Sum-Product Estimates for Well-Spaced Sets

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Abstract

We obtain a new bound for the δ-discretized version of the sum-product problem. We prove that for the $\Theta(|A|^{-1})$-spaced set $A \subset [1,2]$: $\mathcal{N}(A+A,\delta)\cdot\mathcal{N}(AA,\delta) \gtrsim_\epsilon |A|\delta^{-1+\epsilon}$ (Theorem 1). Next we prove two bounds for the problems related to the sum-product problem: Theorem 2 considers the difference in the structure of arithmetic and geometric progressions and Theorem 3 investigates the structure of the ratio set of an arithmetic progression.

1 Introduction

1.1 Background for Theorem 1

Sumsets and productsets are the key terms in additive combinatorics and are defined as the sets of values of all possible pairwise sums and products of a set respectively. More formally, for a set A, we can define a sumset and the productset as follows:

Definition 1.1 (Sumset and Productset)**.** For a set A define a sumset as $A+A = \{a_1 + a_2,\ a_1, a_2 \in A\}$ and a productset $AA = \{a_1 a_2,\ a_1, a_2 \in A\}$.

When operating with these terms, the question of minimizing the cardinality of the sumset or the productset appears to be interesting.

This problem was posed by Erdös and Szemerédi, and in 1983 they conjectured that for any set of real numbers, the sumset and the productset cannot both be small. Their conjecture is the following:

Conjecture 1 (Sum-Product Problem)**.** *There exist such positive constants c and ϵ, such that for any finite subset of real numbers A:*

$$\max\left(|A+A|, |AA|\right) \geq cA^{1+\epsilon}.$$

The bound was improving as new tools were applied to tackle the problem. One of the approaches uses the tools from combinatorial geometry. György Elekes in his paper [2] applies the Szemerédi-Trotter theorem to prove that in terms of conjecture 1: $\epsilon \geq \frac{1}{4}$. We will adapt his ideas and state an analog of the Szemerédi-Trotter theorem (from [6]) to prove the main result of this paper stated further in this section.

Another improvement on the Erdős-Szemerédi conjecture 1 was presented by József Solymosi. He used multiplicative energy to improve the bound and proved that at least one of the sumset and the productset of any finite set of real numbers, A, is at least $|A|^{\frac{4}{3}-\epsilon}$ ([9]).

In their paper [8] Katz and Tao first formulated a δ-discretized version of the Erdős-Szemerédi conjecture along with δ-discretized versions of the Falocner's problem and Fustenberg set's dimension problem. They proved that the three problems are equivalent at a critical dimension, and therefore, new bounds for discretized variation of the sum-product bound imply new results in other fields.

Bourgain solved the conjecture for the Katz and Tao discretized version of the conjecture ([8]) in the paper [1]. Guth, Katz, and Zahl obtained an explicit bound for the Katz and Tao conjecture in [7]. Their work considers a set with weaker spacing conditions than we do and adapts the Garaev's argument for the analogous problem in finite fields [3].

Underlining the strength of the interconnections between sum-product problem and other problems, we provide one more example which is the work by Bourgain, Katz, and Tao [5] on the sum-product problem in finite fields, that provides a new sum-product estimate and shows how the estimate can be applied to get new bounds for the Szemerédi-Trotter theorem, distance problem, and Kakeya problem in finite fields.

In this paper, we will consider a variation of the conjecture 1 which assumes stronger spacing conditions on the set. We show that for any set A with strong spacing conditions (A is $\Omega(|A|^{-1})$ spaced) the following bound holds:

$$\max(\mathcal{N}(A+A,\delta),\mathcal{N}(AA,\delta)) \gtrsim \delta^{-1/2}|A|^{1/2},$$

where δ is a number $\ll |A|^{-1}$, and $\mathcal{N}(B,\delta)$ is a maximum δ-separated subset of B. Formal notations are provided in the section 2. To prove the result, we use the argument from Elekes' work [2] together with the analog of Szemerédi-Trotter Theorem for well-spaced δ-tubes and δ-balls proven by Guth, Solomon and Wang in [6].

A close relation between our setup and the finite field case suggests our result may be tight in some sence. In [4] (page 2), Garaev gives an example: for any integer $N \in [1,p]$ one can construct a subset $A \subset \mathbb{F}_p$ with $|A| = N$ such that

$$\max(|A+A|,|AA|) \lesssim p^{1/2}|A|^{1/2}.$$

This bound fit with our result.

We will formally state and prove our result in section 3 and section 4.3. However, we believe a stronger result may hold, which is our conjecture 2.

1.2 Background for Theorem 2

Returning to the original sum-product problem (conjecture 1) it is evident that each of the two terms $|A+A|$ and $|AA|$ can be minimized separately. Indeed an arithmetic progression of size N has a sumset of cardinality only $2N-1 = O(N)$. Similarly, geometric progression minimizes the productset. However, in these examples, the second set is very large. Thus, the conjecture 1 is a statement that a set A cannot display the properties of an arithmetic progression and a geometric progression simultaneously.

This idea is captured in theorem 2 in the section 5. The theorem shows how geometric and arithmetic progressions cannot intersect "a lot".

1.3 Background for Theorem 3

One more object related to the sum-product problem is a ratio set:

Definition 1.2 (Ratio Set)**.** For a set A, $0 \notin A$, a ratio set is $A/A = \{a_1/a_2,\ a_1, a_2 \in A\}$.

The ratio set is tightly connected to the productset. The result provided in section 5, that is the theorem 3, provides some insights on the structure of the ratio set of an arithmetic progression.

1.4 Acknowledgements

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2 Notation

In our disretized version of the sum-product problem, we will need a notion of the δ-spaced set:

Definition 2.1 (δ-spaced Set)**.** A set A is called a δ-spaced set if

$$\forall a_1, a_2 \in A, a_1 \neq a_2 : \ |a_1 - a_2| \geq \delta.$$

Definition 2.2. For a set of real numbers A define $\mathcal{N}(A, \delta)$ to be the number of elements in the maximum possible δ-spaced subset of A, that is

$$\mathcal{N}(A, \delta) = \max_{A'} |A'|, \ A' \in A \ \text{ s.t. } \ \forall a_1, a_2 \in A' \ |a_1 - a_2| \geq \delta$$

3 Statement of the Theorem 1

Theorem 1. *Fix a number $\alpha \in (1, \frac{3}{2})$. For any subset $A \subset [1, 2]$, with $|A| = N$ and A is $\Omega(|A|^{-1})$-separated, let $\delta = |A|^{-\alpha}$, a scale much smaller than the separation of the set. Then we have:*

$$\mathcal{N}(A + A, \delta) \cdot \mathcal{N}(AA, \delta) \gtrsim_\epsilon |A|\delta^{-1+\epsilon} = N^{1+\alpha-\epsilon'},$$

where $\epsilon > 0$ and $\epsilon' = \epsilon\alpha$.

As an immediate corollary:

$$max(\mathcal{N}(A + A, \delta), \mathcal{N}(AA, \delta)) \gtrsim_\epsilon \delta^{-1/2+\epsilon}|A|^{1/2} = N^{\frac{1+\alpha}{2}-\epsilon'}.$$

Our estimate for the product $\mathcal{N}(A+A, \delta) \cdot \mathcal{N}(AA, \delta)$ is the best possible for if we consider an arithmetic progression A, then

$$\mathcal{N}(A + A, \delta) \cdot \mathcal{N}(AA, \delta) \lesssim |A| \cdot \delta^{-1}.$$

Therefore, our bound on the product is tight.

Actually, we believe one of the sumset or the productset should have full size δ^{-1}. We think it should be a hard problem, since the analog in finite fields fails. We state our conjecture as follows:

Conjecture 2 (Full-Size Conjecture)**.** *There exists an $\alpha > 1$ such that the following is true. For any subset $A \subset [1, 2]$, with $|A| = N$ and A is $\Omega(|A|^{-1})$-separated, let $\delta = |A|^{-\alpha}$, then we have:*

$$\max(\mathcal{N}(A + A, \delta), \mathcal{N}(AA, \delta)) \gtrsim \delta^{-1+\epsilon},$$

for any $\epsilon > 0$.

The next section contains a preparation for the proof in subsections 4.1 and 4.2 and the proof itself in the subsection 4.3.

4 Proof of the Theorem 1

4.1 Analogue of the Szemerédi-Trotter Theorem

In our proof we will need a variation of the Szemerédi-Trotter Theorem for spaced δ-tubes and δ-balls proved in [6].

To state the result, we will first provide some notation from the paper [6].

Definition 4.1 (Intersection of a δ-ball and δ-tube)**.** We will say that a δ-ball intersects a δ-tube if the center of the δ-ball lies inside the δ-tube.

Definition 4.2 (Rich Balls)**.** Consider a set of δ-balls and a set of $\mathbb{T}$ δ-tubes. We define $P_r(\mathbb{T})$ as a set of r-rich δ-balls, that is the balls that intersect about r tubes. (Here, "about r" means the number lies in $[r, 2r)$.)

In the context of the definitions above, the following analog of the Szemerédi-Trotter Theorem is stated and proved in the [6]:

Theorem (Analogue of S-T Theorem)**.** *Suppose that $\delta \leq W \leq 1$. Suppose that $\mathbb{T}$ is a set of δ-tubes in $[0,1]^2$ with $\lesssim 1$ δ-tube of $\mathbb{T}$ in each $W \times 1$ rectangle.*

$$If\ r > \max(\delta^{1-\epsilon}W^{-2}, 1),$$

$$then\ |P_r(\mathbb{T})| \lesssim_\epsilon \delta^{-\epsilon} r^{-3} W^{-4}.$$

Remark. *The original theorem requires $|\mathbb{T}|$ to have full size $\sim W^{-2}$. Of course we can drop this requirement. To see this, we add some tubes to our $\mathbb{T}$ to get $\mathbb{T}'$, which still satisfies the spacing conditions and with $|\mathbb{T}'| \sim W^{-2}$. We see $P_r(\mathbb{T}) \leq \sum_{s \geq r, s \text{ dyadic}} |P_s(\mathbb{T}')| \lesssim_\epsilon \delta^{-\epsilon} r^{-3} W^{-4}$.*

This theorem estimates the number of the r-rich δ-balls. For our purposes, we will derive an estimate for the number of intersections between the tubes and balls, and as an immediate consequence, an estimate for the number of the r-rich δ-tubes.

Lemma 4.1 (Consequence of the Analogue of the S-T Theorem)**.** *Consider the set of δ-tubes and δ-balls that satisfy all the conditions in Theorem 4.1. Then if $\mathbb{B}$ is a set of δ-balls, $I(\mathbb{T}, \mathbb{B})$ is a number of the intersections between δ-balls and δ-tubes, $P_r(\mathbb{B})$ is a set of r-rich tubes, then:*

$$I(\mathbb{T}, \mathbb{B}) \lesssim_\epsilon \delta^{1-\epsilon}|\mathbb{B}|W^{-2} + \delta^{-2+\epsilon},$$

and as a consequence:

$$P_r(\mathbb{B}) \lesssim_\epsilon \delta^{1-\epsilon}|\mathbb{B}|W^{-2}r^{-1} + \delta^{-2+\epsilon}r^{-1}.$$

Proof. Denote m_i as a number of δ-balls that intersect with at least 2^{i-1} tubes and less than 2^i tubes. Then:

$$I(\mathbb{T}, \mathbb{B}) \lesssim \sum_{i=1}^{\log_2 W^{-2}} m_i 2^i = \sum_{i=1}^{\log_2 \delta^{1-\epsilon}W^{-2}} m_i 2^i + \sum_{i=\log_2 \delta^{1-\epsilon}W^{-2}}^{\log_2 W^{-2}} m_i 2^i.$$

For all i, we have $m_i \leq |\mathbb{B}|$ and therefore

$$\sum_{i=1}^{\log_2 \delta^{1-\epsilon} W^{-2}} m_i 2^i \leq |\mathbb{B}| \sum_{i=0}^{\log_2 \delta^{1-\epsilon} W^{-2}} 2^i \lesssim \delta^{1-\epsilon} |\mathbb{B}| W^{-2}.$$

Moreover, for all i, we have $m_i \leq P_{2^{i-1}}(\mathbb{T})$ and therefore

$$\sum_{i=\log_2 \delta^{1-\epsilon} W^{-2}}^{\log_2 W^{-2}} m_i 2^i \lesssim \sum_{i=\log_2 \delta^{1-\epsilon} W^{-2}}^{\log_2 W^{-2}} P_{2^{i-1}}(\mathbb{T}) 2^i$$

$$\lesssim_{\text{Thm 4.1}} \sum_{i=\log_2 \delta^{1-\epsilon} W^{-2}}^{\log_2 W^{-2}} W^{-4} 2^{-2i} \delta^{-\epsilon} \lesssim W^{-4} (\delta^{1-\epsilon} W^{-2})^{-2} \delta^{-\epsilon} = \delta^{-2+\epsilon}.$$

Combining these two inequalities, we get:

$$I(\mathbb{T}, \mathbb{B}) \lesssim \delta^{1-\epsilon} |\mathbb{B}| W^{-2} + \delta^{-2+\epsilon}.$$

r- rich tubes contribute at least $rP_r(\mathbb{B})$ to $I(\mathbb{T}, \mathbb{B})$, and thus

$$P_r(\mathbb{B}) \lesssim \delta^{1-\epsilon} |\mathbb{B}| W^{-2} r^{-1} + \delta^{-2+\epsilon} r^{-1}.$$

□

4.2 Discretization of the Set

We can simplify the proof of the theorem by discretizing the set A in the following way.

First, consider the δ-lattice L of the interval $[1, 2]$. Let A' be the discretization of A, that is $A' = \{x \in L : \text{dist}(x, A) \leq \delta/2\}$. A' is the result of moving all the point of A to the closest lattice points. In our problem $\delta \ll N^{-1}$, and therefore A' is $\sim N^{-1}$-separated and $|A'| = |A| = N$.

Consider the sumset of A'. For all $a_1', a_2' \in A'$ there exist corresponding $a_1, a_2 \in A$, s.t. $a_1 + a_2 = a_1' + a_2' + O(\delta)$ with a fixed constant. Thus,

$$\mathcal{N}(A + A, \delta) \gtrsim \mathcal{N}(A' + A', \delta).$$

For the same reasons

$$\mathcal{N}(AA, \delta) \gtrsim \mathcal{N}(A'A', \delta).$$

Thus, in order to prove Theorem 1, it is sufficient to prove the following simplified theorem:

Theorem 1 (simplified version). *Let $A \subset [1, 2]$ be a N^{-1}-separated set and $|A| \sim N$. Also, fix an $\alpha \in [1, \frac{3}{2}]$ and let $\delta = N^{-\alpha}$. Then,*

$$\mathcal{N}(A + A, \delta) \cdot \mathcal{N}(AA, \delta) \gtrsim_\epsilon N^{1+\alpha-\epsilon}.$$

for any $\epsilon > 0$.

Note that we can always assume $A \subset \delta\mathbb{Z}$ by replacing any element in A with its nearest $\delta-$lattice point, since by doing so remains $\mathcal{N}(A+A, \delta)$ and $\mathcal{N}(AA, \delta)$ essentially unchanged. In the following subsection, we will prove this version of the theorem.

4.3 Proof of the Theorem 1

The idea of the proof is roughly the same as in Elekes' work [2].

Let $A = \{a_i,\ 1 \le i \le N\}$. First, consider N^2 lines $y = a_j(x - a_i),\ \forall a_i, a_k \in A$. Let L be the set of δ-tubes built on the segments of these lines that correspond to the interval $x \in [2, 4]$.

Now define the set of our δ-balls. Consider the set of points $(A+A) \times Q$, where Q is the maximum possible δ-separated subset of AA (then $|Q| = \mathcal{N}(AA, \delta)$). Consider the set of all δ-balls with the centers in this set and denote it M. We will first prove that all the tubes in L contain the centers of at least N balls from M (to apply later Theorem 4.1 with $r = N$):

Lemma 4.2. *Each line in L contains the centers of at least N δ-balls from M.*

Proof. Consider any tube and assume it is built on the line $y = a_j(x - a_i)$. Consider the set of N elements of $A+A$: $\{a_i + a_k,\ 1 \le k \le N\}$. For each element $a_i + a_k$ in this set, the point $(a_i + a_k, a_i a_j)$ lies in the line $y = a_j(x - a_i)$, and therefore, the center of the δ-ball with the center at this point is in the tube. If $a_i a_j \in Q$, then this ball is in M. If $a_i a_j \notin Q$, there exists $a_s a_t \in Q$, s.t $|a_s a_t - a_i a_j| < \delta$. Otherwise we could add $a_i a_j$ to the set Q, and it would remain δ-separated. However, we have already chosen Q as the biggest possible δ-separated set, so it is impossible to add an element to Q.

Then $(a_i + a_k, a_s a_t) \in (A+A) \times Q$ and $(a_i + a_k, a_s a_t)$ lies in the corresponding tube. Thus, we have found N distinct balls from M (at least one for all $a_i + a_k$.) $\square$

To apply Theorem 4.1 it is left to show that the set of tubes we consider is well-spaced. This condition is satisfied because the initial separation of A is $\gg \delta$.

Indeed, the slopes of our tubes are N^{-1}-spaces numbers - $a_1, a_2, ..., a_N$. Thus, any two tubes with different slopes have an angle between them equal to $\sim N^{-1}$, and two tubes of the same slope are $O(N^{-1})$ apart. Thus, our N^2 δ-tubes lie in N^2 essentially distinct rectangles of size $N^{-1} \times 2$, and therefore we can apply Theorem 4.1 with $W = N$. This gives us:

$$N^2 = |A|^2 < P_N(M) \lesssim \delta^{1-\epsilon} |M| N^2 N^{-1} + \delta^{2+\epsilon} N^{-1} \lesssim N^{1-\alpha+\epsilon} |M| + N^{2\alpha - 1 - \epsilon'} \tag{1}$$
$$\Rightarrow |A+A||Q| \gtrsim N^{1+\alpha-\epsilon}.$$

Here we used the condition that $\alpha \le \frac{3}{2}$.

From the definition $|Q| = \mathcal{N}(AA, \delta)$, and because $A + A \in \delta\mathbb{Z}$, $|A + A| = \mathcal{N}(A + A, \delta)$, which gives us the result of the Theorem.

5 Other Results

5.1 Arithmetic and Geometric Progressions

The key idea of the Erdős-Szemerédi conjecture 1 is a significant difference in the structure of arithmetic and geometric progressions. We can capture this idea in the following theorem:

Theorem 2. *Let's fix an $\alpha \in [1, 3/2]$ and let $\delta = N^{-\alpha}$. Consider length-n arithmetic progression $A = \{1 + iN^{-1}, 1 \leq i \leq N\}$ and geometric progression $G = \{q^i, 1 \leq i \leq N\}$ with $q^N - 1 \gtrsim 1$. Then*

$$|G \cap \mathcal{E}_\delta(A)| \lesssim_\epsilon N^{\max\{\alpha-1/2,(3-\alpha)/2\}+\epsilon}$$

for any $\epsilon > 0$ ($\mathcal{E}_\delta(A)$ denotes a δ-neighborhood of the set A).

Proof. Let $B = G \cap \mathcal{E}_\delta(A)$, and assume that $|B| \gtrsim N^{\max\{\alpha-1/2,(3-\alpha)/2\}+\epsilon}$. The condition $q^N - q = \Omega(1)$ implies that there cannot be two elements from G that are in the δ-neighborhood of the same element of A. Actually, $q^N = 1 + \Omega(1)$ implies $q = (1 + \Omega(1))^{1/N} > 1 + \frac{\Omega(1)}{N}$, and hence $q^{i+1} - q^i \geq q - 1 \gtrsim \frac{1}{N}$.

A is an arithmetic progression, and therefore, $\mathcal{N}(A + A, \delta) \sim |A + A| \leq 2N$. Similarly $\mathcal{N}(GG, \delta) = |GG| \leq 2N$. Thus, because $B \subset \mathcal{E}_\delta(A), B \subset G$,

$$\mathcal{N}(B + B, \delta) \lesssim \mathcal{N}(A + A, \delta), \mathcal{N}(BB, \delta) \leq \mathcal{N}(GG, \delta),$$

so

$$\mathcal{N}(B + B, \delta) \cdot \mathcal{N}(BB, \delta) \lesssim N^2,$$

We will obtain a lower bound for $\mathcal{N}(B+B, \delta) \cdot \mathcal{N}(BB, \delta)$ to get a contradiction.

We put $A = B$ in Theorem 1. We do not necessarily have $|B| \sim N$, but Equation(5.2) still holds. Let's write down here:

$$|B|^2 \lesssim \delta^{1-\epsilon}|M|N^2N^{-1} + \delta^{2+\epsilon}N^{-1} \lesssim N^{1-\alpha+\epsilon}|M| + N^{2\alpha-1-\epsilon'}$$

Here, $|M| = \mathcal{N}(B + B, \delta) \cdot \mathcal{N}(BB, \delta)$. By our assumption, $|B| \gtrsim N^{\alpha-1/2}$, so we have

$$|B|^2 \lesssim N^{1-\alpha+\epsilon}|M| \lesssim N^{3-\alpha+\epsilon}$$

which is a contradiction. □

5.2 Ratio Set Structure

One more object we can consider is a ratio set defined in section 1. In order to better understand the properties of the ratio set on some scale δ, we can consider the following problem.

Consider an arithmetic progression of length N: $A =_i= 1 + iN^{-1}, 1 \leq i \leq N\} \subset [1, 2]$ and consider its ratio set $A/A = \{b_i, 1 \leq i \leq |A/A|\}$, $b_1 < b_2 < \cdots < b_{|A/A|}$.

Then we can estimate a number of big "gaps" in the ratio set:

Theorem 3. *In the setup above, for all $\epsilon > 0$,*

$$\#\{b_{i+1} - b_i > N^{-1-\epsilon}\} < O(N^{2\epsilon}),$$

And for $\epsilon = 0$,

$$\#\{b_{i+1} - b_i > cN^{-1}\} < 2c^2.$$

and as an immediate consequence: for each scale $N^{-2} \ll \delta \ll N^{-1}$:

$$\mathcal{N}(A/A, \delta) \sim \delta^{-1}.$$

Proof. A geometric interpretation of a ratio set is a set of lines passing through point $(0,0)$ with their slopes being equal to b_i that cover all the points of $A \times A$. (Figure 1). The lines are situated symmetrically to the line $x = y$, so without

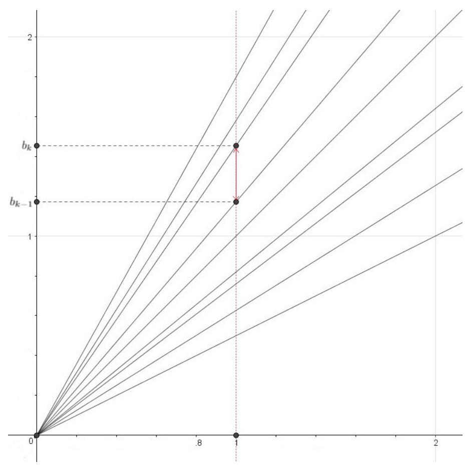

Figure 1: Geometric Interpretation of the Ratio Set

loss of generality, we can only consider the upper half, that is only $b_i > 1$. The number of such b_i is $\frac{1}{2}|A/A|$, so it is sufficient to prove the statement for only $b_i > 0$ The gaps we consider in the theorem $b_{i+1} - b_i$ are the distances between the intersections of two consequent lines with a vertical line passing through $(1,0)$.

Consider a line with a slope $b_k = a_j/a_i$ ($b_k \in A/A, b_k > 1$). We will first find a formula for the "gap" under this line, that is $b_k - b_{k-1}$, then using the formula, we will understand how many "big gaps" are there and where they are situated.

Lemma 5.1. *In the setup above:*

$$b_k - b_{k-1} \leq \min_{1 \leq i \leq N}^{\{b_k(N+i)\} \neq 0} \{b_k(N+i)\} \cdot N^{-1},$$

where $\{x\}$ is a fractional part of X.

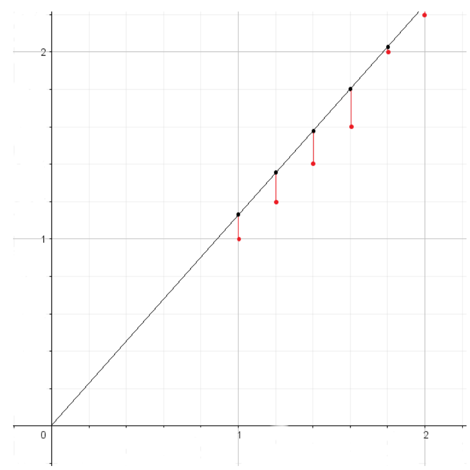

Figure 2: "Gap"

Proof. Consider all vertical lines passing through points $(a_m, 0)$. The line with the slope b_k intersects this vertical line at the point $(a_m, b_k a_m) = (1+mN^{-1}, b_k\frac{N+m}{N})$. Thus the closest point of the grid $A \times A$ is $(1 + mN^{-1}, [b_k(N + m)]N^{-1})$. For each point of the grid, we have a line with a slope from the ratio set that passes through this line. Therefore the gap just under b_k is $< b_k\frac{N+m}{N} - [b_k(N + m)]N^{-1} = \{b_k(N + m)\}N^{-1}$. This is true in the case when this value is $\neq 0$, because in this case a point of the grid lies on the line with the slope b_k. (Figure 2).

Considering all possible a_m, we get that

$$b_k - b_{k-1} < \min_{1\leq i\leq N}^{\{b_k(N+i)\}\neq 0} \{b_k(N + i)\} \cdot N^{-1}.$$

□

Now that we have a formula for the gap, we can investigate when $b_k - b_{k-1} > N^{-1-\epsilon}$. Recall that

$$b_k = \frac{a_j}{a_i} = \frac{1 + jN^{-1}}{1 + iN^{-1}} = \frac{N + j}{N + i}.$$

Assume $b_k = \frac{p}{q}$ in lowest terms. Then $1 \leq q \leq 2N$. Consider several cases depending on the interval where q lies:

1. $1 \leq q \leq N$. Our goal is to estimate the number of such b_k that

$$\min_{1\leq i\leq N}^{\{b_k(N+i)\}\neq 0} \{b_k(N + i)\} > N^{-\epsilon}.$$

$b_k(N + i) = \frac{p(N+i)}{q}$. Because $(p, q) = 1$,

$\{p(N+1) \mod q, p(N+2) \mod q, \ldots, p(N+q) \mod q\} = \{0, 1, 2, \ldots, q\},$

and therefore, among the fractional parts there are all possible fractions of the type $\frac{t}{q}$ including $\frac{1}{q}$. Thus a big gap will appear if and only if $\frac{1}{q} > N^{-\epsilon} \iff q < N^{\epsilon}$. All elements of the ratio set we consider are > 1 and ≤ 2 (because $A \subset [1, 2]$. Therefore for each q, there are at most q elements in A/A with gaps $> N^{-1\epsilon}$. Thus, there are no more than $O(N^{2\epsilon})$ elements of the ratio set with big gaps in this case.

2. $N < q \leq N + N^{\epsilon}$. For the same reasons as in case 1, the fractional parts we are choosing minimum from are N distinct fractions of the type $\frac{t}{q}$ among q possible. Therefore, the minimum non-zero element among them is $\leq \frac{q-N+1}{q} \lesssim \frac{N^{\epsilon}}{N} = N^{-1-\epsilon}$, so in this case we do not have big gaps.

3. $N + N^{\epsilon} < q < 2N - N^{\epsilon}$. Consider $\{b_k q\}, \{b_k(q + 1)\}, \ldots, \{b_k(q + N^{\epsilon})\}$. Among these N^{ϵ} numbers $\in [0, 1)$ there are two such that the difference between them is $\leq N^{-\epsilon}$:

$$\exists\ 1 \leq t_1, t_2 \leq N^{\epsilon} :\ |\{b_k(q + t_1)\} - \{b_k(q + t_2)\}| \leq N^{-\epsilon}.$$

   Then if $t = |t_1 - t_2|$, then $\{b_k(q + t)\} < N^{-\epsilon}$ or $\{b_k(q + t)\} > 1 - N^{-\epsilon}$. In the former case, $b_k - b_{k-1} < N^{-\epsilon}$, and in the latter, because $b_k q \in \mathbb{Z}$, $\{b_k(q - t)\} < N^{-\epsilon}$, and therefore $b_k - b_{k-1} < N^{-\epsilon}$. Thus, there are no big gaps in this case.

4. $2N - N^{\epsilon} < q \leq 2N$. Because we only consider $p/q > 1$, there are $< O(N^{2\epsilon})$ fractions in this case.

Thus, the number of b_k such that $b_k - b_{k-1} > N^{-1-\epsilon}$ is $< O(N^{2\epsilon})$. Note that for $\epsilon = 0$ the proof is similar to the one presented here. □

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