follows easily from the fact that $B\Omega_m B^T = \Omega_m$ for all $B \in Sp(W)$).

Suppose, on the other hand, that $U = V_1 \otimes W_1 \oplus V_2 \otimes W_2$, where $\dim V_1 = \dim V_2 =: n$ and $\dim W_1 = \dim W_2 =: m$. Then the embedding $GL(V_1) \hookrightarrow Sp(V_1 \otimes W_1 \oplus V_2 \otimes W_2)$ is defined by mapping $A \in GL(V_1)$ as follows:

$$A \overset{\iota}{\mapsto} \begin{pmatrix} a_{11}I_m & \cdots & a_{1n}I_m \\ \vdots & \ddots & \vdots \\ a_{n1}I_m & \cdots & a_{nn}I_m \end{pmatrix} \overset{\tau}{\mapsto} \begin{pmatrix} \iota(A) & \\ & (\iota(A)^T)^{-1} \end{pmatrix}$$

$$B \overset{\kappa}{\mapsto} \begin{pmatrix} B & & \\ & \ddots & \\ & & B \end{pmatrix} \overset{\tau}{\mapsto} \begin{pmatrix} \kappa(B) & \\ & (\kappa(B)^T)^{-1} \end{pmatrix}$$

It is straightforward to check that both $\tau(\iota(A))$ and $\tau(\kappa(B))$ preserve the standard symplectic form Ω_{2mn}.

If $U = \bigoplus_{i=1}^{s}(V_i \otimes W_i \oplus V_i' \otimes W_i') \oplus \bigoplus_{i=s+1}^{r} V_i \otimes W_i$ (with $\dim V_i = \dim V_i'$ and $\dim W_i = \dim W_i'$), these embeddings can be extended to product groups such as

$$\prod_{i=1}^{s} GL(V_i) \prod_{i=s+1}^{t} Sp(V_i) \prod_{i=t+1}^{r} O(V_i),$$

where the V_i in the second factor are symplectic and the V_i in the third factor are orthogonal. Here, ι acts on each of the orthogonal factors, κ acts on each of the symplectic factors, and $\tau \circ \iota$ or $\tau \circ \kappa$ acts on each of the $GL(V_i)$ factors. The symplectic form on U is the block-diagonal matrix with blocks comprised of the symplectic forms on each tensor factor. For a detailed example, see Example 4.2.

2.3. $O(U)$**.** Let U, V, and W be finite dimensional complex orthogonal vector spaces such that $U = V \otimes W$. Define $n := \dim V$ and $m := \dim W$. Over $\mathbb{C}$, all orthogonal forms are isomorphic, so assume that U, V, and W have the standard orthogonal forms.

Restricting the map ι defined above to $O(V)$ gives an embedding $O(V) \hookrightarrow O(V \otimes W)$. This simply follows from checking that $\iota(A)\iota(A)^T = I_{mn}$ for $A \in O(V)$ (which follows easily from the fact that $AA^T = I_n$ for all $A \in O(V)$). Similarly, restricting the map κ to $O(W)$ gives an embedding $O(W) \hookrightarrow O(V \otimes W)$. This follows from checking that $\kappa(B)\kappa(B)^T = I_{mn}$ for $B \in O(W)$ (which follows easily from the fact that $BB^T = I_m$ for all $B \in O(W)$). We omit the description of the embedding of $Sp(V)$ and $Sp(W)$ into $O(V \otimes W)$ in the case that V and W are both symplectic; this embedding is more difficult to understand and will not be required in this paper.

Suppose, on the other hand, that $U = V_1 \otimes W_1 \oplus V_2 \otimes W_2$, where $\dim V_1 = \dim V_2 =: n$ and $\dim W_1 = \dim W_2 =: m$. Assume U has orthogonal form given by $\begin{pmatrix} & I_{mn} \\ I_{mn} & \end{pmatrix}$. The $GL(V_1) \hookrightarrow O(V_1 \otimes W_1 \oplus V_2 \otimes W_2)$ is given by $\tau \circ \iota$, and $GL(W_1) \hookrightarrow O(V_1 \otimes W_1 \oplus V_2 \otimes W_2)$ is given by $\tau \circ \kappa$. It is straightforward to check that the images under these embeddings in fact preserve the orthogonal form $\begin{pmatrix} & I_{mn} \\ I_{mn} & \end{pmatrix}$.

If U has more tensor factors, these embeddings can be extended in the same way as in the case of $Sp(U)$.

3. Dual Pairs in $GL(U)$

Let U be a finite dimensional complex vector space, and let H be a complex reductive algebraic group. Since we are working in characteristic zero, every algebraic representation of H is completely reducible. Therefore, for any algebraic representation $\varphi : H \to GL(U)$, Schur's lemma gives the decomposition

$$U \simeq \bigoplus_i V_i \otimes \mathrm{Hom}_H(V_i, U),$$

where the V_i's are the nonisomorphic irreducible subrepresentations of U. This decomposition will be crucial for our classification of dual pairs in $GL(U)$. Another essential tool for our classification is the irreducibility of the *standard representation* $\rho : GL(U) \to GL(U)$; $M \mapsto M$ of $GL(U)$:

Lemma 3.1. *The standard representation of $GL(n, \mathbb{C})$ is irreducible for $n \geq 1$.*

Proof. This follows immediately from the stronger claim that $GL(n, \mathbb{C})$ acts transitively on the nonzero vectors of $\mathbb{C}^n$ for any $n \geq 1$. To see why this stronger claim is true, we note that for a given nonzero $v \in \mathbb{C}^n$, any $g \in GL(n, \mathbb{C})$ with first column equal to v satisfies $g \cdot e_1 = v$. □

Theorem 3.2. *Let U be a finite dimensional complex vector space. Then the dual pairs of $GL(U)$ are exactly the pairs of groups of the form*

$$\left(\prod_{i=1}^{r} GL(V_i),\ \prod_{i=1}^{r} GL(\mathrm{Hom}_H(V_i, U)) \right),$$

where H is a complex reductive algebraic group and $V_1, \ldots, V_r$ are the nonisomorphic irreducible subrepresentations of U (viewed as a representation of H).

Proof. Let $\varphi : H \to GL(U)$ be an algebraic representation with nonisomorphic irreducible subrepresentations $\varphi_1, \ldots, \varphi_r$, where $\varphi_i : H \to GL(V_i)$. Set $W_i := \mathrm{Hom}_H(V_i, U)$.

Step 1: $C_{GL(U)}\left(\prod_{i=1}^{r} GL(V_i)\right) = \prod_{i=1}^{r} GL(W_i) = GL(U)^{\varphi}$.

Let $t \in C_{GL(U)}\left(\prod_{i=1}^{r} GL(V_i)\right)$. Note that t commutes with any element of $\varphi(H)$ (since $\varphi(H) \subseteq \prod_{i=1}^{r} GL(V_i)$). In other words, t is H-linear. Since $V_1, \ldots, V_r$ are nonisomorphic irreducible representations, Schur's lemma gives that there are no nontrivial H-linear maps between them. Therefore, t necessarily preserves each $V_i \otimes W_i$, and hence can be decomposed as $t = t_1 \oplus \cdots \oplus t_r$, where $t_i : V_i \otimes W_i \to V_i \otimes W_i$ is H-linear. Applying Schur's lemma again gives that the action of t_i on each V_i is given by $\lambda \cdot \mathrm{id}_{V_i}$ for some $\lambda \in \mathbb{C}^*$. It follows that $t_i \in GL(W_i)$, giving that $t \in \prod_{i=1}^{r} GL(W_i)$ and hence that $C_{GL(U)}\left(\prod_{i=1}^{r} GL(V_i)\right) \subseteq \prod_{i=1}^{r} GL(W_i)$. On the other hand, it is clear that $\prod_{i=1}^{r} GL(W_i) \subseteq C_{GL(U)}\left(\prod_{i=1}^{r} GL(V_i)\right)$. This same argument also shows that $\prod_{i=1}^{r} GL(W_i)$ consists exactly of the H-linear elements of $GL(U)$, completing Step 1.

Step 2: Each W_i is an irreducible representation of $\prod_{i=1}^{r} GL(W_i)$ with multiplicity space V_i.

Set $H' := \prod_{i=1}^{r} GL(W_i)$ and consider the representation $\rho : H' \hookrightarrow GL(U)$. We will show that the W_i's are precisely the irreducible subrepresentations of ρ, and that W_i has multiplicity space V_i. Certainly, each W_i is a subrepresentation of U, since $\rho(h)w \in W_i$ for each $h \in H'$ and $w \in W_i$. Moreover, each W_i is irreducible by Lemma 3.1. Therefore, we now have two decompositions of U — one as an H-representation and one as an H'-representation. Combining these gives that

$$U \simeq \bigoplus V_i \otimes \mathrm{Hom}_H(V_i, U) \simeq \bigoplus W_i \otimes \mathrm{Hom}_{H'}(W_i, U).$$

Since $W_i = \mathrm{Hom}_H(V_i, U)$, this shows that $V_i = \mathrm{Hom}_{H'}(W_i, U)$, completing Step 2.

Step 3: $C_{GL(U)}\left(\prod_{i=1}^r GL(W_i)\right) = \prod_{i=1}^r GL(V_i)$.

Step 2 shows that we can repeat Step 1 with H' in place of H and with the roles of V_i and W_i to get that $C_{GL(U)}(\prod_{i=1}^r GL(W_i)) = \prod_{i=1}^r GL(V_i)$.

Step 4: All dual pairs of $GL(U)$ are of this form.

It was shown in Step 1 that $\prod_{i=1}^r GL(W_i) = GL(U)^\varphi$. Therefore, Step 4 follows from Remark 1.3. □

Corollary 3.3. *Let U be a finite dimensional complex vector space. Then the dual pairs of $GL(U)$ are exactly the pairs of groups of the form*

$$\left(\prod_{i=1}^r GL(V_i),\ \prod_{i=1}^r GL(W_i)\right),$$

where $U = \bigoplus_{i=1}^r V_i \otimes W_i$ is a vector space decomposition of U.

Proof. Let $U = \bigoplus_{i=1}^r V_i \otimes W_i$ be a vector space decomposition of U. Set $H := \prod_{i=1}^r GL(V_i)$, and let $\varphi : H \to GL(U)$ be the standard representation of H. Then by Lemma 3.1, the V_i's are the nonisomorphic irreducible subrepresentations of U, with the W_i's as corresponding multiplicity spaces. Theorem 3.2 therefore gives that $(\prod_{i=1}^r GL(V_i),\ \prod_{i=1}^r GL(W_i))$ is a dual pair in $GL(U)$. Moreover, Theorem 3.2 gives that every dual pair in $GL(U)$ is of this form. □

Remark 3.4. If $\dim(V_i) = n$, then $V_i \simeq \mathbb{C}^n$, and $GL(V_i) \simeq GL_n(\mathbb{C})$. Similarly, if $\dim(W_i) = m$, then $GL(W_i) \simeq GL_m(\mathbb{C})$. Recalling that $GL_n(\mathbb{C})$ is connected for any $n \in \mathbb{N}$, and that direct products of connected spaces are connected, we get that both members of any dual pair in $GL(U)$ are connected.

4. Dual Pairs in $Sp(U)$

Let U be a finite dimensional complex symplectic vector space. The following lemma sets us up to apply an argument analogous to the proof of Theorem 3.2 in the context of $Sp(U)$, which will allow us classify the dual pairs in $Sp(U)$.

Lemma 4.1. *Let U be a finite dimensional complex symplectic vector space. Let H be a complex reductive algebraic group, and let $\varphi : H \to Sp(U)$ be an algebraic symplectic representation of H with nonisomorphic irreducible subrepresentations $\{V_\gamma\}_\gamma = \{V_\mu, V_\nu\}_{\mu \not\simeq \mu^*,\, \nu \simeq \nu^*}$.*

Set $W_\gamma := \mathrm{Hom}_H(V_\gamma, U)$. *Then*

$$(1)\qquad \left(\prod_\gamma GL(V_\gamma)\right) \cap Sp(U) = \prod_{\mu\not\simeq\mu^*} GL(V_\mu) \prod_{\substack{\nu\simeq\nu^*\\ orthog.}} O(V_\nu) \prod_{\substack{\nu\simeq\nu^*\\ sympl.}} Sp(V_\nu), \text{ and}$$

$$(2)\qquad \left(\prod_\gamma GL(W_\gamma)\right) \cap Sp(U) = \prod_{\mu\not\simeq\mu^*} GL(W_\mu) \prod_{\substack{\nu\simeq\nu^*\\ orthog.}} Sp(W_\nu) \prod_{\substack{\nu\simeq\nu^*\\ sympl.}} O(W_\nu).$$

Proof. To start, note that Schur's lemma gives the decomposition

$$U \simeq \left(\bigoplus_{\mu\not\simeq\mu^*} V_\mu \otimes W_\mu\right) \oplus \left(\bigoplus_{\nu\simeq\nu^*} V_\nu \otimes W_\nu\right).$$

Our strategy in this proof is to understand the structure on each of these summands and tensor factors that is induced by the symplectic structure of U. Then (3) and (4) will follow from a consideration of which elements of $\prod_\gamma GL(V_\gamma)$ and $\prod_\gamma GL(W_\gamma)$ preserve this substructure.

Since U is symplectic and finite dimensional, we have that $U \simeq U^*$, which allows us to write

$$U \simeq \left(\bigoplus_{(\mu,\mu^*)} (V_\mu \otimes W_\mu) \oplus (V_{\mu^*} \otimes W_{\mu^*})\right) \oplus \left(\bigoplus_{\nu\simeq\nu^*} V_\nu \otimes W_\nu\right).$$

Now, note that for each irreducible representation V_γ of H, V_{γ^*} is also irreducible, so

$$\dim(\mathrm{Hom}_H(V_\gamma, V_{\gamma^*})) = \begin{cases} 1 & \text{if } V_\gamma \simeq V_{\gamma^*} \\ 0 & \text{if } V_\gamma \not\simeq V_{\gamma^*} \end{cases}.$$

Note also that H-invariant bilinear forms are in one-to-one correspondence with the elements of $\mathrm{Hom}_H(V_\gamma, V_{\gamma^*})$. Therefore, the V_μ's do not admit H-invariant bilinear forms, whereas each V_ν inherits an H-invariant bilinear form from U (which must be either symplectic or orthogonal on V_ν). It is not hard to see that, by extension, the $V_\mu \otimes W_\mu$'s do not admit H-invariant bilinear forms, whereas the $V_\nu \otimes W_\nu$'s inherit H-invariant symplectic bilinear forms from U. Moreover, since $W_\nu \simeq W_{\nu^*}$ for $\nu \simeq \nu^*$, each W_ν admits an H-invariant bilinear form, which must be symplectic if the form on V_ν is orthogonal and must be orthogonal if the form on V_ν is symplectic (since the symplectic form on $V_\nu \otimes W_\nu$ can be obtained as the product of the forms on V_ν and W_ν).

Although each $V_\mu \otimes W_\mu$ for $\mu \not\simeq \mu^*$ does not admit an H-invariant bilinear form, we claim that each $(V_\mu \otimes W_\mu) \oplus (V_{\mu^*} \otimes W_{\mu^*})$ admits an H-invariant symplectic form. Indeed, write $E := V_\mu \otimes W_\mu$ and $E^* := V_{\mu^*} \otimes W_{\mu^*}$. Then for $e, e' \in E$ and $\mathcal{E}, \mathcal{E}' \in E^*$, we see that

$$\langle (e, \mathcal{E}),\ (e', \mathcal{E}')\rangle := \mathcal{E}'(e) - \mathcal{E}(e')$$

defines an H-invariant symplectic bilinear form on $E \oplus E^*$.

Finally, it is clear that an element of $\prod_\gamma GL(V_\gamma)$ preserves the symplectic bilinear form on U if and only if it preserves the induced bilinear form on each V_γ. Similarly, an element of $\prod_\gamma GL(W_\gamma)$ preserves the symplectic bilinear form on U if and only if it preserves the induced bilinear form on each W_γ. It follows that

$$\left(\prod_\gamma GL(V_\gamma)\right) \cap Sp(U) = \prod_{\mu\not\simeq\mu^*} GL(V_\mu) \prod_{\substack{\nu\simeq\nu^*\\ \text{orthog.}}} O(V_\nu) \prod_{\substack{\nu\simeq\nu^*\\ \text{sympl.}}} Sp(V_\nu), \text{ and}$$

$$\left(\prod_\gamma GL(W_\gamma)\right) \cap Sp(U) = \prod_{\mu\not\simeq\mu^*} GL(W_\mu) \prod_{\substack{\nu\simeq\nu^*\\ \text{orthog.}}} Sp(W_\nu) \prod_{\substack{\nu\simeq\nu^*\\ \text{sympl.}}} O(W_\nu),$$

completing the proof. □

Example 4.2. Let H be a complex reductive algebraic group, and let $\varphi : H \to Sp(U)$ be an algebraic symplectic representation of H with nonisomorphic irreducible subrepresentations V_1, V_1^*, and V_2. Write $U \simeq V_1 \otimes W_1 \oplus V_1^* \otimes W_1^* \oplus V_2 \otimes W_2$, suppose that $V_1 \not\simeq V_1^*$ as representations, that V_2 is orthogonal, and that $\dim V_i = \dim W_i = 2$. The matrix

$$\Omega = \begin{pmatrix} & I_4 & & & & \\ -I_4 & & & & & \\ & & & 1 & & \\ & & -1 & & & \\ & & & & & 1 \\ & & & & -1 & \end{pmatrix}$$

defines a symplectic form on U (so that any matrix $M \in Sp(U)$ satisfies $M\Omega M^T = \Omega$). The symplectic form Ω induces the symplectic form

$$\Omega_1 := \begin{pmatrix} & I_4 \\ -I_4 & \end{pmatrix}$$

on $V_1 \otimes W_1 \oplus V_1^* \otimes W_1^*$. To see that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(V_1)$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in GL(W_1)$ preserve Ω_1, we consider the images of A and B under the embeddings of $GL(V_1)$ and $GL(W_1)$ into $GL(V_1 \otimes W_1 \oplus V_1^* \otimes W_1^*)$:

$$\begin{array}{lll} A & \overset{\iota}{\mapsto} \begin{pmatrix} aI_2 & bI_2 \\ cI_2 & dI_2 \end{pmatrix} & \overset{\tau}{\mapsto} \begin{pmatrix} \iota(A) & \\ & (\iota(A)^T)^{-1} \end{pmatrix} \\ B & \overset{\kappa}{\mapsto} \begin{pmatrix} B & \\ & B \end{pmatrix} & \overset{\tau}{\mapsto} \begin{pmatrix} \iota_1(B) & \\ & (\iota_1(B)^T)^{-1} \end{pmatrix} \end{array}$$

It is straightforward to check that $\tau(\iota(A))$ and $\tau(\kappa(B))$ preserve Ω_1, and hence lie in $Sp(V_1 \otimes W_1 \oplus V_1^* \otimes W_1^*)$. Additionally, the symplectic form Ω induces the symplectic form

$$\Omega_2 := \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}$$

on $Sp(V_2 \otimes W_2)$. Consider the images of $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in GL(V_2)$ and $B' = \begin{pmatrix} e' & f' \\ g' & h' \end{pmatrix} \in GL(W_2)$ under the embeddings of $GL(V_2)$ and $GL(W_2)$ into $GL(V_2 \otimes W_2)$:

$$
\begin{array}{rcll}
A' & \overset{\iota}{\mapsto} & \begin{pmatrix} a'I_2 & b'I_2 \\ c'I_2 & d'I_2 \end{pmatrix} & \in GL(V_2 \otimes W_2) \\
B' & \overset{\kappa}{\mapsto} & \begin{pmatrix} B' & \\ & B' \end{pmatrix} & \in GL(V_2 \otimes W_2)
\end{array}
$$

It is straightforward to check that $\iota(A')$ preserves Ω_2 if and only if $A' \in O(V_2)$ and that $\kappa(B')$ preserves Ω_2 if and only if $B' \in Sp(W_2)$. It follows that in this case,

$$
\begin{aligned}
(GL(V_1) \times GL(V_1^*) \times GL(V_2)) \cap Sp(U) &= GL(V_1) \times GL(V_1^*) \times O(V_2), \text{ and} \\
(GL(W_1) \times GL(W_1^*) \times GL(W_2)) \cap Sp(U) &= GL(W_1) \times GL(W_1^*) \times Sp(W_2),
\end{aligned}
$$

as Lemma 4.1 would suggest.

At this point in our analysis, we require the irreducibility of the standard representations $O(n, \mathbb{C}) \hookrightarrow GL(n, \mathbb{C})$ and $Sp(2n, \mathbb{C}) \hookrightarrow GL(n, \mathbb{C})$:

Lemma 4.3 ([3, Section 5.5])**.** *The standard representations of $O(n, \mathbb{C})$ and $Sp(2n, \mathbb{C})$ are irreducible for $n \geq 1$.*

Theorem 4.4. *Let U be a finite dimensional complex symplectic vector space. Then the dual pairs of $Sp(U)$ are exactly the pairs of groups of the form*

$$
\left(\prod_{\mu \not\simeq \mu^*} GL(V_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \textit{orthog.}}} O(V_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \textit{sympl.}}} Sp(V_\nu), \prod_{\mu \not\simeq \mu^*} GL(W_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \textit{orthog.}}} Sp(W_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \textit{sympl.}}} O(W_\nu) \right),
$$

where H is a complex reductive algebraic group, where $W_\gamma := \mathrm{Hom}_H(V_\gamma, U)$, and where μ and ν together vary over the nonisomorphic irreducible subrepresentations of U.

Proof. Let $\varphi : H \to Sp(U)$ be an algebraic symplectic representation with nonisomorphic irreducible subrepresentations $\gamma : H \to GL(V_\gamma)$. Set $W_\gamma := \mathrm{Hom}_H(V_\gamma, U)$. Write

$$
\begin{aligned}
G_1 &:= \prod_{\mu \not\simeq \mu^*} GL(V_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \text{orthog.}}} O(V_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \text{sympl.}}} Sp(V_\nu), \text{ and} \\
G_2 &:= \prod_{\mu \not\simeq \mu^*} GL(W_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \text{orthog.}}} Sp(W_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \text{sympl.}}} O(W_\nu).
\end{aligned}
$$

Step 1: $C_{Sp(U)}(G_1) = G_2 = Sp(U)^\varphi$.

Let $t \in C_{Sp(U)}(G_1)$. By Lemma 4.1, $\varphi(H) \subseteq G_1$, so we have that t commutes with any element of $\varphi(H)$. In other words, t is H-linear. Applying Schur's lemma in the same way as in Step 1 of the proof of Theorem 3.2 gives that $t \in \prod_\gamma GL(W_\gamma) \cap Sp(U) = G_2$, where γ ranges over the nonisomorphic irreducible subrepresentations of φ, and where we have used Lemma 4.1. It follows that $C_{Sp(U)}(G_1) \subseteq G_2$. On the other hand, it is clear that $G_2 \subseteq C_{Sp(U)}(G_1)$. It also follows from this argument that G_2 consists exactly of the H-linear elements of $Sp(U)$, completing Step 1.

Step 2: Each W_γ is an irreducible representation of G_2 with multiplicity space V_γ.

Consider the representation $\rho : G_2 \hookrightarrow GL(U)$. As in the proof of Theorem 3.2, we will show that the W_γ's are precisely the nonisomorphic irreducible subrepresentations of ρ, and that W_γ has multiplicity space V_γ. To see this, we start by noting that each W_γ is a subrepresentation of U, since $\rho(g)w \in W_\gamma$ for any $g \in G_2$ and $w \in W_\gamma$. Moreover, each W_γ is irreducible by Lemmas 3.1 and 4.3. We therefore obtain two decompositions of U, giving

$$U \simeq \bigoplus V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, U) \simeq \bigoplus W_\gamma \otimes \mathrm{Hom}_{G_2}(W_\gamma, U).$$

Since $W_\gamma = \mathrm{Hom}_H(V_\gamma, U)$, this shows that $V_\gamma = \mathrm{Hom}_{G_2}(W_\gamma, U)$, completing Step 2.

Step 3: $C_{Sp(U)}(G_2) = G_1$.

Step 2 shows that we can repeat Step 1 with G_2 in place of H and with the roles of W_γ and V_γ reversed. Doing so gives that $C_{Sp(U)}(G_2) = G_1$, as desired.

Step 4: All dual pairs of $Sp(U)$ are of this form.

It was shown in Step 1 that $G_2 = Sp(U)^\varphi$. Step 4 therefore follows from Remark 1.3, completing the proof. ☐

Corollary 4.5. *Let U be a finite dimensional complex symplectic vector space. Then the dual pairs of $Sp(U)$ are exactly the pairs of groups of the form*

$$\left(\prod_\mu GL(V_\mu) \prod_\nu O(V_\nu) \prod_\lambda Sp(V_\lambda),\ \prod_\mu GL(W_\mu) \prod_\nu Sp(W_\nu) \prod_\lambda O(W_\lambda)\right),$$

where

$$U = \left(\bigoplus_\mu ((V_\mu \otimes W_\mu) \oplus (V_\mu^* \otimes W_\mu^*))\right) \oplus \left(\bigoplus_\nu V_\nu \otimes W_\nu\right) \oplus \left(\bigoplus_\lambda V_\lambda \otimes W_\lambda\right)$$

is a vector space decomposition of U with $\dim V_\lambda$ even and $\dim W_\nu$ even.

Proof. Let U have such a decomposition, and set $H := \prod_\mu GL(V_\mu) \prod_\nu O(V_\nu) \prod_\lambda Sp(V_\lambda)$. Let $\varphi : H \to Sp(U)$ be the standard representation of H. Then by Lemmas 3.1 and 4.3, the V_μ's, V_ν's, and V_λ's are the nonisomorphic irreducible representations of H, with the W_μ's, W_ν's, and W_λ's as corresponding multiplicity spaces. Theorem 4.4 therefore gives that

$$\left(\prod_\mu GL(V_\mu) \prod_\nu O(V_\nu) \prod_\lambda Sp(V_\lambda),\ \prod_\mu GL(W_\mu) \prod_\nu Sp(W_\nu) \prod_\lambda O(W_\lambda)\right),$$

is a dual pair in $Sp(U)$. Moreover, Theorem 4.4 gives that every dual pair in $Sp(U)$ is of this form. ☐

5. Dual Pairs in $O(U)$

Let U be a finite dimensional complex orthogonal vector space. The classification of dual pairs in $O(U)$ follows from an analysis extremely similar to that for $Sp(U)$ in Section 4. As a result, we omit the proofs in this section. The following three results can be proven in nearly the same way as Lemma 4.1, Theorem 4.4, and Corollary 4.5, respectively.

Lemma 5.1. *Let U be a finite dimensional complex orthogonal vector space. Let H be a complex reductive algebraic group, and let $\varphi : H \to Sp(U)$ be an algebraic orthogonal representation of H with nonisomorphic irreducible subrepresentations $\{V_\gamma\}_\gamma = \{V_\mu, V_\nu\}_{\mu \not\simeq \mu^*,\, \nu \simeq \nu^*}$. Set $W_\gamma := \mathrm{Hom}_H(V_\gamma, U)$. Then*

$$\left(\prod_\gamma GL(V_\gamma)\right) \cap O(U) = \prod_{\mu \not\simeq \mu^*} GL(V_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \text{orthog.}}} O(V_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \text{sympl.}}} Sp(V_\nu), \text{ and} \tag{3}$$

$$\left(\prod_\gamma GL(W_\gamma)\right) \cap O(U) = \prod_{\mu \not\simeq \mu^*} GL(W_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \text{orthog.}}} O(W_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \text{sympl.}}} Sp(W_\nu). \tag{4}$$

Proof. This follows from nearly the same argument as the proof of Lemma 4.1. □

Theorem 5.2. *Let U be a finite dimensional complex orthogonal vector space. Then the dual pairs of $O(U)$ are exactly the pairs of groups of the form*

$$\left(\prod_{\mu \not\simeq \mu^*} GL(V_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \text{orthog.}}} O(V_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \text{sympl.}}} Sp(V_\nu),\ \prod_{\mu \not\simeq \mu^*} GL(W_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \text{orthog.}}} O(W_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \text{sympl.}}} Sp(W_\nu)\right),$$

where H is a complex reductive algebraic group, where $W_\gamma := \mathrm{Hom}_H(V_\gamma, U)$, and where μ and ν together vary over the nonisomorphic irreducible subrepresentations of U.

Proof. This follows from Lemma 5.1 using nearly the same argument as in the proof of Theorem 4.4. □

Corollary 5.3. *Let U be a finite dimensional complex orthogonal vector space. Then the dual pairs of $O(U)$ are exactly the pairs of groups of the form*

$$\left(\prod_\mu GL(V_\mu) \prod_\nu O(V_\nu) \prod_\lambda Sp(V_\lambda),\ \prod_\mu GL(W_\mu) \prod_\nu O(W_\nu) \prod_\lambda Sp(W_\lambda)\right),$$

where

$$U = \left(\bigoplus_\mu ((V_\mu \otimes W_\mu) \oplus (V_\mu^* \otimes W_\mu^*))\right) \oplus \left(\bigoplus_\nu V_\nu \otimes W_\nu\right) \oplus \left(\bigoplus_\lambda V_\lambda \otimes W_\lambda\right)$$

is a vector space decomposition of U with $\dim V_\lambda$ and $\dim W_\lambda$ even.

6. Dual Pairs in Subgroups

Let G be a complex reductive algebraic group. In this section, we discuss how dual pairs in G relate to dual pairs in certain subgroups of G.

Lemma 6.1. *Let G be a complex reductive algebraic group, and suppose that G equals a product of subgroups $G = KH$, where K is central. If G_1 is a subgroup of G containing K, then $G = KH_1$, where $H_1 := G_1 \cap H$.*

Proof. Certainly, $KH_1 \subseteq G_1$. On the other hand, let $g \in G_1$ and write $g = kh$, where $k \in K$ and $h \in H$. Since $K \subseteq G_1$, we see that $h = k^{-1}g \in G_1$, so $h \in G_1 \cap H = H_1$. It follows that $g \in KH_1$ and hence that $G_1 \subseteq KH_1$. □

Lemma 6.2. *Let G be a complex reductive algebraic group, and suppose that G equals a product of subgroups $G = KH$, where K is central. If (G_1, G_2) is a dual pair in G, then $(G_1 \cap H, G_2 \cap H)$ is a dual pair in H.*

Proof. Since G_1 and G_2 are centralizers in G, they each contain K. Therefore, Lemma 6.1 gives that $G_1 = KH_1$ for $H_1 := G_1 \cap H$ and that $G_2 = KH_2$ for $H_2 := G_2 \cap H$. Now, we claim that

$$KC_H(H_1) \subseteq C_G(G_1) \subseteq C_G(H_1) \subseteq KC_H(H_1). \tag{5}$$

For the first inclusion, let $kh \in KC_H(H_1)$, and let $k'h' \in KH_1 = G_1$. Then

$$(kh)(k'h')(kh)^{-1} = k'(hh'h^{-1}) = h',$$

so $kh \in C_G(G_1)$. The second inclusion follows immediately from $H_1 \subseteq G_1$. For the final inclusion, let $g \in C_G(H_1)$ and write $g = kh$ with $k \in K$ and $h \in H$. Then for any $h_1 \in H_1$,

$$(kh)h_1(kh)^{-1} = h_1 \implies hh_1h^{-1} = h_1,$$

so $h \in C_H(H_1)$ and $g \in KC_H(H_1)$. It follows that all of the groups in (5) are equal. The same result holds for G_2 and H_2. It follows that

$$C_H(H_1) = C_G(H_1) \cap H = KC_H(H_1) \cap H = G_2 \cap H = H_2.$$

Similarly, $C_H(H_2) = H_1$. This completes the proof. □

Lemma 6.3. *Let G be a complex reductive algebraic group, and suppose that G equals a product of subgroups $G = KH$, where K is central. If (H_1, H_2) is a dual pair in H, then (KH_1, KH_2) is a dual pair in G.*

Proof. We would like to show that $C_G(KH_1) = KH_2$ and $C_G(KH_2) = KH_1$. Certainly, we have the inclusions $KH_2 \subseteq C_G(KH_1)$ and $KH_1 \subseteq C_G(KH_2)$. We now show that $C_G(KH_1) \subseteq KH_2$. Suppose $t \in C_G(H_1)$. Then since $G = KH$, we can write $t = kh$ with $k \in K$ and $h \in H$. Then for any $h' \in H_1$,

$$h' = (kh)h'(kh)^{-1} \implies h' = k^{-1}h'k = hh'h^{-1} \implies h \in H_2.$$

It follows that $t \in KH_2$, and hence that $C_G(KH_1) \subseteq KH_2$. By a similar argument, we get that $C_G(KH_1) \subseteq KH_1$. Therefore, (KH_1, KH_2) is a dual pair in G, as desired. □

Claim 6.4. Let G, H, K be as in Lemma 6.3. Let (H_1, H_2) be a dual pair of H. Then $KH_1 \cap H = H_1$ and $KH_2 \cap H = H_2$.

Proof. We prove that $KH_1 \cap H = H_1$. Certainly, $H_1 \subseteq KH_1 \cap H$. On the other hand, suppose that $kh \in KH_1 \cap H$. Then by Lemma 6.3, for any $h_2 \in H_2$ we have that

$$(kh)h_2(kh)^{-1} = h_2,$$

which gives that $kh \in C_H(H_2) = H_1$. Hence $KH_1 \cap H = H_1$. The same argument gives that $KH_2 \cap H = H_2$, completing the proof. □

Theorem 6.5. *Let G be a complex reductive algebraic group, and suppose that G equals a product of subgroups $G = KH$, where K is central. Then there exists a bijection*

$$\{\textit{dual pairs of } H\} \leftrightarrow \{\textit{dual pairs of } G\}$$
$$(H_1, H_2) \leftrightarrow (KH_1, KH_2),$$

where $\to$ is given by multiplication by K and $\leftarrow$ is given by restriction to H.

Proof. This follows immediately by putting together the above results. □

7. Dual Pairs in $SL(U)$

Let U be a finite dimensional complex vector space. The following Corollary applies Theorem 6.5 to show that the dual pairs of $GL(U)$ are in bijection with the dual pairs of $SL(U)$. This bijection combines with Corollary 3.3 to give a classification of dual pairs in $SL(U)$ (see Corollary 7.2).

Corollary 7.1. *Let U be a finite dimensional complex vector space. The dual pairs of $GL(U)$ are in bijection with the dual pairs of $SL(U)$.*

Proof. This follows from Theorem 6.5 and the observation that $GL(V) = Z \cdot SL(V)$, where Z is center of $GL(V)$ (i.e. the subgroup of scalar matrices in $GL(V)$). For any $g \in GL(V) = GL_n(\mathbb{C})$, let $r = \operatorname{diag}((1/\delta)^{1/n}, \dots, (1/\delta)^{1/n})$, where $\delta = \det g$. Then $\det(gr) = \det(g)\det(r) = \delta(1/\delta) = 1$. Moreover, $r \in Z$, so we see that $g = r^{-1}gr$. Defining $g' := gr$, we can write $g = r^{-1}g'$, where $g' \in SL(V)$ and $r^{-1} \in Z$. Hence, $GL(V) = Z \cdot SL(V)$, completing the proof. $\square$

Corollary 7.2. *Let U be a finite dimensional complex vector space. Then the dual pairs of $SL(U)$ are exactly the pairs of groups of the form*

$$\left(\left(\prod_{i=1}^{r} GL(V_i)\right) \cap SL(U),\ \left(\prod_{i=1}^{r} GL(W_i)\right) \cap SL(U)\right),$$

where $U = \bigoplus_{i=1}^{r} V_i \otimes W_i$ is a vector space decomposition of U.

Proof. This follows immediately from Corollary 7.1 and Corollary 3.3. $\square$

We have as a consequence of the proof of Corollary 7.1 that $PGL(U) = PSL(U)$. To see this, note that for any $gZ \in PGL(U)$, $g = sz$ for some $s \in SL(U)$ and some $z \in Z$, so $gZ = (sz)Z = sZ$. This justifies our exclusion of $PSL(U)$ from the list of complex projective classical groups that are under consideration.

8. Dual Pairs in $SO(U)$

Let U be a finite dimensional complex vector space. The following Corollary applies Theorem 6.5 to show that the dual pairs of $O(n, \mathbb{C})$ are in bijection with the dual pairs of $SO(n, \mathbb{C})$ when n is odd.

Corollary 8.1. *When n is odd, the dual pairs of $O(n, \mathbb{C})$ are in bijection with the dual pairs of $SO(n, \mathbb{C})$.*

Proof. This follows from Theorem 6.5 and the observation that $O(n, \mathbb{C}) = Z \cdot SO(n, \mathbb{C})$ when n is odd, where Z is the center of $O(n, \mathbb{C})$ (i.e. $Z = \{\pm I\}$). First, note that any orthogonal matrix has determinant ± 1 (since $M^T M = I$ implies $\det M = \det M^{-1}$ which further implies $(\det M)^2 = 1$). Given a matrix $M \in O(V)$, if $\det M = 1$, then there is nothing to show. On the other hand, if $\det M = -1$, then we can write $M = (-I)M(-I)$; since $-I \in Z$ and $M(-I) \in SO(V)$ (because $\det(-I) = (-1)^n = -1$), this completes the proof. $\square$

Corollary 8.1 shows that when n is odd, the dual pairs of $SO(n, \mathbb{C})$ are simply the dual pairs of $O(n, \mathbb{C})$ restricted to $SO(n, \mathbb{C})$. In what follows, we show that this phenomenon in fact holds for all n (see Theorem 8.6). To do so, we require the irreducibility of the standard representation $SO(n, \mathbb{C}) \hookrightarrow GL(n, \mathbb{C})$:

Lemma 8.2 ([3, Section 2.5])**.** *The standard representation of $SO(n,\mathbb{C})$ is irreducible for $n \neq 2$.*

For later use, we prove the following lemma in more generality than is needed here.

Lemma 8.3. *Let U be a finite dimensional complex vector space with bilinear form $\langle,\rangle$ that is either symplectic or orthogonal. Suppose that $U \simeq \bigoplus_\gamma V_\gamma \otimes W_\gamma$, where each V_γ is an orthogonal vector space. Let $G = Sp$ (resp. $G = O$) if $\langle,\rangle$ is symplectic (resp. orthogonal). Then*

$$C_{G(U)}\left(\prod_\gamma SO(V_\gamma)\right) = \prod_\gamma C_{G(V_\gamma \otimes W_\gamma)}(SO(V_\gamma)).$$

Proof. Recall from Section 2 that the embedding $SO(V_\gamma) \hookrightarrow G(V_\gamma \otimes W_\gamma)$ is given by ι if $G = Sp$ and is given by either ι or κ if $G = O$. To view $\prod_\gamma SO(V_\gamma)$ as a subgroup of $G(U)$, form the block-diagonal matrix formed by the images of each $SO(V_\gamma)$ factor under the appropriate embeddings.

It is clear that $\prod_\gamma C_{G(V_\gamma\otimes W_\gamma)}(SO(V_\gamma)) \subseteq C_{G(U)}\left(\prod_\gamma SO(V_\gamma)\right)$. On the other hand, write $\{\gamma\} = \{\gamma_1, \ldots, \gamma_\ell\}$, let $n_i := \dim V_{\gamma_i}$, and let $m_i := \dim W_{\gamma_i}$. Without loss of generality, assume that $n_1 = \cdots = n_k = 2$ and that $n_{k+1}, \ldots, n_\ell \neq 2$. Let $M \in C_{G(U)}\left(\prod_\gamma SO(V_\gamma)\right)$. Consider the set of $\dim U \times \dim U$ block diagonal matrices, where the diagonal blocks have sizes $m_1n_1 \times m_1n_1, \ldots, m_\ell n_\ell \times m_\ell n_\ell$. Let N_i denote the matrix in this set that is the identity on every block except the i-th block, on which it equals $-I_{m_in_i}$. For $1 \leq i \leq k$, $N_i \in \prod_\gamma SO(V_\gamma)$, meaning $MN_i = N_iM$ for all such i. Writing out the entry-wise implications of these relations shows that

$$M \in \left(\prod_{1\leq i\leq k} C_{G(V_{\gamma_i}\otimes W_{\gamma_i})}(SO(V_{\gamma_i}))\right) \times \left(C_{G(\bigoplus_{k+1\leq j\leq \ell} V_{\gamma_j}\otimes W_{\gamma_j})}\left(\prod_{k+1\leq j\leq\ell} SO(V_{\gamma_j})\right)\right). \tag{6}$$

Now, since $n_j \neq 2$ for all $k+1 \leq j \leq \ell$, we have that the standard representation of $SO(V_{\gamma_j})$ is irreducible for all such j. It therefore follows from Schur's lemma that

$$C_{G(\bigoplus_{k+1\leq j\leq \ell} V_{\gamma_j}\otimes W_{\gamma_j})}\left(\prod_{k+1\leq j\leq\ell} SO(V_{\gamma_j})\right) = \prod_{k+1\leq j\leq\ell} C_{G(V_{\gamma_j}\otimes W_{\gamma_j})}(SO(V_{\gamma_j})). \tag{7}$$

Combining (6) and (7) gives that $M \in \prod_\gamma C_{G(V_\gamma\otimes W_\gamma)}(SO(V_\gamma))$, completing the proof. ☐

At this point in our analysis, we require the irreducibility of the standard representation $SL(n,\mathbb{C}) \hookrightarrow GL(n,\mathbb{C})$:

Lemma 8.4. *The standard representation of $SL(n,\mathbb{C})$ is irreducible for $n \geq 1$.*

Proof. If $n = 1$, then the representation in question is one-dimensional and hence irreducible. In the case that $n > 1$, we show that $SL(n,\mathbb{C})$ acts transitively on the nonzero vectors of $\mathbb{C}^n$. Let $v \in \mathbb{C}^n$ be a fixed nonzero vector, and let g be any invertible matrix with first column v. Write $\delta := \det g$, and let g' be the matrix obtained from g by multiplying the second column of g by $1/\delta$. Then g' is invertible with $\det g' = 1$ and satisfies $g \cdot e_1 = v$, completing the proof. ☐

We are now ready to prove that every dual pair in $O(U)$ gives rise to a dual pair in $SO(U)$:

Lemma 8.5. *Let U be a finite dimensional complex orthogonal vector space, and let*

$$\left(G_1 := \prod_\mu GL(V_\mu)\prod_\nu O(V_\nu)\prod_\lambda Sp(V_\lambda),\ G_2 := \prod_\mu GL(W_\mu)\prod_\nu O(W_\nu)\prod_\lambda Sp(W_\lambda)\right)$$

be a dual pair in $O(U)$. Then $(G_1 \cap SO(U), G_2 \cap SO(U))$ is a dual pair in $SO(U)$.

Proof. We will show that $C_{SO(U)}(G_1 \cap SO(U)) = G_2 \cap SO(U)$, and the result will follow by symmetry. Notice that

$$G_2 \cap SO(U) = C_{O(U)}(G_1) \cap SO(U) = C_{SO(U)}(G_1) \subseteq C_{SO(U)}(G_1 \cap SO(U)).$$

It remains to show that $C_{SO(U)}(G_1 \cap SO(U)) \subseteq G_2 \cap SO(U)$. To this end, let $M \in C_{SO(U)}(G_1 \cap SO(U))$. In particular, M commutes with every element of $\prod_\mu SL(V_\mu)\prod_\nu SO(V_\nu)\prod_\lambda Sp(V_\lambda)$. Moreover, by Lemma 8.3,

$$C_{SO(\bigoplus_\nu V_\nu\otimes W_\nu)}\left(\prod_\nu SO(V_\nu)\right) \subseteq C_{O(\bigoplus_\nu V_\nu\otimes W_\nu)}\left(\prod_\nu SO(V_\nu)\right) = \prod_\nu C_{O(V_\nu\otimes W_\nu)}(SO(V_\nu)).$$

Combining this with Lemmas 8.4 and 4.3 gives that

$$\begin{aligned} M \in \prod_\mu & C_{O(V_\mu\otimes W_\mu)}(SL(V_\mu)) \times \prod_\nu C_{O(V_\nu\otimes W_\nu)}(SO(V_\nu)) \times \prod_\lambda C_{O(V_\lambda\otimes W_\lambda)}(Sp(V_\lambda)) \\ & \subseteq \prod_\mu GL(W_\mu) \times \prod_{\nu\setminus\nu'} O(W_\nu) \times \prod_{\nu'} C_{O(V_{\nu'}\otimes W_{\nu'})}(SO(V_{\nu'})) \times \prod_\lambda Sp(W_\lambda), \end{aligned}$$

where $\{\nu'\} := \{\nu \ :\ \dim V_\nu = 2\}$. Now, view M as a block diagonal matrix with a diagonal block for every μ, ν, and λ. Let N be the diagonal block corresponding to a fixed ν'. The above shows that $N \in C_{O(V_{\nu'}\otimes W_{\nu'})}(SO(V_{\nu'}))$. But by our choice of M, we have that M also commutes with every element of $\prod_\mu GL^{(-1)}(V_\mu)\prod_\nu O^{(-1)}(V_\nu)\prod_\lambda Sp(V_\lambda)$, where $GL^{(-1)}(V_\mu)$ (resp. $O^{(-1)}(V_\nu)$) denotes the elements of $GL(V_\mu)$ (resp. $O(V_\nu)$) with determinant -1. This shows that, in fact, $N \in C_{O(V_{\nu'}\otimes W_{\nu'})}(O(V_{\nu'})) = O(W_{\nu'})$, where we have used Theorem 5.2. It follows that

$$M \in \left(\prod_\mu GL(W_\mu)\prod_\nu O(W_\nu)\prod_\lambda Sp(W_\lambda)\right) \cap SO(U) = G_2 \cap SO(U),$$

completing the proof. ☐

To complete the classification of dual pairs in $SO(U)$, we now prove that every dual pair in $SO(U)$ is of the form described in the previous lemma.

Theorem 8.6. *Let U be a finite dimensional complex vector space. Then the dual pairs of $SO(U)$ are exactly the pairs of groups of the form*

$$\left(\left(\prod_\mu GL(V_\mu)\prod_\nu O(V_\nu)\prod_\lambda Sp(V_\lambda)\right) \cap SO(U),\ \left(\prod_\mu GL(W_\mu)\prod_\nu O(W_\nu)\prod_\lambda Sp(W_\lambda)\right) \cap SO(U)\right),$$

where

$$U = \left(\bigoplus_\mu ((V_\mu\otimes W_\mu) \oplus (V_\mu^*\otimes W_\mu^*))\right) \oplus \left(\bigoplus_\nu V_\nu\otimes W_\nu\right) \oplus \left(\bigoplus_\lambda V_\lambda\otimes W_\lambda\right)$$

is a vector space decomposition of U with $\dim V_\lambda$ even and $\dim W_\lambda$ even.

Proof. Corollary 5.3 and Lemma 8.5 give that every pair of groups of the given form is in fact a dual pair in $SO(U)$. It remains to show that every dual pair in $SO(U)$ is of this form. Let (H_1, H_2) be a dual pair in $SO(U)$. By Remark 1.3, $(H_1, H_2) = (C_{SO(U)}(C_{SO(U)}(\varphi(H))), C_{SO(U)}(\varphi(H)))$ for some algebraic representation $\varphi : H \to SO(U)$ of a complex reductive algebraic group H. Therefore, it suffices to show that $C_{SO(U)}(\varphi(H))$ is of the form

$$\left(\prod_{\mu} GL(W_\mu) \prod_{\nu} O(W_\nu) \prod_{\lambda} Sp(W_\lambda)\right) \cap SO(U).$$

To this end, let $t \in C_{SO(U)}(\varphi(H))$ and define $\varphi' : H \to SO(U) \hookrightarrow O(U)$. Writing $\{V_\gamma\}_\gamma = \{V_\mu, V_\nu\}_{\mu \not\simeq \mu^*,\, \nu \simeq \nu^*}$ for the nonisomorphic irreducible subrepresentations of $\varphi' : H \to O(U)$, we note that

$$\varphi(H) = \varphi'(H) \subseteq \left(\prod_{\gamma} GL(V_\gamma)\right) \cap O(U) = \prod_{\mu \not\simeq \mu^*} GL(V_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \text{orthog.}}} O(V_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \text{sympl.}}} Sp(V_\nu),$$

where we have used Lemma 5.1. Since t is $\varphi'(H)$-linear, Schur's lemma gives that

$$t \in \left(\left(\prod_{\mu \not\simeq \mu^*} GL(W_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \text{orthog.}}} O(W_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \text{sympl.}}} Sp(W_\nu)\right) \cap SO(U)\right) =: T.$$

It follows that $C_{SO(U)}(\varphi(H)) \subseteq T$; on the other hand, the inclusion $T \subseteq C_{SO(U)}(\varphi(H))$ is clear, and the result follows. $\square$

9. Dual Pairs in Quotients

Let G be a complex reductive algebraic group and let U be a finite dimensional complex vector space. Having discussed the relationship between dual pairs in G and dual pairs in certain subgroups of G (which helped us classify dual pairs in $SL(U)$ and $SO(U)$), we now turn to the relationship between dual pairs in G and dual pairs in certain quotients of G.

Theorem 9.1. *Let G be a complex reductive algebraic group, let H be a connected subgroup of G, and let N be a normal subgroup of G. Let $\pi : G \to G/N$ denote the canonical projection, and define $K_{H,N} := \{tht^{-1}h^{-1} \ : \ t \in \pi^{-1}(C_{G/N}(\pi(H))),\ h \in H\}$. If $K_{H,N}$ is discrete, then $C_{G/N}(\pi(H)) = \pi(C_G(H))$.*

Proof. The inclusion $C_{G/N}(\pi(H)) \supseteq \pi(C_G(H))$ is clear. On the other hand, let $t \in \pi^{-1}(C_{G/N}(\pi(H)))$. We would like to show that $t \in C_G(H)$. By choice of t, we have that for any $h \in H$,

$$tht^{-1} = n_t(h)h \ \text{ for some } n_t(h) \in K_{H,N}.$$

Since multiplication and inversion of elements are continuous operations in an algebraic group, we see that n_t defines a continuous function

$$\begin{aligned} n_t : H &\to K_{H,N} \\ h &\mapsto tht^{-1}h^{-1}. \end{aligned}$$

Now, since n_t defines a continuous map from a connected group to a discrete group, the image of n_t must be a single point in $K_{H,N}$. Since $n_t(1) = 1$, it follows that n_t is trivial, and hence that $t \in C_G(H)$, completing the proof. $\square$

Remark 9.2. Let G, H, N, and $K_{H,N}$ be as in Theorem 9.1. Note that $K_{H,N} \subseteq N$. Therefore, a weaker version of Theorem 9.1 is that $C_{G/N}(\pi(H)) = \pi(C_G(H))$ whenever N is discrete.

Corollary 9.3. *Let G be a complex reductive algebraic group, let H be a subgroup of G with identity component H°, and let N be a normal subgroup of G such that $K_{H,N} := \{tht^{-1}h^{-1} : t \in \pi^{-1}(C_{G/N}(\pi(H))),\ h \in H\}$ is discrete, where $\pi : G \to G/N$ is the canonical projection. Then*

$$\pi^{-1}(C_{G/N}(\pi(H))) \subseteq C_G(H^\circ).$$

Proof. This follows from the inclusions

$$\pi^{-1}(C_{G/N}(\pi(H))) \subseteq \pi^{-1}(C_{G/N}(\pi(H^\circ))) \subseteq C_G(H^\circ),$$

where the second inclusion comes from Theorem 9.1. □

Corollary 9.4. *Let G be a complex reductive algebraic group, and let (G_1, G_2) be a dual pair in G. Let N be a normal subgroup of G such that $K_{G_1,N} := \{tg_1t^{-1}g_1^{-1} : t \in \pi^{-1}(C_{G/N}(\pi(G_1))),\ g_1 \in G_1\}$ and $K_{G_2,N} := \{tg_2t^{-1}g_2^{-1} : t \in \pi^{-1}(C_{G/N}(\pi(G_2))),\ g_2 \in G_2\}$ are discrete, where $\pi : G \to G/N$ denotes the canonical projection. If G_1 and G_2 are connected, then $(\pi(G_1), \pi(G_2))$ is a dual pair in G/N.*

Proof. Applying Theorem 9.1 with $H = G_1$ gives that $C_{G/N}(\pi(G_1)) = \pi(C_G(G_1)) = \pi(G_2)$. Similarly, applying Theorem 9.1 with $H = G_2$ gives that $C_{G/N}(\pi(G_2)) = \pi(G_1)$, completing the proof. □

Proposition 9.5. *Let G be a complex reductive algebraic group, and let K be a central subgroup of G. Set $H = G/K$. If*

$$1 \to K \to G \to H \to 1$$

splits, then the dual pairs in H are in bijection with the dual pairs of G.

Proof. By definition, the short exact sequence $1 \to K \to G \to H \to 1$ splits if there exists a group homomorphism $\alpha : H \to G$ such that $\pi \circ \alpha = \mathrm{id}_H$. It is straightforward to see that $H \simeq \alpha(H)$, showing that H can be identified with a subgroup of G. Moreover, we have by the splitting lemma that $G = K \rtimes H$, so we have in particular that $G = KH$. Therefore, the hypotheses of Theorem 6.5 are satisfied, and the proposition follows. □

10. Dual Pairs in $PGL(U)$

Let U be a finite dimensional complex vector space. In this section, we (i) classify the connected dual pairs in $PGL(U)$, (ii) construct two classes of disconnected dual pairs in $PGL(U)$, and (iii) discuss an approach for determining whether there are other classes of disconnected dual pairs in $PGL(U)$.

10.1. Connected Dual Pairs in $PGL(U)$.

Lemma 10.1. *Suppose $ABA^{-1} = cB$, where $A, B \in GL(n, \mathbb{C})$ and $c \in Z(GL(n, \mathbb{C})) \simeq \mathbb{C}^*$. Then c is an n-th root of unity.*

Proof. Using that the determinant is multiplicative, we see that $\det(ABA^{-1}) = \det(B)$. Therefore, the relation $ABA^{-1} = cB$ gives that

$$\det(B) = \det(cB) = c^n \det(B).$$

This shows that $c^n = 1$, or that c is an n-th root of unity (not necessarily primitive). □

Theorem 10.2. *Let (G_1, G_2) be a dual pair in $GL(U)$, and let $\pi : GL(U) \to PGL(U)$ be the canonical projection. Then $(\pi(G_1), \pi(G_2))$ is a dual pair in $PGL(U)$.*

Proof. By Remark 3.4, G_1 and G_2 are connected. Let n be such that $GL(n, \mathbb{C}) = GL(U)$, and write $Z := Z(GL(U)) \simeq \mathbb{C}^*$. By Lemma 10.1, both $K_{G_1,Z} := \{tg_1t^{-1}g_1^{-1} : t \in \pi^{-1}(C_{PGL(U)}(\pi(G_1))),\ g_1 \in G_1\}$ and $K_{G_2,Z} := \{tg_2t^{-1}g_2^{-1} : t \in \pi^{-1}(C_{PGL(U)}(\pi(G_2))),\ g_2 \in G_2\}$ contain only n-th roots of unity, and hence are discrete. Therefore, Corollary 9.4 applies, completing the proof. □

Theorem 10.2 shows that every dual pair in $GL(U)$ descends to a dual pair in $PGL(U)$ under the canonical projection. In fact, as the following proposition will show, the dual pairs in $GL(U)$ are in bijection with the connected dual pairs in $PGL(U)$.

Proposition 10.3. *Let (G_1, G_2) be a connected dual pair in $PGL(U)$, and define $\widetilde{G_1} := \pi^{-1}(G_1)$ and $\widetilde{G_2} := \pi^{-1}(G_2)$, where $\pi : GL(U) \to PGL(U)$ denotes the canonical projection. Then $(\widetilde{G_1}, \widetilde{G_2})$ form a dual pair in $GL(U)$.*

Proof. Since $GL(U)$ is a $\mathbb{C}^*$ bundle over $PGL(U)$, we have that $\widetilde{G_1}$ (resp. $\widetilde{G_2}$) is a $\mathbb{C}^*$ bundle over G_1 (resp. G_2). It follows that $\widetilde{G_i}$ has the same component group as G_i for $i = 1, 2$. In particular, $\widetilde{G_1}$ and $\widetilde{G_2}$ are connected. Since (G_1, G_2) is a dual pair in $PGL(U)$, we have that the elements of $\widetilde{G_1}$ and $\widetilde{G_2}$ commute up to a scalar, so if $n := \dim U$, we obtain a map

$$\begin{aligned} z : \ \widetilde{G_1} \times \widetilde{G_2} &\to (n\text{-th roots of unity}) \\ (x, y) &\mapsto xyx^{-1}y^{-1}, \end{aligned}$$

where we have used Lemma 10.1 to determine the codomain. But since $\widetilde{G_1} \times \widetilde{G_2}$ is connected, we see that z is a continuous map from a connected group to a discrete group, and hence must be constant. Since $(1, 1) \in \widetilde{G_1} \times \widetilde{G_2}$ maps to 1 under z, it follows that z is trivial, and hence that $(\widetilde{G_1}, \widetilde{G_2})$ is a dual pair in $GL(U)$. □

Remark 10.4. As explained in the proof of Proposition 10.3, we have that a subgroup G in $PGL(U)$ is connected if and only if its preimage in $GL(U)$ is connected. Recalling from Remark 3.4 that all dual pairs in $GL(U)$ are connected, we see that Theorem 10.2 and Proposition 10.3 give that the dual pairs in $GL(U)$ are in bijection with the connected dual pairs in $PGL(U)$.

10.2. **Disconnected Dual Pairs in $PGL(U)$.** Although all connected dual pairs in $PGL(U)$ arise as the images in $PGL(U)$ of dual pairs in $GL(U)$, we will soon see that not all dual pairs in $PGL(U)$ are connected (see Proposition 10.5, for example). Therefore, to classify the dual pairs in $PGL(U)$, it remains to consider the disconnected dual pairs. The following proposition describes one class of disconnected dual pairs in $PGL(U)$.

Proposition 10.5. *Let A be a finite abelian group of order n, and let $\widehat{A}$ denote its group of characters. Let U be an n-dimensional complex vector space. Then $\pi(\langle A, \widehat{A}\rangle)$ can be realized as a maximal abelian subgroup of $PGL(U)$, where $\pi : GL(U) \to PGL(U)$ denotes the canonical projection.*

Proof. Let us view U as the space of functions $L^2(A, \mathbb{C})$. Then $f \in U$ can be viewed as a column vector $[f(a_1) \ \cdots \ f(a_n)]^T$, where $a_1, \ldots, a_n$ are the elements of A. Each element

$a \in A$ acts by translation (τ_a) on $f \in U$, and each element $\chi \in \widehat{A}$ acts by multiplication (σ_χ) on U. Since multiplying by a^{-1} permutes the elements of A, each

$$\tau_a : U \to U$$
$$f(x) \mapsto f(xa^{-1})$$

can be viewed as a permutation matrix in $GL(U)$. Additionally, we can view each

$$\sigma_\chi : U \to U$$
$$f(x) \mapsto \chi(x)f(x)$$

as a diagonal matrix $\sigma_\chi = \operatorname{diag}(\chi(a_1), \ldots, \chi(a_n))$.

Now, observe that we have the following relations:

$$(\sigma_\chi \tau_a f)(x) = (\sigma_\chi f)(xa^{-1}) = \chi(xa^{-1})f(xa^{-1}), \text{ and}$$
$$(\tau_a \sigma_\chi f)(x) = \chi(x)(\tau_a f)(x) = \chi(x)f(xa^{-1}).$$

Consequently, we get that

$$\sigma_\chi \tau_a \sigma_\chi^{-1} = \chi(a^{-1})\tau_a \quad \text{and} \quad \tau_a \sigma_\chi \tau_a^{-1} = \chi(a)\sigma_\chi. \tag{8}$$

This shows that the actions of A and $\widehat{A}$ commute up to a scalar, and hence that $\pi(\langle A, \widehat{A}\rangle)$ is contained in its own centralizer in $PGL(U)$. We have left to show that $C_{PGL(U)}(\pi(\langle A, \widehat{A}\rangle)) \subseteq \pi(\langle A, \widehat{A}\rangle)$.

Let $t \in \pi^{-1}(C_{PGL(U)}(\pi(\langle A, \widehat{A}\rangle)))$. Then, in particular, $t\sigma_\chi t^{-1} = k_{t,\chi}\sigma_\chi$ for each $\chi \in \widehat{A}$ and for some $k_{t,\chi} \in \mathbb{C}^*$. Recall that $\sigma_\chi = \operatorname{diag}(\chi(a_1), \ldots, \chi(a_n))$, and assume (without loss of generality) that $a_1 = 1$. Since the irreducible characters of a finite group are linearly independent, the linear span of the σ_χ is all diagonal matrices. Consequently, the condition

$$t\sigma_\chi t^{-1} = k_{t,\chi}\sigma_\chi \ \forall \chi \in \widehat{A}$$

gives that t preserves the diagonal matrices. It follows that

$$t \in N_{GL(U)}(\text{diagonal matrices}) = (\text{permutation matrices}) \times (\text{diagonal matrices}).$$

Therefore, t can be written as $t = s^{-1}d$ for some permutation matrix s and some diagonal matrix d. Since conjugation preserves eigenvalues (and hence the set of diagonal entries of σ_χ), and since $\chi(a_1) = 1$, we get that

$$t\sigma_\chi t^{-1} = s^{-1}\sigma_\chi s = \chi(s \cdot 1)\sigma_\chi \ \forall \chi \in \widehat{A}, \tag{9}$$

where s is viewed as a permutation of the elements of A. It follows that $k_{t,\chi} = \chi(s \cdot 1)$, and hence that $s = \tau_{s \cdot 1}^{-1}$.

Next, by our assumption that $t \in \pi^{-1}(C_{PGL(U)}(\pi(\langle A, \widehat{A}\rangle)))$, we have that $t\tau_a t^{-1} = \ell_{t,a}\tau_a$ for all $a \in A$ and for some $\ell_{t,a} \in \mathbb{C}^*$. Now, observe that the τ_a's send class functions on A to class functions on A. (All functions on A are class functions, since A is abelian.) Recalling that the irreducible characters are a basis for the class functions, it follows that the τ_a's can be diagonalized with respect to the irreducible characters $\{\chi_1, \ldots, \chi_n\}$. Since

$$(\tau_a \chi_i)(x) = \chi(xa^{-1}) = \chi(a^{-1})\chi(x),$$

we can write $\tau_a = \operatorname{diag}(\chi_1(a^{-1}), \ldots, \chi_n(a^{-1}))$. Since the irreducible characters of a finite group are linearly independent, the linear span of the τ_a is all diagonal matrices. Therefore, with respect to this basis of irreducible characters, $t = (s')^{-1}d'$ for some permutation matrix

s' and some diagonal matrix d'. Assuming (without loss of generality) that χ_1 is the trivial character, it follows that

$$t\tau_a t^{-1} = (s')^{-1}\tau_a s' = \chi_{s'\cdot 1}(a^{-1})\tau_a \;\forall a \in A, \tag{10}$$

where s' is viewed as a permutation of the indices of the irreducible characters $\{\chi_1, \dots, \chi_n\}$. This gives that $\ell_{t,a} = \chi_{s'\cdot 1}(a^{-1})$, and hence that $s' = \sigma^{-1}_{\chi_{s'\cdot 1}}$.

Combining (8), (9), and (10), we see that the element $\sigma^{-1}_{\chi_{s'\cdot 1}}\tau^{-1}_{s\cdot 1}t$ commutes with every element of $\pi(\langle A, \widehat{A}\rangle)$:

$$(\sigma^{-1}_{\chi_{s'\cdot 1}}\tau^{-1}_{s\cdot 1}t)\sigma_\chi(\sigma^{-1}_{\chi_{s'\cdot 1}}\tau^{-1}_{s\cdot 1}t)^{-1} = \chi(s\cdot 1)\sigma^{-1}_{\chi_{s'\cdot 1}}(\tau^{-1}_{s\cdot 1}\sigma_\chi\tau_{s\cdot 1})\sigma_{\chi_{s'\cdot 1}} = \sigma^{-1}_{\chi_{s'\cdot 1}}\sigma_\chi\sigma_{\chi_{s'\cdot 1}} = \sigma_\chi, \text{ and}$$
$$(\sigma^{-1}_{\chi_{s'\cdot 1}}\tau^{-1}_{s\cdot 1}t)\tau_a(\sigma^{-1}_{\chi_{s'\cdot 1}}\tau^{-1}_{s\cdot 1}t)^{-1} = \chi_{s\cdot 1}(-a)\sigma^{-1}_{\chi_{s'\cdot 1}}(\tau^{-1}_{s\cdot 1}\tau_a\tau_{s\cdot 1})\sigma_{\chi_{s'\cdot 1}} = \chi_{s\cdot 1}(-a)\sigma^{-1}_{\chi_{s'\cdot 1}}\tau_a\sigma_{\chi_{s'\cdot 1}} = \tau_a.$$

Finally, let $H = A.\widehat{A}.S^1$; then H acts on U by unitary operators (as described above), and these actions realize U as a representation of H:

$$H \to GL(U)$$
$$a\cdot\chi\cdot z \mapsto \tau_a\cdot\sigma_\chi\cdot z,$$

where $a \in A$, $\chi \in \widehat{A}$, and $z \in S^1$. Moreover, we have by the Stone-von Neumann theorem (see [7] or [4, Chapter 14]) that U is irreducible as a representation of H. From the above, we have that $\sigma^{-1}_{\chi_{s'\cdot 1}}\tau^{-1}_{s\cdot 1}t$ defines an H-linear map $U \to U$. It follows from Schur's lemma that $\sigma^{-1}_{\chi_{s'\cdot 1}}\tau^{-1}_{s\cdot 1}t = \lambda I$ for some $\lambda \in \mathbb{C}^*$. Therefore, $t = \lambda\tau_{s\cdot 1}\sigma_{\chi_{s'\cdot 1}}$, giving that $\pi(t) \in \pi(\langle A, \widehat{A}\rangle)$, as desired. □

Lemma 10.6. *Let A be a finite abelian group of order n, let $V = L^2(A, \mathbb{C})$, and let W be an m-dimensional vector space. Then there is a natural isomorphism $V \otimes W \xrightarrow{\sim} L^2(A, W)$.*

Proof. To define a map $V \otimes W \to L^2(A, W)$, it suffices to define a map on the simple tensors $(f, w) \in V \otimes W$, and then extend by linearity. With this in mind, define $\psi : V \otimes W \to L^2(A, W)$ as follows:

$$\psi : V \otimes W \to L^2(A, W)$$
$$(f, w) \mapsto f\cdot w := [a \mapsto f(a)w].$$

It is easy to check that ψ is well-defined (i.e. that pairs of simple tensors $(f, w) \sim (f', w')$ map to the same element of $L^2(A, W)$). It remains to show that ψ is injective and surjective. However, since $\dim(V \otimes W) = \dim(L^2(A, W)) = nm$, it suffices to show that ψ is injective. To this end, suppose that $\sum_{i=1}^k f_i \cdot w_i = 0$ for some $f_1, \dots, f_k \in V$ and $w_1, \dots, w_k \in W$. Assume, without loss of generality, that the w_i's are linearly independent. Then for any $a \in A$, $\sum_{i=1}^k f_i(a)w_i = 0$, which gives that $f_i = 0$ for all i. It follows that ψ is injective, completing the proof. □

Theorem 10.7. *Let A be a finite abelian group of order n, and let $\widehat{A}$ denote its group of characters. Let V be an n-dimensional complex vector space, and W an m-dimensional complex vector space (for some $m \in \mathbb{N}$). Let (H_1, H_2) be a dual pair in $GL(W)$. Then*

$$(\pi(\langle A, \widehat{A}, H_1\rangle),\ \pi(\langle A, \widehat{A}, H_2\rangle))$$

can be realized as a dual pair in $PGL(V \otimes W)$, where $\pi : GL(U) \to PGL(U)$ denotes the canonical projection.

Proof. Lemma 10.6 shows that we can view $V \otimes W$ as the space of functions $L^2(A, W)$, where V is viewed as $L^2(A, \mathbb{C})$. A function $f \in L^2(A, W)$ can be viewed as a column vector $[f(a_1) \ \cdots \ f(a_n)]^T$, where each $f(a_i) \in W$. For each $\chi \in \widehat{A}$, we can view

$$\sigma_\chi : V \otimes W \to V \otimes W$$
$$f(x) \mapsto \chi(x) f(x)$$

as a block diagonal matrix $\mathcal{A}_\chi = \operatorname{diag}(\chi(a_1) I_m, \ldots, \chi(a_n) I_m)$. Additionally, since multiplying by a^{-1} permutes the elements of A, each

$$\tau_a : V \otimes W \to V \otimes W$$
$$f(x) \mapsto f(xa^{-1})$$

can be represented as an $(m \times m)$-block permutation matrix $\mathcal{A}_a$. Each matrix $B \in GL(W)$ gets embedded into $GL(V \otimes W)$ as $\mathcal{B} := \operatorname{diag}(B, \ldots, B)$. It is straightforward to see that any $\mathcal{B}$ commutes with any $\mathcal{A}_\chi$ and any $\mathcal{A}_a$. Moreover, the calculations in the proof of Proposition 10.5 show that the $\mathcal{A}_\chi$'s and $\mathcal{A}_a$'s commute up to scalars. Consequently, the inclusions $\pi(\langle A, \widehat{A}, H_1 \rangle) \subseteq C_{PGL(V \otimes W)}(\pi(\langle A, \widehat{A}, H_2 \rangle))$ and $\pi(\langle A, \widehat{A}, H_2 \rangle) \subseteq C_{PGL(V \otimes W)}(\pi(\langle A, \widehat{A}, H_1 \rangle))$ are clear.

Now, let $M \in \pi^{-1}(C_{PGL(V \otimes W)}(\pi(\langle A, \widehat{A}, H_1 \rangle)))$. Then, in particular, M commutes with each $\mathcal{A}_\chi$ up to a scalar. By the same argument as in the proof of Proposition 10.5, this shows that M is a product of an $(m \times m)$-block permutation matrix S and an $(m \times m)$-block diagonal matrix D, and that $S = \mathcal{A}_a$ for some $a \in A$.

Let Z be the group of mn-th roots of unity. By Lemma 10.1 and the argument above, we have that both M and S commute with each $\mathcal{A}_\chi$ and $\mathcal{A}_a$ up to an element of Z; therefore, we see that $D = S^{-1} M$ does as well. Write $D = \operatorname{diag}(D_1, \ldots, D_n)$, where $D_i \in GL(W)$ for $1 \leq i \leq n$. We claim that for each $1 < i \leq n$, $D_i = z_i D_1$ for some $z_i \in Z$. Indeed, $\tau_{a_i^{-1} a_1}$ acts on $f \in V \otimes W$ as follows:

$$\tau_{a_i^{-1} a_1} \cdot [f(a_1), \ldots, f(a_i), \ldots, f(a_n)]^T = [f(a_i), \ldots, f(a_i a_1^{-1} a_i), \ldots, f(a_n a_1^{-1} a_i)]^T.$$

Therefore, conjugating D by $\mathcal{A}_{a_i^{-1} a_1}$ yields $D' = \operatorname{diag}((D_1' = D_i), \ldots, D_i', \ldots, D_n')$. But on the other hand, $\mathcal{A}_{a_i a_1^{-1}} D \mathcal{A}_{a_i a_1^{-1}}^{-1} = z_i D$ for some $z_i \in Z$ (by Lemma 10.1). It follows that $D' = z_i D$ and hence that $D_i = z_i D_1$. Consequently,

$$D = \operatorname{diag}(D_1, z_2 D_1, \ldots, z_n D_1) \text{ for some } z_2, \ldots, z_n \in Z.$$

Therefore, since D commutes with each $\mathcal{A}_\chi$ and $\mathcal{A}_a$ (up to an element of Z), so does $C := \operatorname{diag}(I_m, z_2 I_m, \ldots, z_n I_m)$. But C lies in the image of $GL(V)$ in $GL(V \otimes W)$, so this gives that C must equal $\lambda \cdot \mathcal{A}_\chi$ for some $\lambda \in \mathbb{C}^*$ and some $\chi \in \widehat{A}$. Therefore, $D = \lambda \cdot \mathcal{A}_\chi \cdot \operatorname{diag}(D_1, \ldots, D_1)$, and hence $M = SD = \lambda \cdot \mathcal{A}_a \cdot \mathcal{A}_\chi \cdot \operatorname{diag}(D_1, \ldots, D_1)$. Since $\operatorname{diag}(D_1, \ldots, D_1)$ lies in the image of $GL(W)$ in $GL(V \otimes W)$, we see that D_1 commutes with every element of H_1 (up to a root of unity). But since H_1 is connected (by Remark 3.4), this means that D_1 in fact *commutes* with every element of H_1, giving $D_1 \in C_{GL(W)}(H_1) = H_2$. It follows that $\pi(M) \in \pi(\langle A, \widehat{A}, H_2 \rangle)$, so we get $C_{PGL(V \otimes W)}(\pi(\langle A, \widehat{A}, H_1 \rangle)) = \pi(\langle A, \widehat{A}, H_2 \rangle)$. Reversing the roles of H_1 and H_2 further gives that $C_{PGL(V \otimes W)}(\pi(\langle A, \widehat{A}, H_2 \rangle)) = \pi(\langle A, \widehat{A}, H_1 \rangle)$, completing the proof. □

Theorem 10.8. *Let A be a finite abelian group of order n, and let $\widehat{A}$ denote its group of characters. Let V be an n-dimensional complex vector space, and W an m-dimensional* *complex vector space (for some $m \in \mathbb{N}$). Let (H_1, H_2) be a dual pair in $GL(W)$. Then*

$$\Big(\pi(\langle \widehat{A}, H_1\rangle),\ \pi(\langle A, (H_2)^n\rangle)\Big)$$

can be realized as a dual pair in $PGL(V \otimes W)$, where $\pi : GL(U) \to PGL(U)$ denotes the canonical projection.

Proof. Let $a_1, \ldots, a_n$ denote the elements of A. As in the proof of Proposition 10.7, let us view $V \otimes W$ as the space of function $L^2(A, W)$, where V is viewed as $L^2(A, \mathbb{C})$; additionally, view $\chi \in \widehat{A}$ as $\mathcal{A}_\chi = \operatorname{diag}(\chi(a_1)I_m, \ldots, \chi(a_n)I_m)$, and view each $a \in A$ as an $(m \times m)$-block permutation matrix $\mathcal{A}_a$. Moreover, each $h \in H_1$ can be viewed as $\operatorname{diag}(h, \ldots, h) \in GL(V \otimes W)$, whereas each $(h_1, \ldots, h_n) \in (H_2)^n$ can be viewed as $\operatorname{diag}(h_1, \ldots, h_n)$.

We start by showing that $C_{PGL(V\otimes W)}(\pi(\langle A, (\mathbb{C}^*)^n, H_2\rangle)) = \pi(\langle \widehat{A}, H_1\rangle)$, where $\mathbb{C}^*$ is realized as the set of matrices $\{\operatorname{diag}(c_1 I_m, \ldots, c_n I_m)\ :\ c_1, \ldots, c_n \in \mathbb{C}^*\}$. By Fact 1.2, this will give that $\pi(\langle \widehat{A}, H_1\rangle)$ is a member of a dual pair in $PGL(V \otimes W)$, so to finish the proof it will suffice to show that $C_{PGL(V\otimes W)}(\pi(\langle \widehat{A}, H_1\rangle)) = \pi(\langle A, (H_2)^n\rangle)$.

Let $M \in \pi^{-1}(C_{PGL(V\otimes W)}(\pi(\langle A, (\mathbb{C}^*)^n, H_2\rangle)))$. Since $\widehat{A} \subseteq (\mathbb{C}^*)^n$, we see that

$$C_{PGL(V\otimes W)}(\pi(\langle A, (\mathbb{C}^*)^n, H_2\rangle)) \subseteq C_{PGL(V\otimes W)}(\pi(\langle A, \widehat{A}, H_2)) = \pi(\langle A, \widehat{A}, H_1\rangle),$$

where we have used Theorem 10.7. Hence we can write $M = \lambda \cdot \mathcal{A}_a \cdot \mathcal{A}_\chi \cdot h$ for some $\lambda \in \mathbb{C}^*$, $a \in A$, $\chi \in \widehat{A}$, and $h \in H_1$. But since M commutes (up to scalar) with all of $(\mathbb{C}^*)^n$, we see that $\mathcal{A}_a = I_{mn}$. It follows that $\pi(M) \in \pi(\langle \widehat{A}, H_1\rangle)$, and hence that $C_{PGL(V\otimes W)}(\pi(\langle A, (\mathbb{C}^*)^n, H_2\rangle)) \subseteq \pi(\langle \widehat{A}, H_1\rangle)$; inclusion the other way is clear, so we have $C_{PGL(V\otimes W)}(\pi(\langle A, (\mathbb{C}^*)^n, H_2\rangle)) = \pi(\langle \widehat{A}, H_1\rangle)$.

It remains to show that $C_{PGL(V\otimes W)}(\pi(\langle \widehat{A}, H_1\rangle)) = \pi(\langle A, (H_2)^n\rangle)$. To this end, let $N \in \pi^{-1}(C_{PGL(V\otimes W)}(\pi(\langle \widehat{A}, H_1\rangle)))$. Then, in particular, N commutes with each $\mathcal{A}_\chi$ up to a scalar, and hence must be of the form $\mathcal{A}_a \cdot \operatorname{diag}(D_1, \ldots, D_n)$ for some $a \in A$ and some $D_1, \ldots, D_n \in GL(W)$ (by the same argument as in the proof of Theorem 10.7). Moreover, N commutes (up to a scalar) with $\operatorname{diag}(h, \ldots, h)$ for each $h \in H_1$, meaning each D_i must commute with each $h \in H_1$ (up to a scalar). Since H_1 is connected, we in fact have that $D_i \in C_{GL(W)}(H_1) = H_2$. Hence $D \in (H_2)^n$ and $\pi(N) \in \pi(\langle A, (H_2)^n\rangle)$. It follows that $C_{PGL(V\otimes W)}(\pi(\langle \widehat{A}, H_1\rangle)) \subseteq \pi(\langle A, (H_2)^n\rangle)$; since containment in the other direction is clear, this completes the proof. $\square$

Example 10.9. Let A be a finite abelian group of order 2, and let V be the space of functions on A. Then $\widehat{A} = \langle \sigma_\chi\rangle$, where $\sigma_\chi = \operatorname{diag}(1, -1)$ and $A = \langle \tau_a\rangle$, where $\tau_a = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$. Let $(H_1, H_2) = (GL(2, \mathbb{C}), GL(1, \mathbb{C}))$. Then by Theorem 10.8,

$$\langle \widehat{A}, H_1\rangle = \left\langle \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix},\ \left\{ \begin{pmatrix} a & b & & \\ c & d & & \\ & & a & b \\ & & c & d \end{pmatrix} \ :\ ad - bc \neq 0 \right\} \right\rangle, \text{ and}$$

$$\langle A, (H_2)^2\rangle = \left\langle \begin{pmatrix} & I_2 \\ I_2 & \end{pmatrix},\ \left\{ \begin{pmatrix} x & & & \\ & x & & \\ & & y & \\ & & & y \end{pmatrix} \ :\ x, y \in \mathbb{C}^* \right\} \right\rangle$$

descend to a dual pair in $PGL(V \otimes W)$.

10.3. **Have We Found All of the Dual Pairs in $PGL(U)$?** Let U be a finite dimensional complex vector space, and let $\pi : GL(U) \to PGL(U)$ denote the canonical projection. As previously mentioned, the connected dual pairs in $PGL(U)$ are in bijection with the dual pairs in $GL(U)$. Additionally, Theorems 10.7 and 10.8 describe two classes of disconnected dual pairs in $PGL(U)$. It is currently unknown whether additional classes of disconnected dual pairs in $PGL(U)$ exist. However, there is an approach which, if successful, could help reveal new classes of dual pairs in $PGL(U)$ or prove that all $PGL(U)$ dual pairs have been accounted for. In the remainder of this section, we describe this approach.

Broadly, the aforementioned approach is to take an arbitrary dual pair (G_1, G_2) in $PGL(U)$ and to classify the possible preimages $\widetilde{G_1} := \pi^{-1}(G_1)$ and $\widetilde{G_2} := \pi^{-1}(G_2)$. These preimages appear in the short exact sequences

$$1 \to \widetilde{G_1}^\circ \to \widetilde{G_1} \to \Gamma_1 \to 1 \quad \text{and} \quad 1 \to \widetilde{G_2}^\circ \to \widetilde{G_2} \to \Gamma_2 \to 1, \tag{11}$$

where $\Gamma_1 := \widetilde{G_1}/\widetilde{G_1}^\circ \simeq G_1/G_1^\circ$ and $\Gamma_2 := \widetilde{G_2}/\widetilde{G_2}^\circ \simeq G_2/G_2^\circ$, and where these isomorphisms follow from the explanation given at the beginning of the proof of Proposition 10.3. The following theorem and corollary provide information regarding $\widetilde{G_1}^\circ$, $\widetilde{G_2}^\circ$, Γ_1, and Γ_2 which has the potential to be helpful for classifying the possible $\widetilde{G_1}$ and $\widetilde{G_2}$.

Theorem 10.10 (D. Vogan)**.** *Let (G_1, G_2) be a dual pair in $PGL(U)$, and let $\widetilde{G_1}$, $\widetilde{G_2}$ denote the preimages of G_1 and G_2 (respectively) in $GL(U)$. Then the component groups*

$$\Gamma_1 := G_1/G_1^\circ \simeq \widetilde{G_1}/\widetilde{G_1}^\circ \quad \textit{and} \quad \Gamma_2 := G_2/G_2^\circ \simeq \widetilde{G_2}/\widetilde{G_2}^\circ$$

are dual finite abelian groups.

Proof. Recall that for any algebraic group G, the centralizer in G of any subset of G is algebraic. As a consequence, we have that any member of a dual pair is algebraic, and hence that Γ_1 and Γ_2 are finite. As in the proof of Proposition 10.3, the mutual centralizer relation gives the map

$$\begin{aligned} z:\ \widetilde{G_1} \times \widetilde{G_2} &\to (n\text{-th roots of unity}) \\ (x, y) &\mapsto xyx^{-1}y^{-1}, \end{aligned}$$

where $n := \dim U$. As established in the proof of Proposition 10.3, z must be constant on each connected component of $\widetilde{G_1} \times \widetilde{G_2}$, and hence must descend to a map

$$z : \Gamma_1 \times \Gamma_2 \to (n\text{-th roots of unity}).$$

It is not hard to check from the defining equation that z is actually a group homomorphism for both $\widetilde{G_1}$ and $\widetilde{G_2}$ (i.e. that $z(x_1x_2, y) = z(x_1, y)z(x_2, y)$ and that $z(x, y_1y_2) = z(x, y_1)z(x, y_2)$ for all $x_1, x_2, x \in G_1$ and all $y_1, y_2, y \in G_2$). It follows that z arises from a group homomorphism

$$\begin{aligned} Z:\ \Gamma_1 &\to \widehat{\Gamma_2} \\ a &\mapsto z(a, \cdot). \end{aligned}$$

We will show that Z is in fact an isomorphism. Towards proving injectivity, define $\Gamma_1' := \ker Z$. Then Γ_1' corresponds to the following subgroup of $\widetilde{G_1}$:

$$\widetilde{G_1'} := \{x \in \widetilde{G_1}\ :\ xyx^{-1} = y \text{ for all } y \in \widetilde{G_2}\}.$$

Recall that

$$\widetilde{G_1} = \{t \in GL(U) \ : \ tyt^{-1}y^{-1} \in \mathbb{C}^* \text{ for all } y \in \widetilde{G_2}\}.$$

Therefore, we see that

$$\widetilde{G_1'} = \{t \in GL(U) \ : \ tyt^{-1} = y \text{ for all } y \in \widetilde{G_2}\} = C_{GL(U)}(\widetilde{G_2}).$$

Since $\widetilde{G_1'}$ is a centralizer in $GL(U)$, it is part of a dual pair $(\widetilde{G_1'}, C_{GL(U)}(\widetilde{G_1'}))$, and hence is connected (by Remark 3.4). It follows that Γ_1' is trivial, and hence that Z is injective. Since $\widehat{\Gamma_2}$ is abelian, this gives that Γ_1 is abelian as well. Switching the roles of Γ_1 and Γ_2 in this argument gives that

$$\begin{aligned} Z' :\ \Gamma_2 &\to \widehat{\Gamma_1} \\ \alpha &\mapsto z(\cdot, \alpha) \end{aligned}$$

is injective, and that Γ_2 is abelian.

Finally, since the dual group functor is a contravariant exact functor for locally compact abelian groups, the injectivity of Z' implies the surjectivity of Z, completing the proof. □

As a particular consequence of Theorem 10.10, we obtain information regarding the identity components of preimages of members of $PGL(U)$ dual pairs:

Corollary 10.11. *Let (G_1, G_2) be a dual pair in $PGL(U)$ with preimages $\widetilde{G_1}$ and $\widetilde{G_2}$ in $GL(U)$. Then*

$$\widetilde{G_1}^{\circ} = C_{GL(U)}(\widetilde{G_2}) = \prod_i GL(V_i) \quad \text{and} \quad \widetilde{G_2}^{\circ} = C_{GL(U)}(\widetilde{G_1}) = \prod_j GL(V_j'),$$

where the V_i's and W_j's are finite dimensional complex vector spaces satisfying $U \simeq \bigoplus_i V_i \otimes W_i \simeq \bigoplus_j V_j' \otimes W_j'$ for some sets of finite dimensional complex vector spaces $\{W_i\}$ and $\{W_j'\}$.

Proof. The equalities $\widetilde{G_1}^{\circ} = C_{GL(U)}(\widetilde{G_2})$ and $\widetilde{G_2}^{\circ} = C_{GL(U)}(\widetilde{G_1})$ follow from the proof of Theorem 10.10. Then by Fact 1.2 and Corollary 3.3, we see that $C_{GL(U)}(\widetilde{G_2}) = \prod_i GL(V_i)$ and that $C_{GL(U)}(\widetilde{G_1}) = \prod_j GL(V_j')$ for some $\{V_i\}$ and $\{V_j'\}$ as described in the lemma statement. □

Let (G_1, G_2) be a dual pair in $PGL(U)$ with preimages $\widetilde{G_1}$ and $\widetilde{G_2}$ in $GL(U)$. Define $\Gamma := G_1/G_1^\circ \simeq \widetilde{G_1}/\widetilde{G_1}^{\circ}$. Then by Theorem 10.10, $\widehat{\Gamma} \simeq G_2/G_2^\circ \simeq \widetilde{G_2}/\widetilde{G_2}^{\circ}$. We now require a fact about the algebraic extensions of reductive groups:

Proposition 10.12 ([10, Theorem 1.6])**.** *Let G be a complex reductive algebraic group and Γ a finite group. Then the equivalence classes of algebraic extensions of Γ by G are in bijection with the equivalence classes of algebraic extensions of Γ by $Z(G)$.*

Proof. Since [10] is unpublished and is not yet available online, we outline a proof of this fact. It is a standard fact [2] that the group of algebraic automorphisms $\mathrm{Aut}(G)$ is the semidirect product of the group $G/Z(G)$ of inner automorphisms and the group of automorphisms of the based root datum of G. The sections required for the semidirect product structure can be constructed using a pinning of G [9]. These same ideas can be used to show the desired result. □

Now, since $\widetilde{G_1}^\circ, \widetilde{G_2}^\circ$ are reductive and $\Gamma, \widehat{\Gamma}$ are finite, we have from Proposition 10.12 that giving algebraic extensions as in (11) is equivalent to giving algebraic extensions

$$(12) \qquad 1 \to Z(\widetilde{G_1}^\circ) \to E_1 \to \Gamma \to 1 \quad \text{and} \quad 1 \to Z(\widetilde{G_2}^\circ) \to E_2 \to \widehat{\Gamma} \to 1.$$

Moreover, it is well-known that the extensions in (12) are parametrized by the group cohomologies $H^2(\Gamma, Z(\widetilde{G}^\circ))$ and $H^2(\widehat{\Gamma}, Z(\widetilde{H}^\circ))$. Additionally, it follows from Corollary 10.11 that $Z(\widetilde{G_1}^\circ) = (\mathbb{C}^*)^{r_1}$ and $Z(\widetilde{G_2}^\circ) = (\mathbb{C}^*)^{r_2}$ for some $r_1, r_2 \in \mathbb{N}$. Therefore, to understand the possible $\widetilde{G_1}$ and $\widetilde{G_2}$, it suffices to understand the group cohomology $H^2(\Gamma, (\mathbb{C}^*)^r)$ for $r \in \mathbb{N}$.

Proposition 10.13. *Let $n, r \in \mathbb{N}$. Then for any action of $\mathbb{Z}/n\mathbb{Z}$ on $(\mathbb{C}^*)^r$ by conjugation, we have*

$$H^2(\mathbb{Z}/n\mathbb{Z}, (\mathbb{C}^*)^r) = 0.$$

Proof. Suppose we have

$$(13) \qquad 1 \to (\mathbb{C}^*)^r \to E \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 1,$$

where $\mathbb{Z}/n\mathbb{Z}$ acts on $(\mathbb{C}^*)^r$ by conjugation. It suffices to show that (13) splits.

Let $\mathbb{Z}/n\mathbb{Z} = \langle \mathcal{E} \rangle$, and let $\mathcal{E}'$ be a preimage of $\mathcal{E}$ in E. We have that $E = \langle (\mathbb{C}^*)^r, \mathcal{E}' \rangle$. Note that $\pi((\mathcal{E}')^n) = \pi(\mathcal{E}')^n = \mathcal{E}^n = 1$. Therefore, $(\mathcal{E}')^n \in \ker \pi \simeq (\mathbb{C}^*)^r$.

We would like to show that there exists a group homomorphism $\gamma : \mathbb{Z}/n\mathbb{Z} \to E$ such that $\pi \circ \gamma = \mathrm{id}_{\mathbb{Z}/n\mathbb{Z}}$. To this end, let $z := (\mathcal{E}')^n \in (\mathbb{C}^*)^r$. If $z = (z_1, \ldots, z_n)$, put $z_0 := z^{-1/n} = (z_1^{-1/n}, \ldots, z_n^{-1/n}) \in (\mathbb{C}^*)^r$. Define

$$\begin{aligned} \gamma : \mathbb{Z}/n\mathbb{Z} &\to E \\ 1 &\mapsto 1 \\ \mathcal{E} &\mapsto z_0\mathcal{E}' \\ \mathcal{E}^\ell &\mapsto \gamma(\mathcal{E})^\ell \ \forall \ell \in \mathbb{N}. \end{aligned}$$

We claim that γ is a group homomorphism satisfying $\pi \circ \gamma = \mathrm{id}_{\mathbb{Z}/n\mathbb{Z}}$.

To show that γ is a group homomorphism, it suffices to show that $\gamma(\mathcal{E})^n = 1$. Since $z = (\mathcal{E}')^n$, we have that z commutes with $\mathcal{E}'$; hence, so does z_0. Therefore,

$$\gamma(\mathcal{E})^n = (z_0\mathcal{E}')^n = z_0^n(\mathcal{E}')^n = z^{-1}z = 1,$$

as desired. Finally, note that

$$\pi(\gamma(\mathcal{E}^\ell)) = \pi(\gamma(\mathcal{E})^\ell) = \pi(\gamma(\mathcal{E}))^\ell = \mathcal{E}^\ell,$$

completing the proof. ☐

As a consequence of this proposition, we have that if Γ is cyclic, then $\widetilde{G_1} = \widetilde{G_1}^\circ \rtimes \Gamma$ and $\widetilde{G_2} = \widetilde{G_2}^\circ \rtimes \widehat{\Gamma}$ (with possibly trivial actions by Γ and $\widehat{\Gamma}$). In the case that Γ is cyclic, it remains to consider the possible pairs of actions of Γ and $\widehat{\Gamma}$ on $\widetilde{G_1}^\circ$ and $\widetilde{G_2}^\circ$, respectively, and to understand how the structure of the resulting dual pair (G_1, G_2) relates to those described in Theorems 10.7 and 10.8. In the case that Γ is not cyclic, it remains to answer these same questions, as well as to compute the cohomology $H^2(\Gamma, (\mathbb{C}^*)^r)$.

11. Dual Pairs in $PSp(U)$

Throughout this section, let U be a finite dimensional complex symplectic vector space.

11.1. **Dual Pairs in $PSp(V \otimes W)$.** Let H be a complex reductive algebraic group. Consider an algebraic symplectic representation $\varphi : H \to Sp(U)$, and suppose that $U \simeq V \otimes W$, where V is the unique irreducible subrepresentation of U (up to isomorphism), and where $W := \mathrm{Hom}_H(V, U)$. Moreover, suppose that V is an n-dimensional complex orthogonal vector space, and that W is an m-dimensional symplectic vector space.

From Section 4, we know that $(O(V), Sp(W))$ is a dual pair in $Sp(V \otimes W)$ under the embeddings $\iota : O(V) \hookrightarrow Sp(V \otimes W)$ and $\kappa : Sp(W) \hookrightarrow Sp(V \otimes W)$ described in Section 2. The results that follow in this subsection collectively prove the following theorem:

Theorem 11.1. *Let V be an n-dimensional complex orthogonal vector space and W an m-dimensional complex symplectic vector space. Define $\mathcal{S}(W) := \left\langle \kappa(Sp(W)), \begin{pmatrix} & I_m \\ -I_m & \end{pmatrix} \right\rangle \subseteq Sp(V \otimes W)$. Let $\pi : Sp(V \otimes W) \to PSp(V \otimes W)$ denote the canonical projection. Then the following hold:*

(1) If $n = 2$, then $(\pi(O(V)), \pi(\mathcal{S}(W)))$ is a dual pair in $PSp(V \otimes W)$.
(2) If $n \neq 2$, then $(\pi(O(V)), \pi(Sp(W)))$ is a dual pair in $PSp(V \otimes W)$.

Proof. This follows directly from Proposition 11.2 (which proves (1)), and Proposition 11.4 (which proves (2)). □

11.1.1. $PSp(V \otimes W)$ *with* $\dim V = 2$.

Proposition 11.2. *Let V be a 2-dimensional complex orthogonal vector space and W an m-dimensional complex symplectic vector space. Let $\mathcal{S}(W)$ be as defined above, and let $\pi : Sp(V \otimes W) \to PSp(V \otimes W)$ denote the canonical projection. Then $(\pi(O(V)), \pi(\mathcal{S}(W)))$ is a dual pair in $PSp(V \otimes W)$.*

Proof. We first show $C_{PSp(V\otimes W)}(\pi(O(V))) = \pi(\mathcal{S}(W))$. Let $M \in \pi^{-1}(C_{PSp(V\otimes W)}(\pi(O(V))))$. Then writing out the entry-wise implications of the relation

$$M \begin{pmatrix} I_m & \\ & -I_m \end{pmatrix} = \pm \begin{pmatrix} I_m & \\ & -I_m \end{pmatrix} M \tag{14}$$

gives that M has the block form $\begin{pmatrix} M_{11} & \\ & M_{22} \end{pmatrix}$ if the sign in (14) is positive, and has the 2×2 block form $\begin{pmatrix} & M_{12} \\ M_{21} & \end{pmatrix}$ if the sign in (14) is negative (where $M_{11}, M_{12}, M_{21}, M_{22}$ are $m \times m$ matrices). Subsequently writing out the entry-wise implications of the relation

$$M \begin{pmatrix} (\cos\theta) I_m & (\sin\theta) I_m \\ -(\sin\theta) I_m & (\cos\theta) I_m \end{pmatrix} = \pm \begin{pmatrix} (\cos\theta) I_m & (\sin\theta) I_m \\ -(\sin\theta) I_m & (\cos\theta) I_m \end{pmatrix} M \tag{15}$$

further gives that M has the block form $\begin{pmatrix} M_{11} & \\ & M_{11} \end{pmatrix}$ if the signs in (14) and (15) are both positive, and that M has the block form $\begin{pmatrix} & M_{12} \\ -M_{12} & \end{pmatrix}$ if the signs in (14) and (15) are negative and positive, respectively. Moreover, we see that the remaining sign combinations are impossible. Finally, requiring that $M \Omega M^T = \Omega$ shows that M_{11} and M_{12} are in $Sp(W)$. Consequently, $C_{PSp(V\otimes W)}(\pi(O(V))) \subseteq \pi(\mathcal{S}(W))$. On the other hand, it is straightforward to check that $\pi(\mathcal{S}(W)) \subseteq C_{PSp(V\otimes W)}(\pi(O(V)))$. It follows that $C_{PSp(V\otimes W)}(\pi(O(V))) = \pi(\mathcal{S}(W))$, as desired.

We have left to show that $C_{PSp(V\otimes W)}(\pi(\mathcal{S}(W))) = \pi(O(V))$. To this end, let $N \in \pi^{-1}(C_{PSp(V\otimes W)}(\pi(\mathcal{S}(W))))$. Then, in particular,

$$N\begin{pmatrix} A & \\ & A\end{pmatrix} = \pm\begin{pmatrix} A & \\ & A\end{pmatrix}N \quad \text{for all } A \in Sp(W). \tag{16}$$

Writing out the entry-wise implications of (16) gives that N is of the form $N = \begin{pmatrix} aI_m & bI_m \\ cI_m & dI_m\end{pmatrix}$ for some $a, b, c, d \in \mathbb{C}$. The relation $N\Omega N^T = \Omega$ further gives that $a^2 + b^2 = c^2 + d^2 = 1$ and that $ac + bd = 0$. It follows that $C_{PSp(V\otimes W)}(\pi(\mathcal{S}(W))) \subseteq \pi(O(V))$. On the other hand, it is straightforward to check that both $\begin{pmatrix} (\cos\theta)I_m & (\sin\theta)I_m \\ -(\sin\theta)I_m & (\cos\theta)I_m\end{pmatrix}$ and $\begin{pmatrix} I_m & \\ & -I_m\end{pmatrix}$ commute with $\begin{pmatrix} & I_m \\ -I_m & \end{pmatrix}$ (up to ± 1). Therefore, $C_{PSp(V\otimes W)}(\pi(\mathcal{S}(W))) = \pi(O(V))$. □

11.1.2. *$PSp(V\otimes W)$ with* $\dim V \neq 2$.

Lemma 11.3. *Let V be an n-dimensional complex orthogonal vector space and W an m-dimensional complex symplectic vector space. Assume that $n \neq 2$, and let $\pi : Sp(V\otimes W) \to PSp(V\otimes W)$ denote the canonical projection. Then $C_{Sp(V\otimes W)}(SO(V)) = Sp(W)$.*

Proof. Recall from Lemma 8.2 that the standard representation of $SO(V) \simeq SO(n, \mathbb{C})$ is irreducible for $n \neq 2$. It follows by Schur's lemma that $C_{Sp(V\otimes W)}(SO(V)) \subseteq Sp(W)$. On the other hand, we have that $C_{Sp(V\otimes W)}(SO(V)) \supseteq C_{Sp(V\otimes W)}(O(V)) = Sp(W)$, where we have used Theorem 4.4. □

Proposition 11.4. *Let V be an n-dimensional complex orthogonal vector space and W an m-dimensional complex symplectic vector space. Assume that $n \neq 2$, and let $\pi : Sp(V\otimes W) \to PSp(V\otimes W)$ denote the canonical projection. Then $(\pi(O(V)), \pi(Sp(W)))$ is a dual pair in $PSp(V\otimes W)$.*

Proof. Since $Sp(W)$ is connected and $Z(Sp(V\otimes W)) = \{\pm I_m\}$ is discrete, we can apply Theorem 9.1 to get that

$$C_{PSp(V\otimes W)}(\pi(Sp(W))) = \pi(C_{Sp(V\otimes W)}(Sp(W))) = \pi(O(V)).$$

Moreover, it is clear that $\pi(Sp(W)) \subseteq C_{PSp(V\otimes W)}(\pi(O(V)))$. It remains to show that $C_{PSp(V\otimes W)}(\pi(O(V))) \subseteq \pi(Sp(W))$. But this follows from

$$\pi^{-1}(C_{PSp(V\otimes W)}(\pi(O(V)))) \subseteq C_{Sp(V\otimes W)}(SO(V)) = Sp(W),$$

where the containment comes from Corollary 9.3 and the equality comes from Lemma 11.3. This completes the proof. □

11.2. **Dual Pairs in $PSp(\bigoplus V_\gamma \otimes W_\gamma)$.** Let H be a complex reductive algebraic group. Consider an algebraic symplectic representation $\varphi : H \to Sp(U)$, and suppose that $U \simeq \bigoplus_\gamma V_\gamma \otimes W_\gamma$, where $\{V_\gamma\}_\gamma = \{V_\mu, V_\nu\}_{\mu\not\simeq\mu^*,\, \nu\simeq\nu^*}$ is the set of nonisomorphic irreducible subrepresentations of U, and where $W_\gamma := \mathrm{Hom}_H(V_\gamma, U)$.

Proposition 11.5. *Let U, $\{V_\gamma\}$, $\{W_\gamma\}$ be as defined above, with $|\{\gamma\}| \geq 2$. Suppose that each V_γ is orthogonal. Then*

$$\left(\pi\left(\prod_\gamma O(V_\gamma)\right),\ \pi\left(\prod_\gamma Sp(W_\gamma)\right)\right)$$

is a dual pair in $PSp(U)$, where $\pi : Sp(U) \to PSp(U)$ denotes the canonical projection.

Proof. For convenience, write $G_1 := \prod_\gamma O(V_\gamma)$ and $G_2 := \prod_\gamma Sp(W_\gamma)$. From Section 4, we have that (G_1, G_2) is a dual pair in $Sp(U)$. Moreover, notice that G_2 is connected. Therefore, Theorem 9.1 gives that $C_{PSp(U)}(\pi(G_2)) = \pi(C_{Sp(U)}(G_2)) = \pi(G_1)$.

Next, it is clear that $C_{PSp(U)}(\pi(G_1)) \supseteq \pi(G_1)$. It remains to show that $C_{PSp(U)}(\pi(G_1)) \subseteq \pi(G_2)$. To this end, let $M \in \pi^{-1}(C_{PSp(U)}(\pi(G_1)))$. By Corollary 9.3, $M \in C_{Sp(U)}(G_1^\circ)$, where

$$G_1^\circ = \prod_\gamma SO(V_\gamma).$$

Following Lemma 8.3, write $\{\gamma\} = \{\gamma_1, \dots, \gamma_\ell\}$, $n_i := \dim V_{\gamma_i}$, and $m_i := \dim W_{\gamma_i}$, and assume (without loss of generality) that $n_1 = \cdots = n_k = 2$ and that $n_{k+1}, \dots, n_\ell \neq 2$. By Lemma 8.3, it follows that

$$M \in \prod_\gamma C_{Sp(V_\gamma \otimes W_\gamma)}(SO(V_\gamma)).$$

We can therefore write $M = \operatorname{diag}(M_1, \dots, M_\ell)$, where $M_i \in C_{Sp(V_{\gamma_i} \otimes W_{\gamma_i})}(SO(V_{\gamma_i}))$. By Lemma 11.3, $C_{Sp(V_{\gamma_j} \otimes W_{\gamma_j})}(SO(V_{\gamma_j})) = Sp(W_{\gamma_j})$ for all $k+1 \leq j \leq \ell$. Additionally, we have that $M_i \in \pi^{-1}(C_{PSp(V_{\gamma_i} \otimes W_{\gamma_i}}(\pi(O(V_{\gamma_i}))))) = \mathcal{S}(W_{\gamma_i})$ for $1 \leq i \leq k$, where we have used Theorem 11.1, and where

$$\mathcal{S}(W_{\gamma_i}) := \left\langle \kappa(Sp(W)), \begin{pmatrix} & I_{m_i} \\ -I_{m_i} & \end{pmatrix} =: J_i \right\rangle.$$

Suppose, for the sake of contradiction, that $M_i = J_i$ for some fixed $1 \leq i \leq k$.

Consider the set of $\dim U \times \dim U$ block diagonal matrices, where the diagonal blocks have sizes $m_1 n_1 \times m_1 n_1, \dots, m_\ell n_\ell \times m_\ell n_\ell$. Let N_i denote the matrix in this set that is the identity on every block except the i-th block, on which it equals $\operatorname{diag}(I_{m_i}, -I_{m_i})$. Then $N_i \in G_1$. However, $MN \neq \pm NM$ (since $|\{\gamma\}| \geq 2$), contradicting $M \in \pi^{-1}(C_{PSp(U)}(\pi(G_1)))$. It follows that $M_i \in Sp(W_{\gamma_i})$ for all $1 \leq i \leq \ell$, completing the proof. □

Theorem 11.6. *Let U, $\{V_\gamma\} = \{V_\mu, V_\nu\}$, $\{W_\gamma\} = \{W_\mu, W_\nu\}$ be as defined above, with $|\{\gamma\}| \geq 2$. Then*

$$\left(\pi\left(\prod_{\mu \not\simeq \mu^*} GL(V_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \textit{orthog.}}} O(V_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \textit{sympl.}}} Sp(V_\nu)\right), \pi\left(\prod_{\mu \not\simeq \mu^*} GL(W_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \textit{orthog.}}} Sp(W_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \textit{sympl.}}} O(W_\nu)\right)\right)$$

is a dual pair in $PSp(U)$, where $\pi : Sp(U) \to PSp(U)$ denotes the canonical projection.

Proof. Write

$$G_1 := \prod_{\mu \not\simeq \mu^*} GL(V_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \text{orthog.}}} O(V_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \text{sympl.}}} Sp(V_\nu), \text{ and}$$

$$G_2 := \prod_{\mu \not\simeq \mu^*} GL(W_\mu) \prod_{\substack{\nu \simeq \nu^* \\ \text{orthog.}}} Sp(W_\nu) \prod_{\substack{\nu \simeq \nu^* \\ \text{sympl.}}} O(W_\nu).$$

By Theorem 4.4, we have that (G_1, G_2) is a dual pair in $Sp(U)$. Therefore, we certainly have that $\pi(G_1) \subseteq C_{PSp(U)}(\pi(G_2))$ and that $\pi(G_2) \subseteq C_{PSp(U)}(\pi(G_1))$.

Let $M \in \pi^{-1}(C_{PSp(U)}(\pi(G_1)))$. Then by Corollary 9.3, $M \in C_{Sp(U)}(G_1^\circ)$. From Lemmas 8.3, 3.1, and 4.3, we further have that

$$M \in \prod_{\mu \not\simeq \mu^*} C_\mu(GL(V_\mu)) \times \prod_{\substack{\nu \simeq \nu^* \\ \text{orthog.}}} C_\nu(O(V_\nu)) \times \prod_{\substack{\nu \simeq \nu^* \\ \text{sympl.}}} C_\nu(Sp(V_\nu)), \tag{17}$$

where $C_\gamma(\cdot) := C_{Sp(V_\gamma \otimes W_\gamma)}(\cdot)$. Moreover, we have that $M \in \pi^{-1}(C_{PSp(U)}(\pi(G_1)))$, so (17) gives that

$$\begin{aligned} M &\in \prod_{\mu \not\simeq \mu^*} C_\mu(GL(V_\mu)) \times \pi^{-1}(C_{PSp(U)}(\pi(\prod_{\substack{\nu \simeq \nu^* \\ \text{orthog.}}} O(V_\nu)))) \times \prod_{\substack{\nu \simeq \nu^* \\ \text{sympl.}}} C_\nu(Sp(V_\nu)) \\ &\subseteq \prod_{\mu \not\simeq \mu^*} GL(W_\mu) \times \prod_{\substack{\nu \simeq \nu^* \\ \text{orthog.}}} Sp(W_\nu) \times \prod_{\substack{\nu \simeq \nu^* \\ \text{sympl.}}} O(W_\nu) = G_2, \end{aligned}$$

where we have used Theorem 4.4 and Proposition 11.5. It follows that $C_{PSp(U)}(\pi(G_1)) \subseteq \pi(G_2)$. Reversing the roles of G_1 and G_2 shows that $C_{PSp(U)}(\pi(G_2)) \subseteq \pi(G_1)$, completing the proof. □

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