COMPLEXITY ANALYSIS OF COMPUTATION OF HOMOTOPY GROUPS OF SPHERES

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UROP+ FINAL PAPER, SUMMER 2021
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ABSTRACT. We analyze an algorithm of Brown, which allows for the computation of the stable homotopy groups of spheres. We provide an upper bound on the asymptotic complexity of this algorithm when applied to compute the homotopy groups of odd-dimensional spheres, as well as any space with finite homotopy groups obtained as the realization of a finite simplicial set. We additionally discuss difficulties in providing bounds in the case of spaces with homotopy groups of arbitrary size.

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Date: January 10, 2022.

1. Introduction

In his 1956 paper *Finite Computability of Postnikov Complexes* [Bro57], E.H. Brown provided a procedure which, building on earlier results proven by M. Postnikov [Pos51], yields an algorithmic procedure by which the stable homotopy groups of spheres can be computed. That being said, at the time this procedure was mostly of theoretical importance; Brown himself writes in the paper that *"although the procedures developed for solving these problems are finite, they are much too complicated to be considered practical."* However, computing has evolved enormously since the 1950s, and it may now be possible to compute these groups within a reasonable amount of time. The question of just *how* reasonable then arises, and the purpose of this project is to answer this question.

In the case of a space with finite homotopy groups, obtained as the realization of a simplicial set N finite in each level, we have the following result.

Theorem 1.1. *Given a finite complex N, we can determine $\pi_i(|N|)$ for $1 < i \leq n$ in*

$$O\left(n^2\left[\left(|N_{\max}| + \prod_{j<n} |\pi_j(|N|)|^{\binom{j+n}{n}}\right)^3 + n\binom{2n}{n}\prod_{j<n}|\pi_j(|N|)|^{\binom{j+n}{n}}\right]\right) \tag{1.2}$$

time, where $N_{\max}$ is the level of the simplicial set N with maximal order out of levels 2 through level $n+2$.

Given Stirling's approximation, this roughly tells us that the algorithm runs polynomial in the size of the homotopy groups, but heavily superexponential in the amount of homotopy groups being computed.

Brown's algorithm also gives an process to compute the homotopy groups of a complex N with infinite homotopy groups, but there is a detail in the construction that we have been unable to place a bound on. In lieu, we analyze the case of odd-dimensional spheres, using previous results bounding the order of the homotopy groups to obtain the following result.

Theorem 1.3. *Using Brown's algorithm, we can determine the homotopy groups $\pi_i(S^{2n+1})$, $i < m$ for some $m > 2n+1$, in*

$$O\left(m^2\left[\left(\binom{2n+m}{2n} + e^{m^3\binom{2m}{m}}\right)^3 + m\binom{2m}{m}e^{m^3\binom{2m}{m}}\right]\right) \tag{1.4}$$

time.

2. Acknowledgements

I would like to express much thanks to Robert Burklund, for crucial guidance and patience. I would also like to thank Abraham Corea Diaz, for assistance in translation of some theorems instrumental to the results of this paper.

3. Computational Preliminaries and Model of Computation

Here, we lay out discuss some preliminaries to the analysis of the algorithm, and discuss the computational model which will be the basis of our complexity analysis.

3.1. **Asymptotic Notation: O, Ω, and Θ.** When analyzing the runtime of an algorithm, we do not care too much about the exact time an algorithm might take when run on a particular system with a particular input; there are too many variables that can affect this. Instead, we choose to think about how the runtime scales as the inputs become more complex, which we describe using O, Θ, and Ω notation.

Definition 3.1 (O, Ω, and Θ)**.** Given two functions $f(x)$ and $g(x)$, we say

(1) $g(x) \in O(f(x))$ if there exists some constant $c > 0$ such that $cf(x) \geq g(x)$ for all $x \in \mathbb{R}$,
(2) $g(x) \in \Omega(f(x))$ if there exists some constant $c > 0$ such that $cf(x) \leq g(x)$ for all $x \in \mathbb{R}$, and
(3) $g(x) \in \Theta(f(x))$ if $g(x) \in O(f(x)) \cap \Omega(f(x))$.

As a common abuse of terminology,"$g(x)$ is $O(f(x))$" means that $g(x) \in O(f(x))$, and likewise for Ω and Θ.

One can think of a function $g(x)$ being $O(f(x))$ as saying that it is asymptotically upper bounded by $f(x)$, $\Omega(f(x))$ as being asymptotically lower bounded by $f(x)$, and $\Theta(f(x))$ as being asymptotically "equivalent" to $f(x)$.

Remark 3.2. In this paper (and in the study of algorithms as a whole), most of the time we upper-bound the runtimes of algorithms discussed, and so we will largely be using O. As such, in the discussion below, we will only state facts relating to O; we leave as exercise to verify that the facts also apply to Ω and Θ.

We describe the runtime of an algorithm as a function of some number of parameters, and this is why the notation described above is useful; it allows us to capture information about how our algorithm behaves as inputs become more complicated, which generally will not change drastically between implementations, while throwing away details such as constant factors which cause notational clutter and do not convey too much useful information.

Example 3.3. Mergesort, an efficient algorithm for sorting lists, has a runtime of $O(n \log n)$, where n is the size of the list.

Two useful facts regarding O is the following:

(1) If a function $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then $f_1(x) + f_2(x)$ is $O(g_1(x) + g_2(x))$.
(2) If a function $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, and all functions f_1, f_2, g_1, g_2 are positive, then $f_1(x)f_2(x)$ is $O(g_1(x)g_2(x))$.

In particular, this tells us asymptotic notation behaves how we should want it to when analyzing an algorithm's smaller subparts. If an algorithm is comprised of n subtasks, then the runtime of the algorithm can be obtained by summing the runtimes of the subtasks, and similarly, if you have an algorithm consisting of a subtask which is repeated some amount of times, then the total runtime is obtained by multiplying the runtime of the subtask by the number of times it is being repeated.

3.2. **Runtimes of Basic Operations.** In order to analyze the runtime of algorithms, we need to first define the runtime of some primitive operations.

- *Arithmetic/Comparison* We will assume that all arithmetic operations between two numbers - addition, subtraction, multiplication, and division, in addition to modular arithmetic such as finding remainders, take $O(1)$ or constant time. We will also assume that checking equality/inequality of two numbers can be accomplished in constant time.
- *Boolean Arithmetic/Comparison:* A *boolean value* is simply a **True** or a **False** value. We will assume that we can compute boolean operations (and, or, is and not) in constant time.
- *Memory Access:* We will assume that if we assign a value to a variable, then we can look up the value in constant time. Further, we will assume that we can assign any object/data structure as a value to a variable. (See the section below for a discussion of data structures that will be employed in this paper.)

3.3. **Data Structures.** A *data structure* is a way to arrange data. There are many different data structures, with each one more suited to certain applications. In this section, we discuss the data structures which will be employed in this paper.

3.3.1. *Static Arrays.* An array is an ordered list, where the objects contained can be anything from numbers to other lists. We refer to the object in position i to be at *index* i in the array, and if the array is given by $A = (x_1, x_2, \ldots, x_n)$, then we denote x_i, the object at index i, by $A[i]$. A static array is one where the size is prespecified, and which cannot be changed. We assume the following runtimes for some basic operations with a static array A:

- *Initialization:* We assume that in initializing a length n static array is an $O(n)$ time task.
- *Lookup:* Given an index i, we assume that we can return $A[i]$ in $O(1)$ time.
- *Deletion:* Given an index i, we assume that we can remove the value at $A[i]$ in $O(1)$ time.
- *Write:* Given an index i, we assume we can assign a new value to $A[i]$ in $O(1)$ time.

3.3.2. *Hash Tables.* Hash tables are a data structure which will be employed heavily to describe various constructions. A hash table H has keys, each assigned to a value, and we will say "$x \in H$" to mean that "x exists as a key in H," and refer to the value associated to x as $H(x)$. We assume that we can accomplish the following operations in $O(1)$ time:

- *Lookup*: Given an item x, we will assume that we can check if $x \in H$ in $O(1)$ time, and if $x \in H$, that we can return $H(x)$ also in $O(1)$ time.
- *Deletion*: Given $x \in H$, we will assume that we can remove x and its associated value from H in $O(1)$ time.
- *Insertion*: Given x, we will assume that both newly assign x to a given value and reassign $H(x)$ in $O(1)$ time.

3.4. **Representation of Mathematical Objects.** At the core of our analysis we have some basic mathematical objects. In this section we discuss how we represent them computationally.

3.4.1. *Sets.* We represent sets using hash tables. Specifically, a set S is represented as a hash table S', where $x \in S$ if and only if $S'(x) = \textbf{True}$. We will say $x \in S'$ to mean $S'(x) = \textbf{True}$ in this manner. This allows us to add objects to, remove objects from, and check whether an object is included in a set in $O(1)$ time. It also allows us to implement the following set operations with the following runtimes:

- *Union*: Given two sets S_1, S_2, we can compute the union $S_1 \cup S_2$ as a new set in $O(|S_1| + |S_2|)$ time.
- *Intersection*: Given two sets S_1, S_2, we can compute the intersection $S_1 \cap S_2$ as a new set in $O(\min(|S_1|, |S_2|))$ time.

3.4.2. *Cartesian Products.* Given two sets S_1, S_2, we represent the cartesian product $S_1 \times S_2$ as a new set whose elements are static arrays (x_1, x_2), with $x_1 \in S_1, x_2 \in S_2$. This is computable in $O(|S_1||S_2|)$ time.

Remark 3.4. Note that under this representation, $(S_1 \times S_2) \times S_3$ is not the same computational object as $S_1 \times (S_2 \times S_3)$. However, for our purposes, this does not raise any problems, and so we will employ this representation.

3.4.3. *Functions between sets.* We represent a function $f : S_1 \to S_2$ as a static array

$$(S_1', S_2', F), \tag{3.5}$$

where S_1' is the domain represented as a set, S_2' is the codomain represented as a set, and F is a hash table whose keys are $s_1 \in S_1'$, and where $F(s_1) = f(s_1) \in S_2'$. With this representation, we can determine the image of $s_1 \in S_1$ under our function in $O(1)$ time, and in particular, we can determine the image of our entire domain in $O(|S_1|)$ time. We can also determine the preimage of *all* singletons in $s_2 \in S_2$ at once in $O(|S_1|)$ time.

3.4.4. *Simplicial sets.* A simplicial set N can be thought of as a collection of sets and functions between them. So we can represent the kth level N_k as a list

$$(N_k', \partial_0', \ldots, \partial_k', s_0', \ldots, s_k') \tag{3.6}$$

where N_k' is the computational representation of the set N_k, and ∂_i' and s_i' are the computational representations of the face and degeneracy maps.

3.4.5. *Finitely generated abelian groups.* The Structure Theorem for finitely generated abelian groups tells us that for any finitely generated abelian group G,

$$G \simeq \mathbb{Z}_{d_1} \oplus \ldots \oplus \mathbb{Z}_{d_n} \oplus \mathbb{Z}^r \tag{3.7}$$

where $d_i, r \in \mathbb{N}$; that is, any finitely generated abelian group can be represented as the direct sum of finite cyclic groups and a free abelian group.[1] Thus, representing such a group is simple; all we need is a tuple of integers! So we will use this decomposition to represent a finitely generated abelian group G via a static array

$$(r, d_1, \ldots, d_n), \tag{3.8}$$

and an element $g \in G$ via another static array

$$(v_f, v_1, \ldots, v_n), \tag{3.9}$$

where $v_f = (v_{f_1}, \ldots, v_{f_r}) \in \mathbb{Z}^r$, and $v_i \in \mathbb{Z}_{d_i}$.

[1]Please see [Art10], Ch. 14.7 for a proof of this theorem, in addition to details on how the d_i and r are determined.

3.4.6. *Direct sums.* With the above representation of finitely generated abelian groups, we can represent the direct sum of two such groups G_1, G_2 fairly easily; if $G_1 \simeq \mathbb{Z}_{d_{1_1}} \oplus \ldots \oplus \mathbb{Z}_{d_{n_1}} \oplus \mathbb{Z}^{r_1}$ and $G_2 \simeq \mathbb{Z}_{d_{1_2}} \oplus \ldots \oplus \mathbb{Z}_{d_{m_2}} \oplus \mathbb{Z}^{r_2}$, then we can represent $G_1 \oplus G_2$ simply as

$$(r_1 + r_2, d_{1_1} \ldots, d_{n_1}, d_{1_2} \ldots .d_{m_2}) \tag{3.10}$$

3.4.7. *Abelian group homomorphisms.* We can also represent homomorphisms between finitely generated abelian groups without too much difficulty. We use the fact that the action of a homomorphism is determined by the image of a generating set of the domain. We will represent a homomorphism ϕ between two finitely generated abelian groups G_1, G_2 via a static array (G_1', G_2', φ) where G_1' is the representation of G_1, G_2' is the representation of G_2, and φ is a hash table whose keys are a generating set of G_1' and where the values assigned to each key it its image under ϕ.

In particular, there are many instances where we will want to describe maps out of simplicial chain groups for some simplicial set K. In this case, we will always be describing the action of the map on the 'canonical' basis, given by the elements of K_n.

3.4.8. *Matrices.* Homomorphisms between free abelian groups can be represented via integer matrices, obtained by fixing bases. We will represent an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \tag{3.11}$$

via a "list of lists" A':

$$\begin{aligned} A' = ((&a_{11}, a_{12}, \ldots, a_{1n}), \\ &(a_{21}, a_{22}, \ldots, a_{2n}), \ldots, \\ &(a_{m1}, a_{m2}, \ldots, a_{mn})) \end{aligned} \tag{3.12}$$

With this representation of matrices, we have the following asymptotic runtimes for some basic matrix operations:

- *Multiplication:* Given a $m \times n$ matrix A and an $n \times p$ matrix B, we can compute (the computer representation of) AB in $O(mnp)$ time.
- *Inversion:* Given an invertible $n \times n$ matrix A, we can compute the inverse of A in $O(n^3)$ time.

4. Topological Preliminaries

Brown largely makes use of a certain combinatorial structure in his paper; *simplicial sets*. In this section we will outline some basic definitions surrounding simplicial sets, modernized from Brown's original exposition. We will assume familiarity with basic definitions from general topology and category theory, as in [Mun03] and [Rie16], and draw from exposition in [Hat01], [May67] and [GJ09].

4.1. **Simplicial Sets.** What Brown refers to as complete semi-simplicial complexes are known today as **simplicial sets.** These objects are combinatorial in nature, and can be thought of as generalizing directed graphs.

Definition 4.1 (Simplicial set)**.** A *simplicial set* K is a collection of sets $\{K_n\}_{n\in\mathbb{Z}^+}$, along with *face maps* $\partial_i : K_{n+1} \to K_n$ and *degeneracy maps* $s_i : K_{n-1} \to K_n$, $0 \le i \le n$, which satisfy the following *simplicial identities*:

$$\begin{cases} s_i s_j = s_{j+1} s_i & i \le j \\ \partial_i \partial_j = \partial_{j-1} \partial i & i < j \\ \partial_i s_j = s_{j-1} \partial_i & i < j \\ \partial_j s_j = \mathrm{id} = \partial_{j+1} s_j & \\ \partial_i s_j = s_j \partial_{i-1} & i > j+1 \end{cases} \tag{4.2}$$

We refer to the elements of K_n as *n-simplices*. Given a n-simplex x, $\partial_i(x)$ is called the *ith face of* x, and x is said to be *degenerate* if $x = s_i(y)$ for some i and some $n+1$-simplex y.

Example 4.3. A motivating pseudoexample is a directed graph (V, E), with the condition that the self-directed edge (v, v) is in E for all $v \in V$. We let $K_0 = V$ and $K_1 = E$, and define the face maps $\partial_0, \partial_1 : K_1 \to K_0$ and the degeneracy map $s_0 : K_0 \to K_1$ as follows:

$$\begin{aligned} \partial_0((u, v)) &= v \\ \partial_1((u, v)) &= u \\ s_0(v) &= (v, v) \end{aligned} \tag{4.4}$$

One could complete this pseudoexample to be an actual simplicial set in the following way.

(1) Define higher-dimensional n-simplices as n-tuples $(v_0, v_1, \dots, v_n)$.
(2) Define the higher-dimensional face maps $\partial_i : K_{n+1} \to K_n$ to be the functions where

$$\partial_i(v_0, v_1, \dots, v_n) = (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$$

and the higher-dimensional degeneracy maps $s_i : K_{n-1} \to K_n$ to be the functions where

$$(v_0, v_1, \dots, v_n) = (v_0, \dots, v_i, v_i, \dots, v_n).$$

Notice that these match the given definitions for $n = 0, 1$.
(3) Inductively define K_n to be $\bigcup_i s_i(K_{q-1})$.

One can verify that these maps are well-defined and satisfy the simplicial identities, and therefore this "completion" of a directed graph is a simplicial set.

Definition 4.5. A simplicial set K is *finite* if K_n is finite for all n.

We now define the class of structure-preserving maps between simplicial sets.

Definition 4.6 (Simplicial map)**.** Given two simplicial sets K and L, a collection of maps $f = \{f_q\}_{q\in\mathbb{Z}^+}$, with $f_q : K_q \to L_q$, is called a *simplicial map* if

$$\begin{aligned} f_{q+1} s_i &= s_i f_q \\ f_{q-1} \partial_i &= \partial_i f_q \end{aligned} \tag{4.7}$$

In words, it is said that f *commutes* with the simplicial maps.

In modern day, one typically takes a categorical approach to defining simplicial sets. We describe this in what follows.

Definition 4.8 (Simplex category)**.** The *simplex category* Δ is defined to be the category whose objects are lists of integers $\{0, 1, \ldots, n\}$, denoted as $[n]$, and whose morphisms are monotone increasing functions $\mu : [n] \to [m]$; that is, functions such that $\mu(i) \leq \mu(j)$ if $i < j$.

Definition 4.9 (Simplicial set (categorical definition))**.** A *simplicial set* K is a (covariant) functor

$$K : \Delta^{\text{op}} \to \mathbf{Set}$$

between the opposite category of Δ^{op} and the category of sets **Set**, or equivalently, a contravariant functor $K : \Delta \to \mathbf{Set}$.

More generally, given a category $\mathcal{C}$, one can define a *simplicial object* in the category $\mathcal{C}$ as a functor $K : \Delta^{\text{op}} \to \mathcal{C}$.

To provide intuition as to why this abstract definition matches our more explicit one from before, consider $\sigma_i \in \text{Hom}([n], [n-1])$ such that

$$\sigma_i(j) = \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases} \tag{4.10}$$

and $\delta_i \in \text{Hom}([n], [n+1])$ such that

$$\delta_i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases} \tag{4.11}$$

In Chapter VII.5 of [ML71], the following lemma is proven regarding these maps:

Lemma 4.12. *Any* $f \in \text{Hom}_\Delta([n], [m])$ *has a unique expression of the following form:*

$$f = \delta_{i_k} \ldots \delta_{i_1} \sigma_{j_1} \ldots \sigma_{j_h}, \tag{4.13}$$

such that

$$\begin{cases} n + k = m + h \\ 0 \leq i_1 < \ldots < i_k \leq m \\ 0 \leq j_1 < \ldots < j_h < n \end{cases} \tag{4.14}$$

Further, one can verify that the *cosimplicial identities,* obtained by replacing s_i and ∂_i with σ_i and δ_i respectively in the equations (4.2), hold true. Therefore, to obtain the "classical" simplicial set, according to our first definition, from the categorical one, one simply takes the sets K_n to be $K([n])$, and the face and degeneracy maps ∂_i and s_i to be $K(\delta_i)$ and $K(\sigma_i)$, respectively. We leave it as an exercise to verify that one can also work in reverse; that given a "classical" simplicial set K, there is a unique functor $K : \Delta^{\text{op}} \to \mathbf{Set}$ such that $K([n]) = K_n$, $K(\delta_i) = \partial_i$, and $K(\sigma_i) = s_i$, thus showing us the equivalence of our definitions.

Simplicial maps, defined in categorical terms, offers an example of how our categorical reformulation allows for a streamlining in description.

Definition 4.15 (Simplicial map (categorical definition))**.** A *simplicial map* is a natural transformation between simplicial sets.

Definition 4.16 (**sSet**)**.** The category **sSet** of simplicial sets is the category whose objects are simplicial sets and whose morphisms are simplicial maps.

In the remainder of the paper, we will adopt notation and terminology from the "classical" definition when we require access to more specific data in a simplicial set, such as individual face/degeneracy maps or individual n-simplices, and attempt a categorical view otherwise.

4.2. **Functors.** Given the purely combinatorial nature of simplicial sets, in order to make topological claims we require some way to interface between them and topological spaces. We accomplish this via the **singular** and **geometric realization** functors.

We begin by describing the singular functor.

Definition 4.17 (Singular n-simplex)**.** Given a topological space X, a *singular n-simplex* of X is a continuous map $f : |\Delta^n| \to X$, where

$$|\Delta^n| = \left\{(x_1, \dots, x_n) \in \mathbb{R}^{n+1} : \sum x_i = 1, x_i \geq 0\right\}, \tag{4.18}$$

endowed with the subspace topology from $\mathbb{R}^{n+1}$, is the *topological n-simplex.*

Let $\partial^i : |\Delta^{n-1}| \to |\Delta^n|$ be defined by

$$\begin{aligned} \partial^i(x_1, \dots, x_{n-1}) &= (x_1, \dots, x_{i-1}, 0, x_i \dots, x_{n-1}) \\ s^i(x_1, \dots, x_{n+1}) &= (x_1, \dots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \dots, x_{n+1}) \end{aligned} \tag{4.19}$$

Definition 4.20 (Singular simplicial set)**.** Given a topological space X, the *singular simplicial set* of X, denoted by $S(X)$, is the simplicial set whose n-simplices are the simply the singular n-simplices of X, and where the face maps $\partial_i : S_{n+1}(X) \to S_n(X)$ and degeneracy maps $s_i : S_{n-1}(X) \to S_n(X)$ are described as follows:

$$\begin{cases} \partial_i f = f \circ \partial^i \\ s_i f = f \circ s^i \end{cases} \tag{4.21}$$

We leave it as an exercise to verify that these maps satisfy the simplicial identities.

Definition 4.22 (Singular functor)**.** The *singular functor* $S : \mathbf{Top} \to \mathbf{SSet}$ is the functor which sends a topological space X to its total singular complex $S(X)$, and a continuous map $g : X \to Y$ to the simplicial map $S(g) : S(X) \to S(Y)$ which takes $f : |\Delta_n| \to X \in S_n(X)$ to $g \circ f : |\Delta_n| \to Y$.

Now we know of a way to interface one way, from topological spaces to simplicial sets. Now let's start to think about how we can go in the other direction. We accomplish this via *geometric realization.*

Definition 4.23 (Geometric Realization Functor)**.** Let K be a simplicial set, and endow each K_n with the discrete topology. The *geometric realization* of K, denoted as $|K|$, is the quotient space formed by the discrete union $\bigsqcup_n K_n \times |\Delta_n|$ modulo the equivalence $(\sim)$, under which

$$(\partial_i x, u) \sim (x, \partial^i u), \ (s_i x, u) \sim (x, s^i u) \tag{4.24}$$

As one may suspect, there exists a functor Geom : $\mathbf{SSet} \to \mathbf{Top}$ which sends a simplicial set K to its geometric realization $|K|$.

We then have the following strong relationship between a space X and the realization of its total singular complex $|S(X)|$, first introduced in [Mil57].

Proposition 4.25. *For any topological space X, $\pi_i(X,x) \simeq \pi_i(|S(X)|, |S(e)|)$.*

From a high-level perspective, this means that if we wish to investigate the homotopy groups of some space, we can instead study a simplicial set, a purely combinatorial object. This is the idea behind Brown's algorithm, which we will begin to describe in section 5.

4.3. **The simplicial sets Δ^k.** Next, we move to defining and exploring a certain special simplicial set.

Definition 4.26. The simplicial set Δ^k is defined as $\mathrm{Hom}_\Delta(-,[k])$.

From the purely categorial side, Yoneda Lemma tells us that this simplicial set has a special property:

Corollary 4.27. *Given a simplicial set N, there is a unique simplicial map ι_s between $\mathrm{Hom}_\Delta(-,[k])$ and N taking $\mathrm{id}_{[k]}$ to a given k-simplex $s \in N_k$.* □

We will omit proof of this result here, but revisit the reasoning when it comes to be applied.

This simplicial set is also special geometrically; we have the following theorem given in [Mil57]:

Theorem 4.28. *The geometric realization of Δ^k is homeomorphic to the topological k-simplex.*

Thus, we remain consistent with our previous use of $|\Delta^k|$ to denote the topological n-simplex.

There is a certain subsimplicial set of Δ^k called the boundary $\partial\Delta^k$, with the following property.

Proposition 4.29. *$|\partial\Delta^k|$ is homeomorphic to the $(k-1)$-sphere bounding $|\Delta^k|$.*

We refer to [GJ09], I.1 for a definition of $\partial\Delta^k$ and a proof of the above property.

We also make definitions of a collection of simplicial maps, relating the Δ^k.

Definition 4.30. We define simplicial maps $e_i : \Delta^{k-1} \to \Delta^k$, $t_i : \Delta^{k+1} \to \Delta^k$ via

$$\begin{aligned} (e_i)_{[n]}(\alpha) &= \alpha \circ \sigma_i \\ (t_i)_{[n]}(\alpha) &= \alpha \circ \delta_i, \end{aligned} \tag{4.31}$$

where σ_i and δ_i are the maps given in (4.10) and (4.11), respectively.

Now, first let's try to determine the size of the levels of these simplicial set.

Lemma 4.32.

$$|\mathrm{Hom}_\Delta([n],[m])| = \binom{n+m+1}{m}. \tag{4.33}$$

Proof. Recall that $\mathrm{Hom}_\Delta([n],[m])$ is the set of all functions $\mu : [n] \to [m]$ such that $\mu(i) \leq \mu(j)$ if $i < j$. So we can view an element $\mu \in \mathrm{Hom}_\Delta([n],[m])$ as a length $(n+1)$-tuple

$$(\mu(0), \mu(1), \ldots, \mu(n)), \tag{4.34}$$

with $\mu(i) \in [m]$ and (monotone) increasing across indices, making our problem a purely combinatorial one. Counting shows us the case where $n = 0$;

$$|\operatorname{Hom}_\Delta([0],[m])| = m + 1 = \binom{0+m+1}{m} \tag{4.35}$$

Towards the general case, a way we can imagine determining such a map μ is by choosing $\mu(0)$, and then choosing a monotone increasing map $\mu_1 : \{1,\dots,n\} \to \{\mu(0),\dots,m\}$, which is the same thing as choosing a monotone increasing map $\{0,\dots,n-1\} \to \{0,\dots,m-\mu(0)\}$. Given that that $\mu(0)$ can take any value in $[m]$, we obtain the following recursive relation:

$$|\operatorname{Hom}_\Delta([n],[m])| = \sum_{i=0}^{m} |\operatorname{Hom}_\Delta([n-1],[m-i])| \tag{4.36}$$

We can now induct. Assume $|\operatorname{Hom}_\Delta([n-1],[j])| = \binom{n+j}{j}$ for any j. Then we have

$$\begin{aligned} |\operatorname{Hom}_\Delta([n],[m])| &= \sum_{i=0}^{m} \binom{n+m-i}{m-i} \\ &= \sum_{i=0}^{m} \binom{n+i}{i} \\ &= \binom{n+m+1}{m}, \end{aligned} \tag{4.37}$$

where the last equality follows from a variant of the hockey-stick identity. □

4.4. **Chain complexes and homology.** One meaningful algebraic structure assigned to simplicial sets are chain complexes, which we discuss now.

Definition 4.38 ((Co)chain complex)**.** A *chain complex* C is a sequence of abelian groups C_n (called *chain groups*) and homomorphisms $\partial_n : C_n \to C_{n-1}$ (called *boundary maps*),

$$\cdots \longleftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \longleftarrow \cdots$$

with the property that $\partial_n \partial_{n+1} = 0$. We call the groups C_n *chain groups*, $Z_n(C) := \ker(\partial_n)$ the group of *n-cycles*, $B_n(C) := \operatorname{im}(\partial_{n+1})$ to be group of *n-boundaries*, and $H_n(C) := Z_n(C) / B_n(C)$ the *nth homology group* of the chain complex C.

A *cochain complex* is a structure dual to the chain complex, and is a sequence of abelian groups C^n and homomorphisms ∂^n,

$$\cdots \longrightarrow C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \longrightarrow \cdots$$

with the property that $\partial_n \partial_{n-1} = 0$. We similarly call the groups C^n *cochain groups*, $Z^n(C) := \ker(\partial^n)$ the group of *n-cocycles*, $B^n(C) := \operatorname{im}(\partial^{n-1})$ the group of *n-coboundaries,* and $H^n(C) := Z^n(C) / B^n(C)$ the *nth cohomology group* of the complex C.

Definition 4.39. Given two (co)chain complexes C, D, a *(co)chain map* $f : C \to D$ is a collection of maps $\{f_n : C_n \to D_n\}$ such that $\partial \circ f = f \circ \partial$.

Definition 4.40. If $f : C \to D$ is a chain map, the *mapping cone* of f, denoted by $\mathrm{Cone}(f)$, is a chain complex where $\mathrm{Cone}(f)_n = C_{n-1} \oplus D_n$, and with boundary map ∂ given by

$$\partial(x, y) = (-\partial_C(x), f(x) + \partial_D(y)) \tag{4.41}$$

Definition 4.42 (Simplicial homology/cohomology)**.** The *simplicial chain complex* of a simplicial set K is defined to be the chain complex $C(K)$ where the chain groups $C_n(K)$ are the free abelian groups generated by K_n, and with boundary maps ∂_n whose action on the generators $k \in K_n$ of the free abelian groups is described by $\sum_i (-1)^i \partial_i(k)$, where $\partial_i : K_n \to K_{n-1}$ are the appropriate face maps. The *nth simplicial homology group* $H_n(K)$ is obtained by taking the nth homology of the simplicial chain complex.

Given an abelian group π, one can also consider a *simplicial cochain complex*, obtained through application of the contravariant functor $\mathrm{Hom}(-, \pi)$ to the simplicial chain complex:

$$\begin{array}{ccccccccc}
\cdots & \longleftarrow & C_{n-1}(K) & \xleftarrow{\partial_n} & C_n(K) & \xleftarrow{\partial_{n+1}} & C_{n+1}(K) & \longleftarrow & \cdots \\
 & & & & \Big\Downarrow {\scriptstyle \mathrm{Hom}(-,\pi)} & & & & \\
\cdots & \longrightarrow & C^{n-1}(K;\pi) & \xrightarrow{\mathrm{Hom}(\partial_n,\pi)} & C^n(K;\pi) & \xrightarrow{\mathrm{Hom}(\partial_{n+1},\pi)} & C^{n+1}(K;\pi) & \longrightarrow & \cdots
\end{array}$$

The *nth simplicial cohomology group with coefficients in* π is defined to be the the nth cohomology group of the cochain complex described above.

Remark 4.43*.* One may notice that the definition of simplicial homology relies solely on the face maps. For this reason, for most of the computations we do with simplicial sets in thie paper we will be only concerned with determining face maps. The reason the degeneracy maps can be useful is because they can allow us to compute homology using smaller substructures; there is a theorem that roughly says that "simplicial homology is equal to simplicial homology modulo degeneracies." We refer the reader to [GJ09], III.2 for a precise statement and proof.

Computing (co)homology is an integral part of Brown's algorithm, and so we list here a couple of computational lemmas regarding the complexity of determining these groups and some surrounding structures.

Definition 4.44 (Smith Normal Form)**.** Let A be an $m \times n$ integer matrix. Then there exists invertible integer matrices P, Q such that

$$Q^{-1}AP = \begin{bmatrix}
d_1 & 0 & 0 & & \cdots & & 0 \\
0 & d_2 & 0 & & \cdots & & 0 \\
0 & 0 & \ddots & & & & 0 \\
\vdots & & & d_r & & & \vdots \\
 & & & & 0 & & \\
 & & & & & \ddots & \\
0 & & & \cdots & & & 0
\end{bmatrix}, \tag{4.45}$$

where $d_1 | d_2 | \ldots | d_r$. The product $Q^{-1}AP$ is known as the **Smith Normal Form** (SNF) of A, and the integers d_i are known as the **elementary divisors** of A.

We note that the rank of the matrix A is the same as the rank of its SNF, which can be determined by counting the number of nonzero rows.

Lemma 4.46. *Given a sequence*

$$\mathbb{Z}^p \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^m$$

where A and B are matrices such that $AB = 0$, we can determine the homology group $\ker A \big/ \operatorname{im} B$ up to isomorphism as a direct sum of cyclic groups and free abelian groups

$$O(m^2 n + n^2 p) \tag{4.47}$$

time.

Proof of lemma. According to [AC21], we know that $\ker A \big/ \operatorname{im} B \simeq \bigoplus_{i=1}^{r} \mathbb{Z}_{d_i} \oplus \mathbb{Z}^{n - \operatorname{rank} A - \operatorname{rank} B}$, where the d_i are the elementary divisors of the matrix B. We obtain the elementary divisors through computing the Smith Normal Form of the matrix A, which is accomplished in $O(n^2 p)$ time according to [Sto96]. We can determine the ranks of A and B through the SNF also, which we compute in $O(n^2 p + m^2 n)$ time. □

Lemma 4.48. *Given an $m \times n$ integer matrix A, we can determine a basis for the kernel of A, in addition to an extension of it to a basis for all of $\mathbb{Z}^n$, in*

$$O(m^2 n) \tag{4.49}$$

time.

Proof of lemma. We begin by determining the SNF S of A. This is an $O(m^2 n)$ operation according to [Sto96]. Now, let's try to relate the kernel of A to the kernel of S. Given that $S = Q^{-1} A P$ for some invertible Q, P, we can see that $Sx = 0 \Leftrightarrow QSx = 0 \Leftrightarrow QSP^{-1}(Px) = 0 \Leftrightarrow A(Px) = 0$. Thus, we can see that the kernel of A is the image under P of the kernel of S, and that we can find a basis for the kernel of A by simply computing Px for all x in a basis of S. Further, finding a basis for the kernel of S is easy! Given we know that

$$S = \begin{bmatrix} d_1 & 0 & 0 & & \cdots & & 0 \\ 0 & d_2 & 0 & & \cdots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ \vdots & & & d_r & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & 0 \end{bmatrix}, \tag{4.50}$$

we can see that the elementary basis vectors $\{e_i : r < i \leq n\}$ form a basis for the kernel of S. Therefore, $\{Pe_i : r < i \leq n\}$ form a basis for the kernel of A. These are just the $(r+1)$th through nth columns of P. Further, given that P is invertible, we know that all of its columns are linearly independent when considered as vectors in $\mathbb{Z}^n$. Therefore, we can accomplish our desired task by simply reading off the columns of P, which we can do in $O(n^2)$ time. □

Lemma 4.51. *Given the following sequence*

$$\mathbb{Z}^p \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^m$$

where A, B are matrices such that $AB = 0$, we can determine the action of some extension of the canonical map $\ker A \to \ker A \diagup \operatorname{im} B$ *on the canonical basis $\{e_i\}$ in*

$$O(n^3 + m^2 n + n^2 p) \tag{4.52}$$

time.

Proof of lemma. First, let's discuss how we can determine the action. By definition, we know that B is a presentation matrix for $\ker A \diagup \operatorname{im} B$. This means that if

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}, \tag{4.53}$$

then the following relations hold amongst the residues $\{\overline{e_i}\}$ of the canonical basis vectors $\{e_i\}$ of $\mathbb{Z}^n$:

$$\begin{cases} b_{11}\overline{e_1} + b_{21}\overline{e_2} + \ldots + b_{n1}\overline{e_n} = 0 \\ b_{12}\overline{e_1} + b_{22}\overline{e_2} + \ldots + b_{n2}\overline{e_n} = 0 \\ \vdots \\ b_{1p}\overline{e_1} + b_{2p}\overline{e_2} + \ldots + b_{np}\overline{e_n} = 0 \end{cases} \tag{4.54}$$

The question is then what relations this implies amongst the residues of vectors in a separate basis $\{w_i\}$. The answer is not too complicated; we consider the change of basis matrix

$$Q = \begin{bmatrix} | & & | \\ w_1 & \dots & w_n \\ | & & | \end{bmatrix} \tag{4.55}$$

Then the presentation matrix giving the relations between $\{\overline{w_i}\}$ is given by the matrix QB.

Further, we can also convince ourselves that if we take an invertible $p \times p$ matrix, say P, and consider the matrix QBP, then this also provides a complete set of relations between $\{\overline{w_i}\}$, as the right multiplication by P represents a change of basis in the domain $\mathbb{Z}^p$.

So looking back, 4.44 tells us that there exists invertible matrices Q, P such that

$$Q^{-1}BP = \begin{bmatrix} d_1 & 0 & 0 & & \cdots & & 0 \\ 0 & d_2 & 0 & & \cdots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ \vdots & & & d_r & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & 0 \end{bmatrix}, \tag{4.56}$$

which in turn tells us that there is a basis $\{w_i\}$ such that the following is a complete set of relationships for the module $\ker A \diagup \operatorname{im} B$:

$$\begin{cases} d_1\overline{w_1} = 0 \\ d_2\overline{w_2} = 0 \\ \quad\vdots \\ d_r\overline{w_r} = 0 \end{cases} \tag{4.57}$$

Given that the matrix in 4.56 is how we determined $\ker A \diagup \operatorname{im} B$ to be $\bigoplus_{i=1}^{r} \mathbb{Z}_{d_i} \oplus \mathbb{Z}^{n-\operatorname{rank} A - \operatorname{rank} B}$ (see [AC21]), we know that we can take the the residue $\overline{w_i} = \pi(w_i)$ to be the element $\epsilon_i := (0, \dots, 0, 1, 0, \dots, 0) \in \bigoplus_{i=1}^{r} \mathbb{Z}_{d_i} \oplus \mathbb{Z}^{n-\operatorname{rank} A - \operatorname{rank} B}$, where the 1 is in the ith position. Now we can use properties of homomorphisms to deduce where other elements should map under π! Let's start with the canonical basis $\{e_i\}$. We use the fact that Q represented a change of basis from $\{e_i\}$ to $\{w_i\}$ to reason

$$\begin{aligned} \pi(w_i) &= \pi(Qe_i) \\ \Rightarrow \pi(e_i) &= \pi(Q^{-1}w_i) \end{aligned} \tag{4.58}$$

We know that $Q^{-1}w_i = \sum_k (Q^{-1})_{ki} w_k$; therefore, we obtain

$$\pi(e_i) = \pi(Q^{-1}w_i) = \pi\left(\sum_k Q^{-1}_{ki} w_k\right) = \sum_k \left[Q^{-1}_{ki} \mod d_k\right] \epsilon_k \tag{4.59}$$

We then want to take this a step further; we want to describe what this implies for the residues of a basis $\{v_i\}$, where $\{v_i\}_{i=1}^{r}$ is a basis for $\ker A$. So let $\{v_i\}$ be such a basis, and R be the matrix representing a change of basis from $\{e_i\}$ to $\{v_i\}$; then we can repeat reasoning similar to above to see that

(4.60)

$$\pi(v_i) = \pi(R(Q^{-1})w_i) = \pi\left(\sum_k (R(Q^{-1}))_{ki} w_k\right) = \sum_k \left[(R(Q^{-1}))_{ki} \mod d_k\right] \epsilon_k$$

It is here where things might get a bit confusing. What we have succeeded in describing at the moment is finding out where our basis $\{v_i\}$ of Z^n is taken under the canonical map $\pi : Z^n \to Z^n \diagup \operatorname{im} B$. However, this isn't quite an extension of the canonical map from $\ker A \to \ker A \diagup \operatorname{im} B$, as in general $Z^n \diagup \operatorname{im} B$ is slightly "bigger" than $\ker A \diagup \operatorname{im} B$; $\ker A \diagup \operatorname{im} B \simeq \bigoplus_{i=1}^{r} \mathbb{Z}_{d_i} \oplus \mathbb{Z}^{n-\operatorname{rank} A - \operatorname{rank} B}$, while $\mathbb{Z}^n \diagup \operatorname{im} B \simeq \bigoplus_{i=1}^{r} \mathbb{Z}_{d_i} \oplus \mathbb{Z}^{n-\operatorname{rank} B}$. Thus, as it stands we run the risk of describing a map where the basis elements $\{v_i\}_{i=r+1}^{n}$ are mapped to elements outside of our group of interest. To remedy this, we will instead consider a minor modification to the map π we currently have.

$$\begin{array}{ccccc} \mathbb{Z}^n & \xrightarrow{\quad r \quad} & \mathbb{Z}^n & \xrightarrow{\quad \pi \quad} & \mathbb{Z}^n \diagup \operatorname{im} B \\ v_i & \longmapsto & \begin{cases} v_i & i \le r \\ 0 & i > r \end{cases} & \longmapsto & \begin{cases} \pi(v_i) & i \le r \\ 0 & i > r \end{cases} \end{array}$$

FIGURE 4.1.

By considering the composition $\pi \circ r$, we obtain an extension of the canonical map $\ker A \to \ker A \diagup \operatorname{im} B$. Now, this map is easy to describe for our basis $\{v_i\}$, but our end goal is to describe it for our basis $\{e_i\}$. Let's work this out. The matrix for the transformation r in terms of the basis $\{v_i\}$ is easy to write down; one verifies that it is the following;

$$\begin{bmatrix} [I_r] & \\ & 0 \end{bmatrix}, \tag{4.61}$$

where I_r is the $r \times r$ identity. Thus, to describe its action in terms of the basis e_i, we conjugate; the matrix for r in terms of the basis $\{e_i\}$ is given by

$$R \begin{bmatrix} [I_r] & \\ & 0 \end{bmatrix} R^{-1} \tag{4.62}$$

That is, the image of e_i under r, written in the coordinates given by $\{e_i\}$, is the ith column of the matrix $R \begin{bmatrix} [I_r] & \\ & 0 \end{bmatrix} R^{-1}$. We can then use our knowledge of where π takes $\{e_i\}$ to find out where this element is taken, and thus we have finally determined the image of $\{e_i\}$ under the composition $\pi \circ r$, which is what we wanted.

With that, let us see how long each step in this process takes.

(1) First, we want to find the Smith Normal Form of the $n \times p$ matrix B, along with the matrix Q representing the change of basis between $\{e_i\}$ to $\{w_i\}$. This is accomplished in $O(n^2 p)$ time according to [Sto96]
(2) Next, we want to determine where our canonical basis e_i maps to under π, which we accomplish by computing Q^{-1} and computing the sum $\sum_k \left[Q^{-1}_{ki} \mod d_k\right] \epsilon_k$. The inversion is accomplished via Gaussian elimination, which we can count to be done in $O(n^3)$ time, while computing the sum involves $O(n)$ arithmetic operations for each e_i, giving us an $O(n^2)$ runtime total.
(3) Then, we want to find the matrix R representing change of basis between $\{e_i\}$ and $\{v_i\}$. This is obtained in the process of determining the kernel of A, which is accomplished in $O(m^2 n)$ by our work in Lemma 4.48.
(4) Then, we want to compute the product $R \begin{bmatrix} [I_r] & \\ & 0 \end{bmatrix} R^{-1}$, and then do more appropriate arithmetic to determine the image of $\{e_i\}$ under $\pi \circ r$. We can count the matrix multiplication to be accomplished in $O(n^3)$ time, while the arithmetic can again be seen to take $O(n^2)$ time.

Thus, taking the sum across all steps, we obtain that the total runtime is $O(n^3 + m^2 n + n^2 p)$. □

5. Analysis of Postnikov Construction with Finite Coefficient Groups

5.1. **The Postnikov Construction.** Brown makes use of a construction with simplicial sets, originally detailed by M. Postnikov in [Pos51], which has particularly nice properties when considering the homotopy groups of the realization of a simplicial set. The description of these constructions has been streamlined since Brown's day, tending to sacrifice explicitness for elegance. For the purposes of this paper, we require the former, and so we will describe the construction largely following Brown's original exposition.

We begin by defining some objects which will be used in what follows, and determining the complexity of determining them. First, we describe a categorical construction.

Definition 5.1. For three objects A, B, C and two morphisms $f : A \to B$, $g : C \to B$ in a category $\mathcal{C}$, the *fiber product* of f and g is defined to be an object, denoted by $A \times_B C$, along with two maps $p_1 : A \times_B C \to A$, $p_2 : A \times_B C \to C$ which is universal with respect to the following square:

$$\begin{array}{ccc} A \times_B C & \xrightarrow{\quad p_1 \quad} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[\quad f \quad]{} & B \end{array}$$

The motivation for the name is that if $\mathcal{C} = \mathbf{Set}$, then $A \times_B C$ can be obtained by taking the product of the fibers of f and g.

$$A \times_B C = \bigcup_{x \in B} \{f^{-1}(x) \times g^{-1}(x)\} \tag{5.2}$$

Then the maps p_1, p_2 are the projection maps onto the first and second coordinates, respectively. Although formally a fiber product of two maps refers to the triple $(A \times_B C, p_1, p_2)$, we will often use it to refer to only the object $A \times_B C$. It will be of interest of us to compute the fiber product of two maps of sets, so we will state the following lemma here.

Lemma 5.3. *Given functions $f : A \to B$, $g : C \to B$ between sets A, B and C, we can determine the set $A \times_B C$ in $O(|A||C|)$ time.*

Proof. Our process is the following:

(1) Compute the collection of singleton fibers $\{f^{-1}(x)\}_{x \in B}$, $\{g^{-1}(x)\}_{x \in B}$.
(2) Compute the collection of products $\{f^{-1}(x) \times g^{-1}(x)\}_{x \in B}$.
(3) Compute the union $\bigcup_{x \in b} f^{-1}(x) \times g^{-1}(x)$.

Step (1) is accomplished in $O(|A| + |B|)$ time according to 3.4.3. Step (2) is accomplished in $O\left(\sum_{x \in B} |f^{-1}(x)||g^{-1}(x)|\right)$ time according to 3.4.2. However, the inequality $\sum_{x \in B} |f^{-1}(x)||g^{-1}(x)| \leq \left(\sum_{x \in B} |f^{-1}(x)|\right)\left(\sum_{x \in B} |g^{-1}(x)|\right) = |A||C|$ tells us that we can further upper bound this to be $O(|A||B|)$ time. Step (3) is accomplished in $O\left(\left|\bigcup_{x \in B} f^{-1}(x) \times g^{-1}(x)\right|\right)$ time; we use the same inequality as above to conclude that this is also an $O(|A||C|)$ time task. Summing yields our desired result. □

With this, we move to defining the following structures, obtained from Δ^k.

Definition 5.4 $(K(\pi, n), E(\pi, \lambda, n))$**.** Given an abelian group π and n, we define $K(\pi, n)$ to be the simplicial set where the set of k-simplices is given as

$$K(\pi, n)_k = Z^n(\Delta^k; \pi) \tag{5.5}$$

and where the face/degeneracy maps are the maps induced via the simplicial maps e_i, t_i in (4.30).

Given $\lambda \subset \pi$, we can also define $E(\pi, \lambda, n)$ to be the simplicial set where the set of k-simplices is given as

$$E(\pi, \lambda, n)_k = \{u \in C^n(\Delta^k; \pi) : \operatorname{im} u \subset \lambda\} \tag{5.6}$$

and where the face/degeneracy maps are the same maps discussed for $K(\pi, n)$. We let $E(\pi, n) = E(\pi, \pi, n)$.

We define

$$\delta : E(\pi, \lambda, n-1) \to K(\pi, n) \tag{5.7}$$

to be the map sending u in $E(\pi, \lambda, n-1)_k \subset C^{n-1}(\Delta^k, \pi)$ to its coboundary in $K(\pi, n)_k = Z^n(\Delta^k, \pi)$.

Lemma 4.32 gives us the following:

Lemma 5.8. *Given finite* π,

$$\begin{aligned} |E(\pi, n)_k| &= |\pi|^{\binom{n+k+1}{k}} \\ |K(\pi, n)_k| &= O\left(|\pi|^{\binom{n+k+1}{k}}\right) \end{aligned} \tag{5.9}$$

Proof of lemma. Δ^k is defined to be the simplicial set $\operatorname{Hom}_\Delta(-, [k])$. Therefore given that $\dim(C_n(N)) = |N_n|$ for any simplicial set N, $\dim\left(C_n(\Delta^k)\right) = |\operatorname{Hom}_\Delta([n], [k])| = \binom{n+k+1}{k}$. Since $E(\pi, n)_k = C^n(\Delta^k; \pi) = \operatorname{Hom}_{\mathbf{Ab}}(C_n(\Delta^k), \pi)$, all that remains is to count the number of unique homomorphisms $\varphi : C_n(\Delta^k) \to \pi$. Given that a homomorphism from a free abelian group into an abelian group is determined by the images of a basis of the source, we count that there are $|\pi|^{\binom{n+k+1}{k}}$ unique homomorphisms, as desired. The second bound is obtained by corollary since $K(\pi, n)_k \subset E(\pi, n)_k$. ☐

It will be necessary to determine the action of the face maps of the simplicial sets E and K.

Lemma 5.10. *Let* π *be a finite abelian group. Then we can determine the action of all of the* d_i *on* $E(\pi, n)_k$ *in*

$$O\left(n\binom{n+k}{k}|\pi|^{\binom{n+k}{k}}\right) \tag{5.11}$$

time.

Proof. For a fixed $f \in E(\pi, n)_k$, we need to first determine $d_i f$ for all i. $d_i f$ is given by $f \circ (e_i)_*$ where e_i is the map in (4.30). We can determine what this map is in $O(|E(\pi, n)_{k-1}|) = O\left(\binom{n+k}{k}\right)$ time. Repeating this for the $O(n)$ d_i and for all $f \in E(\pi, n)_k$ gives us the desired runtime. ☐

It is also of interest of us to see how we can determine the map $\delta : E(\pi, \lambda, n) \to K(\pi, n+1)$. Towards this, we have the following lemma:

Lemma 5.12. *Given* Δ^k*, we can determine the action of* δ *on* $E(\pi, n)_k$ *in*

$$O\left(n\binom{n+k+2}{k}|\pi|^{\binom{n+k+1}{k}}\right) \tag{5.13}$$

time.

Proof of lemma. For a basis element $f \in \operatorname{Hom}(C_n(\Delta^k), \pi)$, we have that $\delta f = \sum_i f \circ \sigma_i \in Z^{n+1}(\Delta^k, \pi)$. We can determine the action of $f \circ \sigma_i$ on $C_{n+1}(\Delta^k)$ in $O(|(\Delta^k)_{n+1}|) = O(\binom{n+k+2}{k})$ time. Given that there are $O(n)$ σ_i, we can determine δf in $O\left(n\binom{n+k+2}{k}\right)$ time. Repeating this for all $f \in \operatorname{Hom}(C_n(\Delta^k), \pi)$ gives us the above runtime. □

We have one more structure left to describe before we can describe the Postnikov construction.

Definition 5.14. A map $z \in C^n(N; \pi)$ induces a map $\hat{z} : N \to K(\pi, n)$ such that for a k-simplex $s \in N_k$,

$$\hat{z}(s) = z \circ (\iota_s)_n^*, \tag{5.15}$$

where $(\iota_s)_n^* : C_n(\Delta^k) \to C_n(N)$ is the map induced by $(\iota_s)_n : (\Delta^k)_n \to N_n$.

Maps defined in this way play a key role in the Postnikov construction, and it will be of interest to use to see the complexity of determining $\hat{z}$ given z. We will do this in steps.

Lemma 5.16. *Fix $s \in N_k$, and let $\iota_s : \Delta^k \to N$ be the simplicial map from 4.27. For any $u \in (\Delta^k)_n$, we can determine $(\iota_s)_n(u)$ in $O(n + k)$ time.*

Proof. First, let's see how fixing $\mathrm{id}_{[k]}$ maps to determines the entire simplicial map. For any $u \in (\Delta^k)_n = \operatorname{Hom}([n], [k])$ we can see where u must map from the following commutative square, obtained from the fact that ι_s is a natural tranformation.

$$\begin{array}{ccc}
\operatorname{Hom}([k],[k]) & \xrightarrow{\operatorname{Hom}(u,[k])} & \operatorname{Hom}([n],[k]) \\
\downarrow{\scriptstyle (\iota_s)_k} & & \downarrow{\scriptstyle (\iota_s)_n} \\
N_k & \xrightarrow[Nu]{} & N_n
\end{array}$$

Analyzing starting from the point $\mathrm{id}_{[k]} \in \operatorname{Hom}([k], [k])$, we obtain the following pointed diagram

$$\begin{array}{ccc}
\mathrm{id}_{[k]} & \xmapsto{\operatorname{Hom}(u,[k])} & u \\
\downarrow{\scriptstyle (\iota_s)_k} & & \downarrow{\scriptstyle (\iota_s)_n} \\
s & \xmapsto[Nu]{} & (\iota_s)_n(u) = Nu(s)
\end{array}$$

So we see that it suffices to determine $Nu(s)$. So let us figure out how we can determine this. Lemma 4.12 tells us that u is the composition of $O(n + k)$ face/degeneracy maps, and functoriality tells us that Nu is the composition of the corresponding face/degeneracy maps. So given N, we can determine the appropriate composition in $O(n + k)$ time, and then determine where s maps under the determined composition in $O(n+k)$ time. Summing yields our desired runtime. □

Corollary 5.17. *We can determine the action of* $(\iota_s)_n^*$ *on* $C_n(\Delta^k)$ *in*

$$O\left((n+k)\binom{n+k+1}{k}\right) \tag{5.18}$$

time.

Proof. We can determine $(\iota_s)_n(u)$ in $O(n+k)$ time; repeating over all $u \in (\Delta^k)_n$ yields that we can determine the action on all of $(\Delta^k)_s$ in $O\left((n+k)|(\Delta^k)_n|\right)$ time. Since $(\Delta^k)_n$ is a basis of $C_n(\Delta^k)$, we therefore have determined the action on $C_n(\Delta^k)$. □

Lemma 5.19. *For a fixed* $s \in N_k$ *and* $z \in C^k(N;\pi)$*, we can determine* $\hat{z}(s) = z \circ (\iota_s)_n^*$ *in*

$$O\left((|N_k| + (n+k))\binom{n+k+1}{k}\right) \tag{5.20}$$

time.

Proof. The process is to first determine $(\iota_s)_n^*$, and then determine $(z \circ (\iota_s)_n^*)(u)$ for all $u \in (\Delta^k)_n$. The first step has runtime $O\left((n+k)\binom{n+k+1}{k}\right)$, and the second step has runtime $|N_k||(\Delta^k)_n| = O\left(|N_k|\binom{n+k+1}{k}\right)$ time. Summing yields the desired runtime. □

Lemma 5.21. *For* $z \in C^k(N;\pi)$*, we can determine the action of* $\hat{z}$ *on* N_k *in*

$$O\left(|N_k|(|N_k| + (n+k))\binom{n+k+1}{k}\right) \tag{5.22}$$

Proof. We need to determine $\hat{z}(s)$ for all $s \in N_k$; multiplying using the lemma above yields the desired runtime. □

We this, we are finally well-equipped to describe the Postnikov Construction.

Definition 5.23 (The Postnikov Construction)**.** Let N be a simplicial set, π an abelian group, $\lambda \subset \pi$, and $A^n \in Z^n(N;\pi)$. The *Postnikov Construction*, denoted as $P(N,\pi,\lambda,A^n)$, is the subsimplicial set of $N \times E(\pi,\lambda,n-1)$, where

$$P(N,\pi,\lambda,A^n)_k = \delta_k \times_{K(\pi,n)_k} \hat{A^n}_k, \tag{5.24}$$

and with face/degeneracy maps inherited from N and $E(\pi,\lambda,n-1)$. We define $P(N,\pi,A^n)$ to be the complex $P(N,\pi,\pi,A^n)$.

First, a rudimentary bound on the size of this complex;

Lemma 5.25. $|P(N,\pi,A^n)_k| = O\left(|N_k||\pi|^{\binom{n+k}{k}}\right)$

Proof of lemma. $P(N,\pi,A^n)_k \subset N_k \times E(\pi,n)_k$, and $|N_k \times E(\pi,n)_k| = |N_k||E(\pi,n)_k| = |N_k||\pi|^{\binom{n+k}{k}}$ from (5.8). □

Now, let us determine how hard it is to compute each level of the Postnikov construction.

Lemma 5.26. *Let N be a simplicial set, π a finite abelian group, and $A^n \in Z^n(N;\pi)$. Given finite N_k, π, and $(A^n)_k : C_k(N) \to \pi$, we can determine $P(N,\pi,A^n)_k$ in*

$$O\left(\left[|N_k| + n\binom{n+k}{k}\right]|\pi|^{\binom{n+k}{k}} + |N_k|\left(|N_k| + (n+k)\right)\binom{n+k}{k}\right) \tag{5.27}$$

time.

Proof of lemma. One procedure to determine the Postnikov construction is the following.

1. Determine the maps $\hat{A}^n$ and δ.
2. Compute the fiber product between $\hat{A}^n$ and δ.

Using the lemmas above and summing yields the above runtime. □

In our later applications, we will require a little more information than just the individual sets; we will also need some of the data of the face maps. So we end this section with the following theorem:

Theorem 5.28. *Let N be a simplicial set, π a finite abelian group, and $A^n \in Z^n(N;\pi)$. Given the data in the sequence*

$$N_{k+1} \underset{d_{k+1}}{\overset{d_0}{\rightrightarrows}}\; \vdots \;\; N_k \underset{d_k}{\overset{d_0}{\rightrightarrows}}\; \vdots \;\; N_{k-1}\ ,$$

π, and $A^n \in Z^n(N;\pi)$, we can determine the data in the sequence

$$P(N,\pi,A^n)_{k+1} \underset{d_{k+1}}{\overset{d_0}{\rightrightarrows}}\; \vdots \;\; P(N,\pi,A^n)_k \underset{d_k}{\overset{d_0}{\rightrightarrows}}\; \vdots \;\; P(N,\pi,A^n)_{k-1}$$

in

$$\begin{aligned} O\Big(&\left[\left(\max_{i\in I}|N_i|\right) + n\binom{n+k}{k}\right]|\pi|^{\binom{n+k}{k}} \\ &+ \left(\max_{i\in I}|N_i|\right)\left(\left(\max_{i\in I}|N_i|\right) + (n+k)\right)\binom{n+k}{k}\Big) \end{aligned} \tag{5.29}$$

time, where $I = \{k-1, k, k+1\}$.

Proof. Follows from (5.10) and (5.26). □

6. Analysis of Iterated Postnikov Construction

We now move to analyzing what Brown dubs the "iterated Postnikov Construction," or the construction of a Postnikov tower for a simplicial set N. The high-level overview of the construction is to inductively define a triple $(P_n(N), p_n(N), g_n)$, where $P_n(N)$ is a simplicial set, $p_n(N)$ is an abelian group, and $g_n : N \to P_n(N)$ is a simplicial map, for each $n \in \mathbb{N}$. We then have the following powerful theorem:

Theorem 6.1 (Brown 5.1)**.** *If N is a complex such that $|N|$ is simply connected, then $p_n(N) \simeq \pi_n(|N|)$.*

which tells us that in order to calculate the higher homotopy groups $\pi_n(X)$, it suffices to first determine some representation of our space X as a simplicial set - that is, a simplicial set N such that $|N|$ is homotopy equivalent to X - and then inductively determine $p_n(N)$.

6.1. **Iterated Postnikov Construction.** With this in mind, it remains to describe this construction.

Definition 6.2 (Iterated Postnikov Construction)**.** Given a simplicial set N, we define a simplicial set $P_n(N)$, an abelian group $p_n(N)$, and simplicial map $g_n : N \to P_n(N)$ inductively as follows:

- $P_1(N)$ is the "trivial simplicial set," where each level of the simplicial set contains 1 element, and where the all of the face/degeneracy maps are the (unique) map between two one-element sets. $p_1(N)$ is the trivial group, and $g_1 : N \to p_1(N)$ is the map which takes each N_k into the appropriate one-element set.
- The inductive step is a bit more involved. Assume $P_{n-1}(N)$, $p_{n-1}(N)$ and $g_{n-1} : N \to P_{n-1}(N)$ have been defined. Let $\hat{g}_{n-1}$ be the mapping cone of the chain map $C(N) \to C(P_{n-1}(N))$ induced by g_{n-1}. We then define

$$p_n(N) := H_{n+1}(\hat{g}_{n-1}) \tag{6.3}$$

  Moving on, we let $E^{n+1} \in Z^{n+1}(\hat{g}_{n-1}; p_n(N))$ be some extension of the quotient map from $Z_{n+1}(\hat{g}_{n-1})$ into $H_{n+1}(\hat{g}_{n-1}) = p_n(N)$. We use this to define the pair of maps;
  - $A^{n+1} \in Z^{n+1}(P_n(N); p_n(N))$ is defined by letting $A^{n+1}(s) = E^{n+1}(0, s)$, and
  - $B^n \in C^n(N; p_n(N))$ is defined by letting $B^n(r) = E^{n+1}(r, 0)$

  We can now finally make the definitions of the last two objects we need;

$$P_n(N) := P(P_{n-1}(N), p_n(N), A^{n+1}) \tag{6.4}$$

$$g_n := (g_{n-1}, B^n) \tag{6.5}$$

6.2. **Complexity of determining P_n from P_{n-1}.** With this in mind, we move to analyzing the difficulty of going a "level" upwards in the Postnikov construction: the complexity of determining $P_n(N), p_n(N)$ and g_{n-1} given $P_{n-1}(N), p_{n-1}(N)$ and g_{n-1}. It is here where a lot of the linear algebra work we did in the Preliminaries section comes in handy.

Lemma 6.6. *Given the data in the sequences*

$$N_{n+1} \underset{d_{n+1}}{\overset{d_0}{\rightrightarrows}} \vdots \; N_n \underset{d_n}{\overset{d_0}{\rightrightarrows}} \vdots \; N_{n-1}$$

and

$$P_{n-1}(N)_{n+2} \underset{d_{n+2}}{\overset{d_0}{\rightrightarrows}} \vdots \; P_{n-1}(N)_{n+1} \underset{d_{n+1}}{\overset{d_0}{\rightrightarrows}} \vdots \; P_{n-1}(N)_n \; ,$$

in addition to $(g_{n-1})_{n+1}$ *and* $(g_{n-1})_n$ *we can determine* $p_n(N)$ *in*

$$O[(|N_{n-1}| + |P_{n-1}(N)_n|)^2 (|N_n| + |P_{n-1}(N)_{n+1}|) + (|N_n| + |P_{n-1}(N)_{n+1}|)^2 (|N_{n+1}| + |P_{n-1}(N)_{n+2}|)] \tag{6.7}$$

time.

Proof of lemma. Recall that $p_n(N)$ is defined to be $(n+1)$th homology of $\mathrm{Cone}(g_{n-1})$. So the data we care about in order to compute the homology is the following sequence:

$$N_{n+1} \oplus P_{n-1}(N)_{n+2} \xrightarrow{\partial} N_n \oplus P_{n-1}(N)_{n+1} \xrightarrow{\partial} N_{n-1} \oplus P_{n-1}(N)_n$$

So from Lemma (4.46), we retrieve the above runtime. □

Our next task is to determine the quotient map from $Z_{n+1}(\hat{g}_{n-1})$ into $p_n(N)$, and then determine an extension of it to a map from $C_{n+1}(\hat{g}_{n-1})$ into $p_n(N)$. Some effort is required here to determine the action of this map on the "canonical" basis of $C_{n+1}(\hat{g}_{n-1})$;

(1) First, we must find a basis for $Z_{n+1}(\hat{g}_{n-1})$, and then extend it to a basis for $C_{n+1}(\hat{g}_{n-1})$.
(2) Then, we must find out where the basis of $Z_{n+1}(\hat{g}_{n-1})$ is taken under the canonical map; that is, what remains after "killing off" $\mathrm{im}(\partial_{n+2})$. (We can just send the basis vectors spanning the rest of $C_{n+1}(\hat{g}_{n-1})$ to 0 arbitrarily.)
(3) Finally, with the information about how our map of interest behaves on this basis, we must reconstruct how it behaves on our "canonical" basis.

Luckily, we can accomplish each of these steps with a bit of linear algebra.

Lemma 6.8. *Given the data given in (6.6), we can find a basis of* $C_{n+1}(\hat{g}_{n-1})$ *which is an extension of a basis of* $Z_{n+1}(\hat{g}_{n-1})$ *in*

$$O\left[(|P_{n-1}(N)_{n+1}| + |N_n|)^2 (|P_{n-1}(N)_{n+2}| + |N_{n+1}|)\right] \tag{6.9}$$

time.

Proof of lemma. This follows directly from Lemma (4.48). □

Lemma 6.10. *Given the data given in (6.6), we can determine* $E^{n+1} \in Z^{n+1}(\hat{g}_{n+1}; p_n(N))$ *in*

$$O[(|P_{n-1}(N)_{n+1}| + |N_n|)^3 + (|N_{n-1}| + |P_{n-1}(N)_n|)^2 (|N_n| + |P_{n-1}(N)_{n+1}|) + (|N_n| + |P_{n-1}(N)_{n+1}|)^2 (|N_{n+1}| + |P_{n-1}(N)_{n+2}|)] \tag{6.11}$$

time.

Proof of lemma. This follows directly from Lemma (4.51). □

With this, we essentially have all the information we need to make the final construction; we omit proof of the following lemma.

Lemma 6.12. *Given* E^{n+1}*, we can determine* A^{n+1} *in* $O(|P_{n-1}(N)_{n+1}|)$ *time and* B^n *in* $O(|N_n|)$ *time*

Lemma 6.13. *Given the data given in 6.6, in addition to $P_{n-1}(N)_k$ and $(g_{n-1})_k$, we can determine $P_n(N)_k, p_n(N)$ and $(g_n)_k$ in*

$$\begin{aligned}
O\Big(&\left[(|P_{n-1}(N)_k|) + n\binom{n+k}{k}\right] |p_n(N)|^{\binom{n+k}{k}} \\
&+ (|P_{n-1}(N)_k|)\left((|P_{n-1}(N)_k|) + (n+k)\right)\binom{n+k}{k} \\
&+ \max_{i\in I}(|N_i| + |P_{n-1}(N)_{i+1}|)^3\Big)
\end{aligned} \tag{6.14}$$

time, where $I = \{n-1, n, n+1\}$, assuming $|p_n(N)| < \infty$.

Proof. Follows from the lemmas in this section, in addition to (5.26) □

Proposition 6.15. *Given the data given in (6.6), in addition to the data in the sequence*

$$P_{n-1}(N)_{k+1} \underset{d_{k+1}}{\overset{d_0}{\rightrightarrows}} \vdots \; P_{n-1}(N)_k \underset{d_k}{\overset{d_0}{\rightrightarrows}} \vdots \; P_{n-1}(N)_{k-1} \ ,$$

$(g_{n-1})_{k-1}, (g_{n-1})_k$ and $(g_{n-1})_{k+1}$, we can determine $p_n(N)$, $(g_n)_{k-1}$, $(g_n)_k$, $(g_n)_{k+1}$, and the data in the sequence

$$P_n(N)_{k+1} \underset{d_{k+1}}{\overset{d_0}{\rightrightarrows}} \vdots \; P_n(N)_k \underset{d_k}{\overset{d_0}{\rightrightarrows}} \vdots \; P_n(N)_{k-1}$$

in

$$\begin{aligned}
O\Big(&\left[\left(\max_{i\in I'}|P_{n-1}(N)_i|\right) + n\binom{n+k}{k}\right] |p_n(N)|^{\binom{n+k}{k}} \\
&+ \left(\max_{i\in I'}|P_{n-1}(N)_i|\right)\left(\left(\max_{i\in I}|P_{n-1}(N)_i|\right) + (n+k)\right)\binom{n+k}{k} \\
&+ \max_{i\in I}(|N_i| + |P_{n-1}(N)_{i+1}|)^3\Big)
\end{aligned} \tag{6.16}$$

time, where $I = \{n-1, n, n+1\}$ and $I' = \{k-1, k, k+1\}$, assuming that $|p_n(N)| < \infty$.

Proof. Follows from the lemmas in this section, in addition to (5.28). □

6.3. **Complexity of determining** P_i, $i < n$**.** Now, let N be a complex where $|N|$ is simply connected, and $|\pi_i(|N|)| < \infty$ for all i. Let us work towards determining the complexity of determining $\pi_i(|N|)$ for $1 < i \leq n$.

Lemma 6.17. *If N is a simplicial set such that $|\pi_i(|N|)| < \infty$, then*

$$|P_n(N)_k| = O\left(\prod_{i\leq n} |\pi_i(|N|)|^{\binom{i+k}{k}}\right) \tag{6.18}$$

Proof. This follows inductively from (5.25) and (6.1). □

Theorem 6.19. *Given a finite complex N with finite homotopy groups $\pi_i(|N|)$, we can determine $\pi_i(|N|)$ for $1 < i \leq n$ in*

$$O\left(n^2\left[\left(|N_{\max}| + \prod_{j<n} |\pi_j(|N|)|^{\binom{j+n}{n}}\right)^3 + n\binom{2n}{n}\prod_{j<n}|\pi_j(|N|)|^{\binom{j+n}{n}}\right]\right) \tag{6.20}$$

time, where $N_{\max}$ is the level of the simplicial set N with maximal order out of levels 2 through level $n+2$.

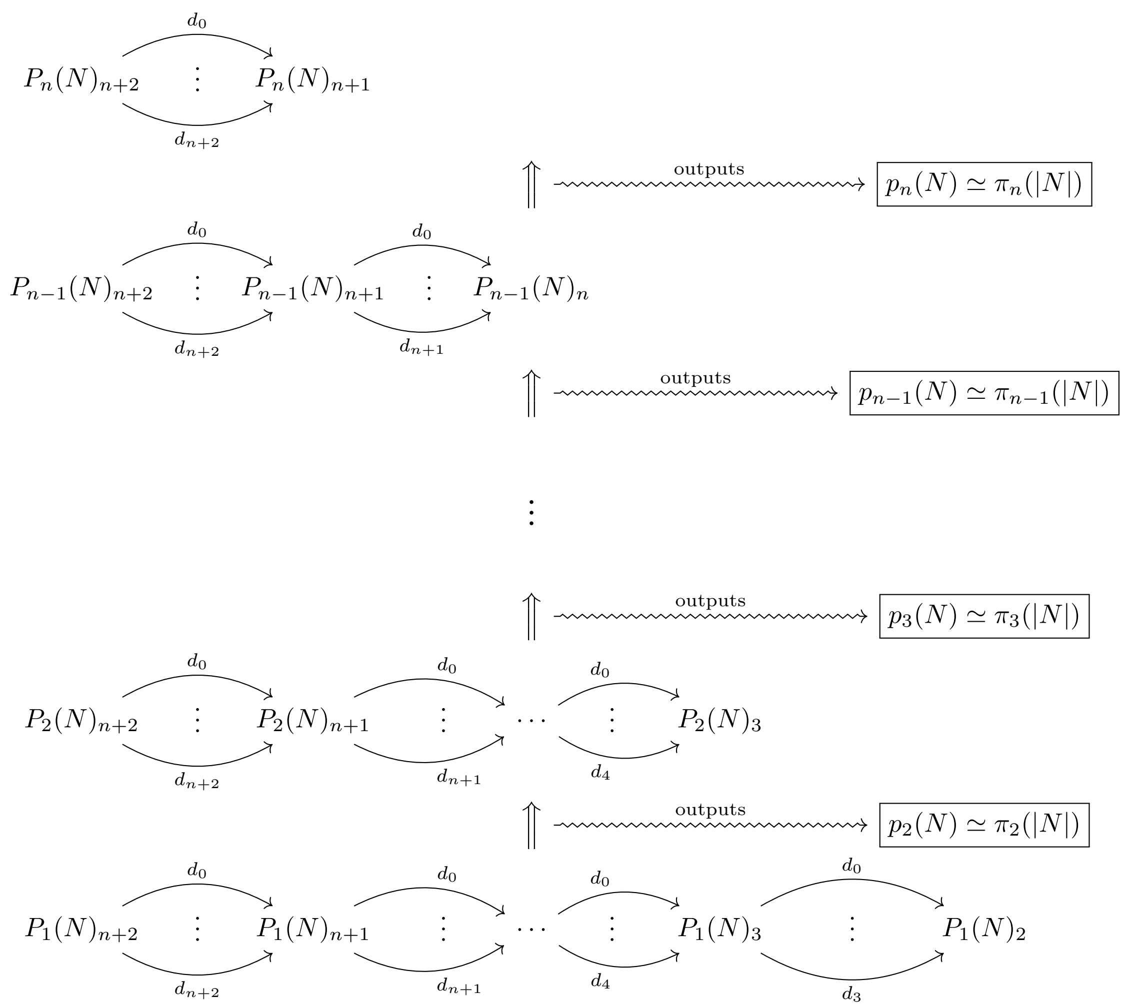

Figure 6.1.

Proof of theorem. We illustrate the necessary computations in Figure 6.1. For $i < n$, we must compute the necessary data (sets and face maps) of the simplicial set $P_i(N)$ from level $i+1$ to level $n+2$, in addition to the necessary auxiliary maps

g_i and the object of interest, $p_i(N)$. Using 6.15, 6.17, and the fact that $\binom{n+k}{k}$ is strictly increasing in k, we can compute this data for fixed i in

(6.21)
$$O\left((n-i)\left[\left(|N_{\max}|+\prod_{j<i}|\pi_j(|N|)|^{\binom{j+n}{n}}\right)^3+(i+n)\binom{i+n}{n}\prod_{j<i}|\pi_j(|N|)|^{\binom{j+n}{n}}\right]\right)$$

time. Summing over i and upper-bounding using $i \leq n$ gives the runtime above. □

7. Computation of Higher Homotopy Groups of Spheres

In the previous section, we described the iterated Postnikov construction, which provides us a systematic procedure for determining the higher homotopy groups $\pi_k(X)$ of a simply connected space X. However, in order to guarantee that this procedure, as described, is finitely computable, we require crucially that the groups $\pi_k(X)$ are finite. In this section, we first describe how Brown circumvents this requirement, and then apply it to analyze the complexity of determining the higher homotopy groups of spheres.

7.1. **q-Deformation Retracts and the Modified Iterated Postnikov Construction.** Brown begins by making a certain homotopical definition for simplicial sets.

Definition 7.1 (q-deformation retract)**.** Let N be a simplicial set, and let M be a subsimplicial set of N. M is a *q-deformation retract* of N if M satisfies the following conditions:

- $N^0 \subset M^0$.
- Given a simplicial set K and a subsimplicial set $L \subset K$ such that $\dim K \leq q$ and $\dim L < q$, if $f : K \to N$ is a simplicial map taking L into M, then there is a simplicial set $\overline{K}$ and a subsimplicial set $\overline{L} \subset \overline{K}$ such that $K \subset \overline{K}$ and $L \subset \overline{L}$, along with a simplicial map $\overline{f} : \overline{K} \to N$ such that
  - $\overline{f}$ takes $\overline{L}$ into M,
  - $\dim \overline{K} \leq q+1$ and $\dim \overline{L} \leq q$,
  - $\overline{f}|_K = f$, and
  - $(\overline{K}, K)$ and $(\overline{K}, \overline{L})$ are acyclic.

He then moves to describing a family of q-deformation retracts for the Postnikov construction.

Definition 7.2. Let π be a finitely generated abelian group, with decomposition such that $\pi = \bigoplus_{i=1}^{\ell} \pi_i \oplus \pi_f$, where the π_i are infinite cyclic groups and π_f is finite. Let $p_i : \pi \to \mathbb{Z}$ be the composition of the projection $\pi \to \pi_i$ and an isomorphism $\pi_i \to \mathbb{Z}$. We will call $\mathcal{P} = \{p_i\}_{i=1}^{\ell}$ a *projective decomposition* for π. Let $a = \{a_i\}_{i=1}^{\ell}$ be a collection where a_i is either a positive integer or ∞. We then let

$$\lambda(\mathcal{P}, a) = \{x \in \pi : |p_i(x)| < a_i \text{ for all } 1 \leq i \leq \ell\}, \tag{7.3}$$

where given a map $\varphi : G \to \mathbb{Z}$, we define

$$|\varphi| = \begin{cases} \max\{|\varphi(x)| : x \in G\} & \max\{|\varphi(x)| : x \in G\} \text{ exists} \\ \infty & \text{else} \end{cases} \tag{7.4}$$

Lemma 7.5 (Brown 7.6)**.** *Given a simplicial set N, abelian group π, and $A^n \in Z^n(N;\pi)$, $P(N,\pi,\lambda(\mathcal{P},a),A^n)$ is a q-deformation retract of $P(N,\pi,A^n)$ if $a_i \geq (q+1)(|p_i \circ A^n|+1)$ for all i.*

With this, we are finally prepared to describe the "modified iterated Postnikov construction" which allows us to compute homotopy groups in a finite amount of time.

Definition 7.6 (Modified Iterated Postnikov Construction)**.** Given a simplicial set N and integer q, we define a simplicial set $P_{n,q}(N)$ *which is finite in each dimension*, an abelian group $p_n(N)$, and simplicial map $g_{n,q} : N \to P_{n,q}(N)$ inductively as follows:

- $P_{1,q}(N) = P_1(N)$, $p_{1,q}(N) = p_1(N)$, and $g_{1,q} = g_1$.
- Towards induction, assume that $P_{n-1,q}(N)$, $p_{n-1,q}(N)$, and $g_{n-1,q}$ have been defined appropriately. Let $\hat{g}_{n-1,q}$ be the mapping cone of the chain map $C(N) \to C(P_{n-1,q}(N))$ induced by $g_{n-1,q}$. We define

$$p_{n,q}(N) := H_{n+1}(\hat{g}_{n-1,q}) \tag{7.7}$$

Moving on, we let $R^{n+1} \in Z^{n+1}(\hat{g}_{n-1,q}; p_{n,q}(N))$ be some extension of the canonical map from $Z_{n+1}(\hat{g}_{n-1,q})$ into $H_{n+1}(\hat{g}_{n-1,q}) = p_{n,q}(N)$. We use this to define the pair of maps;
  - $S^{n+1} \in Z^{n+1}(P_{n-1,q}(N); p_n(N))$ is defined by letting $S^{n+1}(s) = R^{n+1}(0,s)$, and
  - $T^n \in C^n(N; p_{n,q}(N))$ is defined by letting $T^n(t) = R^{n+1}(t,0)$

Next, take $\mathcal{P} = \{p_i\}_{i=1}^{\ell}$ to be a projective decomposition of $p_{n,q}(N)$.[2] For $1 \leq i \leq \ell$, let $a_i = \max\left\{(q+1)(|p_i \circ S^{n+1}|+1),\ |p_i \circ T^n|\right\}$, $a = \{a_i\}_{i=1}^{\ell}$, and $\lambda_{n,q} = \lambda(\mathcal{P},a)$. Then we define

$$P_{n,q}(N) := P(P_{n-1,q}(N), p_{n,q}(N), \lambda_{n,q}, S^{n+1}) \tag{7.8}$$

$$g_{n,q} := (g_{n-1,q}, T^n) \tag{7.9}$$

We note that the process is nearly identical to the iterated Postnikov construction described in the previous section, with the only added step being the computation of the $a = \{a_i\}$, and the use of the finite subset $\lambda(\mathcal{P},a)$ instead of $p_i(N) \simeq \pi_i(|N|)$ in the inductive step if the homotopy group is infinite. As a result, we have the following modification of 6.19.

Theorem 7.10. *Given a finite complex N, we can determine $\pi_i(|N|)$ for $1 < i \leq n$ in*

$$O\left(n^2\left[\left(|N_{\max}| + \prod_{j<n} |\pi_j^*(|N|)|^{\binom{j+n}{n}}\right)^3 + n\binom{2n}{n}\prod_{j<n}|\pi_j^*(|N|)|^{\binom{j+n}{n}}\right]\right) \tag{7.11}$$

time, where $N_{\max}$ is the level of the simplicial set N with maximal order out of levels 2 through level $n+2$, and

$$\pi_i^*(|N|) := \begin{cases} \pi_i(|N|) & |\pi_i(|N|)| < \infty \\ \lambda(\mathcal{P},a) & \textit{else} \end{cases} \tag{7.12}$$

[2]The existence of such a decomposition is guaranteed by the finiteness of N, and the finiteness in each dimension of $P_{n-1,q}(N)$.

The author has not yet been able to determine whether the complexity of this process can in general be bounded; there may be families of spaces where the determining a projective decomposition can take arbitrarily long. We instead analyze the particular example of using this algorithm to determine the higher homotopy groups of odd-dimensional spheres.

7.2. **Modified Iterated Postnikov Construction on Odd-Dimensional Spheres.**

We begin with a discussion of various results regarding the growth of the homotopy groups of spheres. First, a basic result found in [Hat01]:

Theorem 7.13. *$\pi_i(S^n) = 0$ if $i < n$.*

Next, a consequence of the Hurewicz theorem;

Theorem 7.14. *$\pi_n(S^n) \simeq \mathbb{Z}$*

It is the groups $\pi_i(S^n)$, $i > n$ where things get complicated. First, a couple of results from Serre's seminal 1951 paper [Ser51], which tell us that most of these groups are finite, and which gives us a bound on the primes p where $\pi_i(S^n)$ has nontrivial p-torsion.

Theorem 7.15. *$\pi_i(S^n)$ $(i > n)$ are finite with the exception of $\pi_{4n-1}(S^{2n})$, which is the direct sum of $\mathbb{Z}$ and a finite group.*

Theorem 7.16. *The p-torsion of $\pi_i(S^n)$, $(n \geq 3, p$ prime$)$, is zero if $i < n+2p-3$, and the p-primary component of $\pi_{n+2p-3}(S^n)$ is a cyclic group of order p^j, $j \geq 1$.*

Next, we have some more classical results, from James ([Jam57]) and Cohen-Moore-Neisendorfer ([CMN79]), giving us a bound on the order of the p-torsion elements at primes p.

Theorem 7.17. *The homotopy groups of S^{2n+1} $(n \geq 1)$ contain no elements of order 2^{2n+1}. That is, all $\mathbb{Z}_{2^j}$ summands are with $j \leq 2n$.*

Theorem 7.18. *The homotopy groups of S^{2n+1} contain no elements of order p^{n+1} for an odd prime p. That is, all $\mathbb{Z}_{p^j}$ summands are with $j \leq n$.*

Finally, a result which lets us bound on the number of p-torsion summands in these homotopy groups, from [HW19].

Theorem 7.19. *Let N be a simply connected finite complex. Then the p-torsion of $\pi_*(|N|)$ has at most exponential growth, i.e.,*

$$\limsup_k \frac{\ln T_k}{k} < \infty, \tag{7.20}$$

where T_k is the number of $\mathbb{Z}_{p^r}$-summands in $\bigoplus_{i \leq k} \pi_i(|N|)$, $r > 1$.

Now, let us begin combining these results.

Proposition 7.21. *For any prime p, the order of the p-torsion of $\pi_*(S^{2n+1})$ has at most exponential growth, i.e.,*

$$\limsup_k \frac{\ln \left| \bigoplus_{i \leq k} \pi_i(S^{2n+1})_p \right|}{k} < \infty, \tag{7.22}$$

or equivalently,

$$\left| \bigoplus_{i \leq k} \pi_i(S^{2n+1})_p \right| = O\left(e^k\right) \tag{7.23}$$

where $\pi_i(S^{2n+1})_p$ denotes the p-torsion of $\pi_i(S^{2n+1})$.

Proof of proposition. Follows from (7.17), (7.18), and (7.19). □

Proposition 7.24. *For $n \geq 1$ and $i > 2n+1$,*

$$|\pi_i(S^{2n+1})| = O\left(e^{i^2}\right) \tag{7.25}$$

Proof of proposition. (7.16) tells us that the only primes p for which the p-torsion of $\pi_i(S^{2n+1})$ can be nonzero are $p < \frac{i-2n+2}{2}$. In particular, the number of primes p with nonzero p-torsion is $O(i)$. (7.15) tells us that $\pi_i(S^{2n+1}) = \bigoplus_p \pi_i(S^{2n+1})_p$, and so (7.21) gives us the bound above. □

With this, we obtain the following bound on determining the homotopy groups of odd spheres:

Theorem 7.26. *Using Brown's algorithm, we can determine the homotopy groups $\pi_i(S^{2n+1})$, $i < m$ for some $m > 2n+1$, in*

$$O\left(m^2\left[\left(\binom{2n+m}{2n} + e^{m^3\binom{2m}{m}}\right)^3 + m\binom{2m}{m}e^{m^3\binom{2m}{m}}\right]\right) \tag{7.27}$$

time.

Proof of theorem. Recall that the geometric realization of $\partial\Delta^{k+1}$ is homeomorphic to the k-sphere S^k. So in order to use Brown's algorithm to compute $\pi_i(S^{2n+1})$, we can use the simplicial set $\partial\Delta^{2n+2}$, contained within $\Delta^{2n+2} = \mathrm{Hom}(-, [2n+2])$. We then use (7.10) to craft our bound. First, given $\pi_{2n+1}(S^{2n+1}) \simeq \mathbb{Z}$, we have that $\pi^*_{2n+1}(S^{2n+1}) = 0$. Further, we have that $|\partial\Delta^{2n+2}_{\max}| \leq |\Delta^{2n+2}_{\max}| = O\left(\binom{2n+m}{2n}\right)$ and so using 7.24, we obtain a bound of

$$O\left(m^2\left[\left(\binom{2n+m}{2n} + \prod_{2n+1<j<m} e^{j^2\binom{j+m}{m}}\right)^3 + m\binom{2m}{m}\prod_{2n+1<j<m} e^{j^2\binom{j+m}{m}}\right]\right) \tag{7.28}$$

Using monotonicity $\left(e^{j^2\binom{j+m}{m}} \leq e^{m^2\binom{2m}{m}}\right)$ gives us the bound above. □

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