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Associated varieties for real reductive groups

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In fond memory of our teacher and friend, Bert Kostant.

Abstract: We describe an algorithm for computing the associated variety of any finite length Harish-Chandra module for a real reductive group $G(\mathbb{R})$. The algorithm has been implemented in the `atlas` software package.

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A great guiding principles of infinite-dimensional representation theory is the *method of coadjoint orbits* of Alexandre Kirillov and Bertram Kostant. It says that there should be a close relationship

(0.1a)
$$\begin{array}{ccl} i\mathfrak{g}^*_{\mathbb{R}}/G_{\mathbb{R}} & = & \text{orbits of a real Lie group on the imaginary dual of its Lie algebra} \\ \big\downarrow \Pi & & \\ \left(\widehat{G_{\mathbb{R}}}\right)_{\text{unitary}} & = & \text{irreducible unitary representations.} \end{array}$$

Here

$$\mathcal{O}_{i\mathbb{R}} \mapsto \Pi(\mathcal{O}_{i\mathbb{R}})$$

is informal notation for the desired construction attaching a unitary representation to a coadjoint orbit. This map Π is not intended to be precisely defined or even definable: in the cases where such a correspondence is known, the domain of Π consists of just *certain* coadjoint orbits (satisfying integrality requirements), and endowed with some additional structure (something like local systems). We introduce the name Π just to talk about the problem.

We will be concerned here with the case of real reductive groups. For the remainder of this introduction, we therefore assume

(0.1b)
$$\begin{aligned} G &= \text{complex connected reductive algebraic group defined over } \mathbb{R}, \\ G(\mathbb{R}) &= \text{group of real points of } G. \end{aligned}$$

The status of the orbit method for real reductive groups is discussed in some detail for example in [22]. There it is explained that

(0.1c) the construction of a map Π (from orbits to representations) reduces to the case of *nilpotent* coadjoint orbits.

This nilpotent case remains open in general. We write

(0.1d)
$$\mathcal{N}^*_{i\mathbb{R}} = \text{nilpotent elements in } i\mathfrak{g}(\mathbb{R})^*.$$

(A precise definition appears in Section 1.) We write informally

(0.1e)
$$\widehat{G(\mathbb{R})}_{\text{unip}} = \text{representations corresponding to nilpotent coadjoint orbits,}$$

the *unipotent representations*; this is not a definition, because the Kirillov-Kostant orbit correspondence Π has not been defined.

Harish-Chandra found that the study of irreducible unitary representations could proceed more smoothly inside the larger set

$$\widehat{G(\mathbb{R})} \supset \widehat{G(\mathbb{R})}_{\text{unitary}} \tag{0.1f}$$

of irreducible *quasisimple* representations. These are the irreducible objects of the category introduced in (0.2a) below. (These are irreducible topological representations on nice topological vector spaces. "Quasisimple" means that the center of the enveloping algebra is required to act by scalars, as Schur's lemma (not available in this topological setting) suggests that it should.)

The present paper is concerned with how to tell whether a proposed map Π is reasonable. The idea comes from [13], [6], and [21]. To each coadjoint orbit we can attach an *asymptotic cone*, a closed $G_{\mathbb{R}}$-invariant cone (Definition 2.3

$$\mathcal{O}_{i\mathbb{R}} \in i\mathfrak{g}(\mathbb{R})^*/G(\mathbb{R}) \longrightarrow \mathrm{Cone}_{\mathbb{R}}(\mathcal{O}_{i\mathbb{R}}) \subset \mathcal{N}^*_{i\mathbb{R}}. \tag{0.1g}$$

An easy but important property is that the asymptotic cone of a nilpotent orbit is just its closure:

$$\mathrm{Cone}_{\mathbb{R}}(\mathcal{O}_{i\mathbb{R}}) = \overline{\mathcal{O}_{i\mathbb{R}}}, \qquad \mathcal{O}_{i\mathbb{R}} \in \mathcal{N}^*_{i\mathbb{R}}/G(\mathbb{R}). \tag{0.1h}$$

In a parallel way, to each irreducible quasisimple representation [13] and [6] attached a *wavefront set*, a closed cone

$$\pi \in \widehat{G(\mathbb{R})} \longrightarrow \mathrm{WF}_{\mathbb{R}}(\pi) \subset \mathcal{N}^*_{i\mathbb{R}}/G(\mathbb{R}). \tag{0.1i}$$

(The definition of the wavefront set comes from distribution theory. Because we are going to prove theorems in an algebraic setting, we will not recall the definition.) One of the desiderata of the orbit method is that the asymptotic cone and wavefront set constructions should be compatible with the proposed map Π of (0.1a): that if $\mathcal{O}_{i\mathbb{R}}$ is a coadjoint orbit, then

$$\mathrm{WF}_{\mathbb{R}}(\Pi(\mathcal{O}_{i\mathbb{R}})) \stackrel{?}{=} \mathrm{Cone}_{\mathbb{R}}(\mathcal{O}_{i\mathbb{R}}). \tag{0.1j}$$

When $\mathcal{O}_{i\mathbb{R}}$ is *nilpotent*, (0.1h) shows that this desideratum simplifies to

$$\mathrm{WF}_{\mathbb{R}}(\Pi(\mathcal{O}_{i\mathbb{R}})) \stackrel{?}{=} \overline{\mathcal{O}_{i\mathbb{R}}} \qquad (\mathcal{O}_{i\mathbb{R}} \text{ nilpotent}). \tag{0.1k}$$

Our motivation (not achieved) is the construction of a Kirillov-Kostant orbit-to-representation correspondence Π as in (0.1a). According to (0.1c), it is enough to construct $\Pi(\mathcal{O}_{i\mathbb{R}})$ for each *nilpotent* orbit $\mathcal{O}_{i\mathbb{R}}$. A common method to do this has been to construct a candidate representation π, and then to test whether the requirement (0.1k) is satisfied. That is,

$$\text{if } \mathcal{O}_{i\mathbb{R}} \text{ is nilpotent, candidates for } \Pi(\mathcal{O}_{i\mathbb{R}}) \text{ must satisfy } \mathrm{WF}_{\mathbb{R}}(\pi) = \overline{\mathcal{O}_{i\mathbb{R}}}. \tag{0.1l}$$

In order to use this idea to guide the construction of Π, we therefore need to know how to

$$\text{compute the wavefront set of any quasisimple irreducible representation.} \tag{0.1m}$$

That is the problem solved in this paper.

Everything so far has been phrased in terms of real nilpotent coadjoint orbits, but all of the ideas to be used come from complex algebraic geometry. In Section 1 we recall fundamental results of Kostant-Sekiguchi [17] and Schmid-Vilonen [16] allowing a reformulation of (0.1m) in complex-algebraic terms.

We do not know even how properly to formulate the main results except in this complex-algebraic language, so a proper summary of them will appear only in Section 6. For the moment we will continue as if it were possible to make a real-groups formulation of the solution to (0.1m). The reader can take this as an outline of an interesting problem: to make precise sense of the statements in the rest of the introduction.

We continue with the assumption (0.1b) that $G(\mathbb{R})$ is a real reductive algebraic group. Write

$$\mathcal{F}_{\mathrm{mod}}(G(\mathbb{R})) = \begin{array}{l}\mathfrak{Z}(\mathfrak{g})\text{-finite finite length smooth Fréchet}\\ \text{representations of moderate growth}\end{array} \tag{0.2a}$$

(see [25, Chapter 11.6]). Casselman and Wallach proved that this is a nice category; the irreducible objects are precisely the irreducible quasisimple representations $\widehat{G(\mathbb{R})}$, so the Grothendieck group of the category is

$$K_0(\mathcal{F}_{\mathrm{mod}}(G(\mathbb{R}))) = \mathbb{Z} \cdot \widehat{G(\mathbb{R})}, \tag{0.2b}$$

a free abelian group with basis the irreducible quasisimple representations.

We do not know a good notion of equivariant K-theory for real algebraic groups. But such a notion ought to exist; and there ought to be an "associated graded" map

$$\mathrm{gr}_{\mathbb{R}} \colon K_0(\mathcal{F}_{\mathrm{mod}}(G(\mathbb{R}))) \stackrel{?}{\to} K^{G(\mathbb{R})}(\mathcal{N}^*_{i\mathbb{R}}). \tag{0.2c}$$

Each element of $K^{G(\mathbb{R})}(\mathcal{N}^*_{i\mathbb{R}})$ should have a well-defined "support," which should be a closed $G(\mathbb{R})$-invariant subset of $\mathcal{N}^*_{i\mathbb{R}}$. In the case of a quasisimple representation π of finite length, this support should be the wavefront set of (0.1i):

$$\operatorname{supp}_{\mathbb{R}} \operatorname{gr}_{\mathbb{R}}([\pi]) \stackrel{?}{=} \mathrm{WF}_{\mathbb{R}}(\pi). \tag{0.2d}$$

The question mark is included because the left side is for the moment undefined; in the algebraic geometry translation of Section 6, this equality will become meaningful and true. The problem (0.1m) becomes

$$\text{compute explicitly the map } \operatorname{supp}_{\mathbb{R}} \circ \operatorname{gr}_{\mathbb{R}}. \tag{0.2e}$$

Evidently this can be done in two stages: to compute explicitly the map $\operatorname{gr}_{\mathbb{R}}$, and then to compute explicitly the map $\operatorname{supp}_{\mathbb{R}}$.

Here is the first step. Just as in the case of highest weight representations, each irreducible quasisimple representation π is described by the Langlands classification as the unique irreducible quotient of a "standard representation." Standard representations have very concrete parameters

$$\Gamma = (\Lambda, \nu), \qquad \Gamma \in \mathcal{P}_{\mathrm{L}}(G(\mathbb{R})) \tag{0.2f}$$

which we will explain in Section 9 (see in particular (9.5). For the moment, the main points are that

$$\Lambda \in \mathcal{P}_{\mathrm{disc}}(G(\mathbb{R})) \tag{0.2g}$$

runs over a countable *discrete* set, and

$$\nu \in \mathfrak{a}^*(\Lambda) \tag{0.2h}$$

runs over a complex vector space associated to the discrete parameter Λ. Attached to each parameter Γ we have

$$I(\Gamma) \twoheadrightarrow J(\Gamma), \tag{0.2i}$$

a standard representation and its unique irreducible quotient.

"Proposition" 0.3. *Suppose we are in the setting of* (0.2)*.*

1. *The irreducible modules*

$$\{J(\Gamma) \mid \Gamma \in \mathcal{P}(G(\mathbb{R}))\}$$

are a $\mathbb{Z}$ basis of the Grothendieck group $K_0(\mathcal{F}_{\mathrm{mod}}(G(\mathbb{R})))$.

2. *The standard modules*
$$\{I(\Gamma) \mid \Gamma \in \mathcal{P}(G(\mathbb{R}))\}$$
are a $\mathbb{Z}$ basis of $K_0(\mathcal{F}_{\mathrm{mod}}(G(\mathbb{R})))$.
3. *The change of basis matrix*
$$J(\Gamma) = \sum_{\Xi} M(\Xi, \Gamma) I(\Xi)$$
is computed by Kazhdan-Lusztig theory ([15]*).*
4. *The image*
$$\mathrm{gr}_{\mathbb{R}}(I(\Lambda, \nu)) \in K^{G(\mathbb{R})}(\mathcal{N}^*_{i\mathbb{R}})$$
is independent of the continuous parameter $\nu \in \mathfrak{a}(\Lambda)^*$.
5. *The classes*
$$\{\mathrm{gr}_{\mathbb{R}}(I(\Lambda, 0)) \mid \Lambda \in \mathcal{P}_{\mathrm{disc}}(G(\mathbb{R}))\}$$
are a $\mathbb{Z}$-basis of the equivariant K-theory $K^{G(\mathbb{R})}(\mathcal{N}^*_{i\mathbb{R}})$.

The quotation marks are around the proposition for two reasons. First, we do not have a definition of $G(\mathbb{R})$-equivariant K-theory; we will actually prove algebraic geometry analogues of (4) and (5) (Corollary 9.9 and Proposition 9.11). Second, the description of the Langlands classification above is slightly imprecise; the corrected statement is just as concrete and precise, but slightly more complicated.

One way to think about (4) is that K-theory is a topological notion, which ought to be invariant under homotopy. Varying the continuous parameter in a standard representation is a continuous deformation of the representations, and so does not change the class in K-theory.

This proposition is a complete computation of $\mathrm{gr}_{\mathbb{R}}$: it provides $\mathbb{Z}$ bases for the range and domain, and says that the map is given by identifying certain continuous families of basis vectors. Furthermore it explains how to write each irreducible module in the specified basis.

We turn next to the explicit computation of $\mathrm{supp}_{\mathbb{R}}$. Again the key point is a change of basis: this time from the representation-theoretic basis of equivariant K-theory given by Proposition 0.3(5) to one related to the geometry of $\mathcal{N}^*_{i\mathbb{R}}$.

Suppose that $H(\mathbb{R})$ is any real algebraic subgroup of $G(\mathbb{R})$. Assuming that there is a reasonable notion of equivariant K-theory for real algebraic groups, it ought to be true that

$$K^{G(\mathbb{R})}(G(\mathbb{R})/H(\mathbb{R})) \simeq K^{H(\mathbb{R})}(\text{point}), \tag{0.4}$$

The right side in turn should be a free abelian group with natural basis indexed by the irreducible representations of a maximal compact subgroup $H_K(\mathbb{R})$. Combining these facts with the notion of support in equivariant K-theory, we get

"Proposition" 0.5. *Suppose Y is a closed $G(\mathbb{R})$-invariant subset of $\mathcal{N}^*_{i\mathbb{R}}$ (a union of orbit closures). Write*

$$\{Y_1, \cdots, Y_r\}, \quad Y_j \simeq G(\mathbb{R})/H_j(\mathbb{R})$$

for the open orbits in Y, and

$$\partial Y = Y - \bigcup_j Y_j$$

for their closed complement. Write finally

$$K_Y^{G(\mathbb{R})}(\mathcal{N}^*_{i\mathbb{R}})$$

for the subspace of classes supported on Y, and ∂Y for the boundary of Y (the complement of the open orbits in Y).

1. *There is a natural short exact sequence*

$$0 \to K_{\partial Y}^{G(\mathbb{R})}(\mathcal{N}^*_{i\mathbb{R}}) \to K_Y^{G(\mathbb{R})}(\mathcal{N}^*_{i\mathbb{R}}) \to \sum_j K^{G(\mathbb{R})}(Y_j) \to 0.$$

2. *There are natural isomorphisms*

$$K^{G(\mathbb{R})}(Y_j) \simeq K^{H_j(\mathbb{R})}(\text{point}) \simeq \mathbb{Z} \cdot \widehat{H_{j,K}(\mathbb{R})},$$

   a free abelian group with basis indexed by irreducible representations of a maximal compact subgroup $H_{j,K}(\mathbb{R}) \subset H_j(\mathbb{R})$.
3. *The equivariant K-theory space $K^{G(\mathbb{R})}(\mathcal{N}^*_{i\mathbb{R}})$ has a $\mathbb{Z}$-basis*

$$\{e(\mathcal{O}_{i\mathbb{R}}, \tau)\}$$

   indexed by pairs $(\mathcal{O}_{i\mathbb{R}}, \tau)$, with

$$\mathcal{O}_{i\mathbb{R}} \simeq G(\mathbb{R})/H(\mathbb{R}) \subset \mathcal{N}^*_{i\mathbb{R}}$$

a nilpotent coadjoint orbit, and $\tau \in \widehat{H_K(\mathbb{R})}$ *an irreducible representation of a maximal compact subgroup of* $H(\mathbb{R})$.

4. *The basis vector* $e(\mathcal{O}_{i\mathbb{R}}, \tau)$ *is supported on* $\overline{\mathcal{O}}_{i\mathbb{R}}$*, and has a well-defined image in*
$$K^{Y}_{G(\mathbb{R})}/K^{\partial Y}_{G(\mathbb{R})};$$
that is, it is unique up to a combination of basis vectors $e(\mathcal{O}'_{i\mathbb{R}}, \tau')$*, with*
$$\mathcal{O}'_{i\mathbb{R}} \subset \partial\overline{\mathcal{O}_{\mathbb{R}}}.$$
5. *Each subspace*
$$\mathrm{span}\left(\{e(\mathcal{O}'_{i\mathbb{R}}, \tau') \mid \mathcal{O}'_{i\mathbb{R}} \subset \overline{\mathcal{O}_{\mathbb{R}}}\}\right)$$
has an explicitly computable spanning set $S(\mathcal{O}_{i\mathbb{R}})$ *expressed in the basis of Proposition 0.3(5).*
6. *Suppose that* $\sigma \in K^{G(\mathbb{R})}(\mathcal{N}^*_{i\mathbb{R}})$ *is a class in equivariant* K*-theory; write*
$$\sigma = \sum_{\mathcal{O}_{i\mathbb{R}}, \tau} m_{\mathcal{O}_{i\mathbb{R}}, \tau} e(\mathcal{O}_{i\mathbb{R}}, \tau).$$
Then
$$\mathrm{supp}_{\mathbb{R}}(\sigma) = \bigcup_{\substack{\mathcal{O}_{i\mathbb{R}} \subset \mathcal{N}^*_{i\mathbb{R}}, \\ \textit{some } m_{\mathcal{O}_{i\mathbb{R}}, \tau} \neq 0}} \overline{\mathcal{O}_{i\mathbb{R}}}.$$
7. *The open orbits* $\mathcal{O}_{i\mathbb{R},j}$ *in* $\mathrm{supp}_{\mathbb{R}}(\sigma)$ *are the minimal ones so that*
$$\sigma \in \sum_j S(\mathcal{O}_{i\mathbb{R},j}).$$

The quotation marks are around this proposition again because we do not know a good definition of $G(\mathbb{R})$-equivariant K-theory, much less whether it has these nice properties; we will actually prove versions in algebraic geometry (Theorem 3.5 and Corollary 11.3). The "explicitly computable" assertion is explained in Algorithm 11.4.

Part (7) of the proposition provides a (linear algebra) computation of the support from the expression of σ in the basis for K-theory of Proposition 0.3(5): we must decide whether a vector of integers (a Kazhdan-Lusztig character formula, computed using deep results about perverse sheaves) is in the span of other vectors of integers (the spanning sets $S(\mathcal{O}_{i\mathbb{R}})$, computed by much more elementary geometry in part (5)).

In case $G(\mathbb{R})$ is a complex group regarded as a real group, these two propositions (and therefore the algorithm for (0.1m)) are closely related to a conjecture of Lusztig, proved by Bezrukavnikov in [7], establishing a bijection between some objects on nilpotent orbits (related to equivariant K-theory) and dominant weights. These ideas of Lusztig and Bezrukavnikov, and especially Achar's work in [1], guided all of our work. This may be clearest in Section 7, which explains (still in the complex case) Achar's ideas for computing the spanning set of Proposition 0.5(5).

Section 8 will explains how to solve (0.1m) for a complex reductive algebraic group.

Section 9 explains the general formalism for extending matters to real groups. Section 10 has some information about the geometry of cohomological induction, needed to relate the geometry of nilpotent orbits to the Langlands classification. This is used in Section 11 to complete the proof of Proposition 0.5 for real groups.

The wavefront set of (0.1i) has a refinement, the *wavefront cycle*:

$$\mathcal{WF}_{\mathbb{R}}(\pi) = \sum_{\substack{\mathcal{O}_{i\mathbb{R}} \text{ open} \\ \text{in } \mathrm{WF}_{\mathbb{R}}(\pi)}} \mu_{\mathcal{O}_{i\mathbb{R}}}(\pi)\mathcal{O}_{i\mathbb{R}}. \tag{0.6a}$$

Here the coefficient $\mu_{\mathcal{O}_{i\mathbb{R}}}(\pi)$ is a genuine virtual representation of a maximal compact subgroup of the isotropy group $G(\mathbb{R})_y$ of a point $y \in \mathcal{O}_{i\mathbb{R}}$. In the formalism explained in Proposition 0.5, this means that there should be a natural definition

$$\mu_{\mathcal{O}_{i\mathbb{R}}}(\pi) \in K^{G(\mathbb{R})}(\mathcal{O}_{i\mathbb{R}}); \tag{0.6b}$$

but we will actually use an algebraic geometry definition (Definition 4.2). In the setting of Proposition 0.5(6), the coefficient $\mu_{\mathcal{O}_{i\mathbb{R}}}(\pi)$ is

$$\mu_{\mathcal{O}_{i\mathbb{R}}}(\pi) = \sum m_{\mathcal{O}_{i\mathbb{R}},\tau}(\pi)\tau; \tag{0.6c}$$

so *the wavefront cycle can be computed from knowledge of the basis vectors* $e(\mathcal{O}_{i\mathbb{R}}, \tau)$. But what we actually know how to compute, as explained in Proposition 0.5(5), is not these individual basis vectors but rather the span of all those attached to a single $\mathcal{O}_{i\mathbb{R}}$. For this reason we cannot compute the full wavefront cycle.

There is a weaker invariant, the *weak wavefront cycle*:

$$\mathcal{WF}_{\mathrm{weak},\mathbb{R}}(\pi) = \sum_{\substack{\mathcal{O}_{i\mathbb{R}}\ \text{open} \\ \text{in } \mathrm{WF}_{\mathbb{R}}(\pi)}} m_{\mathcal{O}_{i\mathbb{R}}}(\pi)\mathcal{O}_{i\mathbb{R}}, \qquad m_{\mathcal{O}_{i\mathbb{R}}}(\pi) = \dim \mu_{\mathcal{O}_{i\mathbb{R}}}(\pi). \tag{0.6d}$$

Here the coefficient $m_{\mathcal{O}_{i\mathbb{R}}}(\pi)$ is just a positive integer instead of a compact group representation. The algorithm computing the spanning set $S(\mathcal{O}_{i\mathbb{R}})$ computes the multiplicity m_s for each of its spanning vectors s; so the algorithm of Proposition 0.5 actually computes the weak wavefront cycle as well as the wavefront set for any finite length representation π.

1. Kostant-Sekiguchi correspondence

We work in the setting (0.1b), with $G(\mathbb{R})$ the group of real points of a complex connected reductive algebraic group G. Write

$$\sigma_{\mathbb{R}}\colon G \to G, \qquad G^{\sigma_{\mathbb{R}}} = G(\mathbb{R}) \tag{1.1a}$$

for the Galois action. As usual we fix also a *compact* real form σ_0 of G, so that

$$\sigma_{\mathbb{R}}\sigma_0 = \sigma_0\sigma_{\mathbb{R}} =_{\mathrm{def}} \theta\colon G \to G \tag{1.1b}$$

is an (algebraic) involutive automorphism of G, the *Cartan involution*. The group

$$K =_{\mathrm{def}} G^{\theta}, \qquad K(\mathbb{R}) = K \cap G(\mathbb{R}) \tag{1.1c}$$

is a (possibly disconnected) complex reductive algebraic group, and $K(\mathbb{R})$ is a compact real form. What Cartan showed is that $K(\mathbb{R})$ is a maximal compact subgroup of $G(\mathbb{R})$. We write

$$\mathfrak{g}(\mathbb{R}) = \mathrm{Lie}(G(\mathbb{R})), \qquad \mathfrak{g} = \mathfrak{g}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{Lie}(G), \tag{1.1d}$$

and use parallel notation for other algebraic groups. The very familiar decomposition

$$\mathfrak{g} = \mathfrak{g}(\mathbb{R}) + i\mathfrak{g}(\mathbb{R}) \tag{1.1e}$$

is the $+1$ and -1 eigenspaces of $\sigma_{\mathbb{R}}$. The analogue for θ is the *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{s}, \qquad \mathfrak{s} = \mathfrak{g}^{-\theta}. \tag{1.1f}$$

In this setting, we can define

$$\mathcal{F}(\mathfrak{g}, K) = \text{finite length } (\mathfrak{g}, K)\text{-modules} \tag{1.1g}$$

([20]). An *invariant Hermitian form* on a $(\mathfrak{g}, K)$-module X is a Hermitian bilinear form $\langle , \rangle$ on X satisfying

$$\begin{aligned} \langle k \cdot x, y\rangle &= \langle x, \sigma_{\mathbb{R}}(k^{-1})y\rangle \qquad (x, y \in X, k \in K)\\ \langle Z \cdot x, y\rangle &= \langle x, \sigma_{\mathbb{R}}(-Z)y\rangle \qquad (x, y \in X, Z \in \mathfrak{g}). \end{aligned} \tag{1.1h}$$

Harish-Chandra showed that many questions about functional analysis and representations of $G(\mathbb{R})$ on Hilbert spaces could be reduced to algebraic questions about $(\mathfrak{g}, K)$-modules. Here are some of his main results, as sharpened by Casselman and Wallach in [25, 11.6.8]. We will use Proposition 1.2(2) to identify the objects of our ultimate interest (irreducible unitary representations) with something easier (irreducible $(\mathfrak{g}, K)$-modules).

Proposition 1.2. *Suppose we are in the setting* (0.2a) *and* (1.1)*.*

1. *The functor*

$$V \mapsto V_{K(\mathbb{R})} =_{\text{def}} \{v \in V \mid \dim \operatorname{span}\langle k \cdot v \mid k \in K(\mathbb{R})\rangle < \infty\}$$

   is an equivalence of categories from $\mathcal{F}_{\text{mod}}(G(\mathbb{R}))$ to $\mathcal{F}(\mathfrak{g}, K)$. (Here we implicitly extend the differentiated action of $\mathfrak{g}(\mathbb{R})$ on V to the complexification $\mathfrak{g}$, and the locally finite representation of the compact group $K(\mathbb{R})$ to an algebraic representation of its complexification K.) In particular, the set $\widehat{G(\mathbb{R})}$ of irreducible quasisimple smooth Fréchet representations of moderate growth is naturally identified with the set of irreducible $(\mathfrak{g}, K)$-modules.
2. *If $(\pi, \mathcal{H})$ is a unitary representation of G of finite length, then*

$$\mathcal{H}^\infty = \{v \in \mathcal{H} \mid G \to \mathcal{H},\ g \mapsto \pi(g)v \textit{ is smooth}\}$$

   is a $\mathfrak{Z}(\mathfrak{g})$-finite finite length smooth Fréchet representation of moderate growth. This functor defines an inclusion

$$\widehat{G(\mathbb{R})}_{\text{unitary}} \subset \widehat{G(\mathbb{R})}.$$

3. *The image of the functor*

$$\mathcal{H} \mapsto \mathcal{H}^\infty_{K(\mathbb{R})}$$

(from finite length unitary representations to $\mathcal{F}(\mathfrak{g}, K)$) consists precisely of those $(\mathfrak{g}, K)$-modules X admitting a positive definite invariant Hermitian form.

We now describe the geometry that we will use to make geometric invariants of finite-length representations.

The *complex nilpotent cone* consists of elements of $\mathfrak{g}^*$ whose orbits are weak (complex) cones (Definition 2.5):

$$\mathcal{N}^* = \{\xi \in \mathfrak{g}^* \mid \mathbb{C}^\times \cdot \xi \subset G \cdot \xi\}. \tag{1.3a}$$

The *imaginary nilpotent cone* consists of elements of $i\mathfrak{g}(\mathbb{R})^*$ whose orbits are (positive real) cones:

$$\begin{aligned} \mathcal{N}^*_{i\mathbb{R}} &= \mathcal{N}^* \cap i\mathfrak{g}(\mathbb{R})^* \\ &= \{-1 \text{ eigenspace of } \sigma_{\mathbb{R}} \text{ on } \mathcal{N}^*\} \\ &= \{i\xi \in i\mathfrak{g}(\mathbb{R})^* \mid \mathbb{R}^\times_+ \cdot i\xi \subset G(\mathbb{R}) \cdot i\xi\}. \end{aligned} \tag{1.3b}$$

It is classical that G acts on $\mathcal{N}^*$ with finitely many orbits; consequently $G(\mathbb{R})$ acts on $\mathcal{N}^*_{i\mathbb{R}}$ with finitely many orbits.

The *K-nilpotent cone* is

$$\begin{aligned} \mathcal{N}^*_\theta &= \mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^* \\ &= \{-1 \text{ eigenspace of } \theta \text{ on } \mathcal{N}^*\} \\ &= \{\xi \in \mathfrak{s}^* \mid \mathbb{C}^\times \cdot \xi \subset K \cdot \xi\}. \end{aligned} \tag{1.3c}$$

Kostant and Rallis proved in [14] that K acts on $\mathcal{N}^*_\theta$ with finitely many orbits.

We wish now to describe the Kostant-Sekiguchi relationship between $\mathcal{N}^*_{i\mathbb{R}}$ and $\mathcal{N}^*_\theta$. It is a gap in our understanding that there is no really satisfactory description of this relationship in terms of orbits on $\mathfrak{g}^*$; rather we need to use an identification of $\mathfrak{g}^*$ with $\mathfrak{g}$. Our reductive algebraic group G may always be realized as a group of matrices, in a way respecting the real form and the Cartan involution:

$$G \subset GL(n, \mathbb{C}), \quad \sigma_{\mathbb{R}}(g) = \overline{g}, \quad \theta(g) = {}^t g^{-1} \quad (g \in G). \tag{1.4a}$$

This provides first of all inclusions

$$\begin{aligned} \mathfrak{g} &\subset \mathfrak{gl}(n, \mathbb{C}), \\ \mathfrak{k}(\mathbb{R}) &\subset \text{real skew-symmetric matrices} \\ \mathfrak{s} &\subset \text{complex symmetric matrices} \end{aligned} \tag{1.4b}$$

(and others of a similar nature) and then an invariant bilinear form on $\mathfrak{g}$,

$$\langle X, Y\rangle = \mathrm{tr}(XY) \tag{1.4c}$$

taking positive real values on $\mathfrak{k}(\mathbb{R})$ and negative real values on $\mathfrak{s}(\mathbb{R})$, and making these spaces orthogonal. It follows that the (complex-valued) form $\langle , \rangle$ is *nondegenerate* on $\mathfrak{g}$ and makes θ orthogonal (so that the Cartan decomposition (1.1f) is orthogonal).

We use the nondegenerate form $\langle , \rangle$ to identify

$$\mathfrak{g} \simeq \mathfrak{g}^*, \quad X \mapsto \xi_X, \quad \xi_X(Y) = \langle X, Y\rangle, \tag{1.4d}$$

and so also to define a nondegenerate form (still written $\langle , \rangle$) on $\mathfrak{g}^*$.

We define *adjoint nilpotent cones* by

$$\begin{aligned} \mathcal{N} &= \{\text{nilpotent } X \in \mathfrak{g}\} \\ \mathcal{N}_{i\mathbb{R}} &= \{\text{nilpotent } X \in i\mathfrak{g}(\mathbb{R})\} \\ \mathcal{N}_\theta &= \{\text{nilpotent } X \in \mathfrak{s}\} \end{aligned} \tag{1.4e}$$

In each line the term "nilpotent" can be interpreted equivalently as "nilpotent in $\mathfrak{gl}(n,\mathbb{C})$ (see (1.4b))" or as "X belongs to $[\mathfrak{g},\mathfrak{g}]$ and $\mathrm{ad}(X)$ is nilpotent."

The identification (1.4d) provides equivariant identifications $\mathcal{N}^* \simeq \mathcal{N}$, $\mathcal{N}_{i\mathbb{R}}^* \simeq \mathcal{N}_{i\mathbb{R}}$, and so on. The choice of form is unique up to a positive scalar on each simple factor of $\mathfrak{g}$, so the identification of nilpotent adjoint and coadjoint orbits that it provides is independent of choices.

Before discussing nilpotent orbits, we record a familiar but critical fact about the form $\langle , \rangle$.

Proposition 1.5. *In the setting of* (1.4)*, suppose that $H \subset G$ is a complex maximal torus, so that $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra. Write $X^*(H)$ for the lattice of weights (algebraic characters) of H, so that*

$$X^*(H) \subset \mathfrak{h}^*, \qquad \mathfrak{h}^* = X^*(H) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Then the bilinear form $\langle , \rangle$ has nondegenerate restriction to $\mathfrak{h}$ and $\mathfrak{h}^*$. It is real-valued and positive on $X^*(H)$, and therefore positive definite on the "canonical real form"

$$\mathfrak{h}^*_{\mathrm{RE}} =_{\mathrm{def}} X^*(H) \otimes_{\mathbb{Z}} \mathbb{R}$$

(see [3, Definition 5.5]*).*

Proposition 1.6 (Jacobson-Morozov)**.** *In the setting of* (1.4)*, suppose that* $\xi \in \mathcal{N}^*$ *is a nilpotent linear functional. Define*

$$E \in \mathcal{N} \subset \mathfrak{g}$$

by the requirement $\xi_E = \xi$ *(*cf. (1.4d)*).*

1. *We can find elements* D *and* F *in* $\mathfrak{g}$ *so that*

   $$[D, E] = 2E, \quad [D, F] = -2F, \quad [E, F] = D.$$

   These elements specify an algebraic map

   $$\phi = \phi_{D,E,F} \colon SL(2, \mathbb{C}) \to G,$$

   $$d\phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = D, \quad d\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = E, \quad d\phi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = F.$$

2. *The element* F *is uniquely determined up to the adjoint action of*

   $$G^E =_{\mathrm{def}} \{g \in G \mid \mathrm{Ad}(g)(E) = E\};$$

   and the elements E *and* F *determine* D *and* ϕ*.*
3. *The Lie algebra grading*

   $$\mathfrak{g}_r =_{\mathrm{def}} \{X \in \mathfrak{g} \mid [D, X] = rX\}$$

   is by integers. Consequently

   $$\mathfrak{q} =_{\mathrm{def}} \sum_{r \geq 0} \mathfrak{g}_r$$

   is a parabolic subalgebra of $\mathfrak{g}$*, with Levi decomposition*

   $$\mathfrak{l} = \mathfrak{g}_0 = \mathfrak{g}^D, \qquad \mathfrak{u} = \sum_{r > 0} \mathfrak{g}_r.$$

   We write

   $$Q = LU, \quad L = G^D$$

   for the corresponding parabolic subgroup.
4. *The centralizer* G^E *(defined in (2)) is contained in* Q*. Consequently* $\mathfrak{q}$ *depends only on* E *(and not on the choice of* F *and* D *used to define it).*

5. *The Levi decomposition* $Q = LU$ *of* Q *restricts to a Levi decomposition*

$$G^E = L^E U^E = G^{\phi(SL(2))} U^E.$$

Here the first factor is reductive (but possibly disconnected), and the second is connected, unipotent, and normal.

6. *The orbit*

$$L \cdot E \simeq L/L^E \subset \mathfrak{g}_2$$

is open and dense. Furthermore

$$Q \cdot E \simeq Q/G^E = (L \cdot E) + \sum_{r>2} \mathfrak{g}_r.$$

A convenient reference for the proof is [9, Theorem 3.3.1].

It is standard to call the elements of the "$SL(2)$ triple" (E, H, F), but we prefer to reserve the letter H for algebraic groups and particularly for maximal tori. The letter D may be taken to stand for "diagonal," or just to be the predecessor of E and F.

Corollary 1.7. *Suppose* E *and* E' *are nilpotent elements of* $\mathfrak{g}$*; choose Lie triples* (E, D, F) *and* (E', D', F') *as in Proposition 1.6. Then* D *is conjugate to* D' *if and only if* E *is conjugate to* E'*.*

Proof. The assertion "if" follows from Proposition 1.6(2). So assume that D and D' are conjugate; we may as well assume that they are equal. Then the parabolic subalgebras $\mathfrak{q}$ and $\mathfrak{q}'$ are equal, along with their gradings. By Proposition 1.6(6), the two orbits $L \cdot E$ and $L \cdot E'$ are both open and Zariski dense in $\mathfrak{g}_2$, so they must coincide. That is, E' is conjugate to E by L. □

Proposition 1.8 (Kostant-Rallis [14], Kostant-Sekiguchi [17])**.** *Use the notation of* (1.1), (1.3)*, and Proposition 1.6.*

1. *Assume that* $i\xi_{\mathbb{R}} \in \mathcal{N}^*_{i\mathbb{R}}$ *is a real nilpotent element, or equivalently that the element* $iE_{\mathbb{R}}$ *belongs to* $\mathcal{N}_{i\mathbb{R}}$*. Then the element* $iF_{\mathbb{R}}$ *may also be chosen to belong to* $\mathcal{N}_{i\mathbb{R}}$ *(so that automatically* $D_{\mathbb{R}} \in \mathfrak{g}(\mathbb{R})$*). Such a choice is unique up to conjugation by* $G(\mathbb{R})^E$*. Equivalently, the map* ϕ *may be chosen to be defined over* $\mathbb{R}$*:*

$$\phi_{\mathbb{R}} \colon SL(2, \mathbb{R}) \to G(\mathbb{R}).$$

$$d\phi_{\mathbb{R}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = D_{\mathbb{R}}, \quad d\phi_{\mathbb{R}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = E_{\mathbb{R}}, \quad d\phi_{\mathbb{R}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = F_{\mathbb{R}}.$$

2. *With choices as in (1), the Jacobson-Morozov parabolic $Q = LU$ of Proposition 1.6(2) is defined over $\mathbb{R}$. The Levi decomposition*
$$Q(\mathbb{R}) = L(\mathbb{R})U(\mathbb{R})$$
restricts to a Levi decomposition
$$G^{iE_{\mathbb{R}}} = L^{iE_{\mathbb{R}}}U^{iE_{\mathbb{R}}}.$$
The first factor is $G(\mathbb{R})^{\phi_{\mathbb{R}}}$, a (possibly disconnected) real reductive algebraic group, and the second factor is connected, unipotent, and normal.
3. *The orbit*
$$L(\mathbb{R}) \cdot iE_{\mathbb{R}} \simeq L(\mathbb{R})/L(\mathbb{R})^{iE_{\mathbb{R}}} \subset i\mathfrak{g}_2(\mathbb{R})$$
is open but not necessarily dense. The open orbits of $L(\mathbb{R})$ on this vector space are in one-to-one correspondence with the $G(\mathbb{R})$ orbits on $\mathcal{N}_{i\mathbb{R}}$ having semisimple part conjugate to $D_{\mathbb{R}}$. Furthermore
$$Q(\mathbb{R}) \cdot iE_{\mathbb{R}} \simeq Q(\mathbb{R})/Q(\mathbb{R})^{iE_{\mathbb{R}}} = (L(\mathbb{R}) \cdot iE_{\mathbb{R}}) + \sum_{r>2} i\mathfrak{g}(\mathbb{R})_r.$$
4. *After replacing $(iE_{\mathbb{R}}, \phi_{\mathbb{R}})$ by a conjugate $(iE_{\mathbb{R},\theta}, \phi_{\mathbb{R},\theta})$ under $G(\mathbb{R})$, we may assume that the map ϕ also respects the Cartan involution:*
$$\phi_{\mathbb{R},\theta}({}^t g^{-1}) = \theta\left(\phi_{\mathbb{R},\theta}(g)\right), \qquad iF_{\mathbb{R},\theta} = -\theta(iE_{\mathbb{R},\theta})$$
5. *Assume that $\xi_\theta \in \mathcal{N}^*_\theta$ is a K-nilpotent element, or equivalently that the element E_θ belongs to $\mathcal{N}_\theta \subset \mathfrak{s}$ (the -1 eigenspace of θ). Then the element F_θ may also be chosen in $\mathfrak{s}$, and in this case D_θ belongs to $\mathfrak{k}$. Such choices are unique up to conjugation by K^{E_θ}. They define an algebraic map*
$$\phi_\theta \colon SL(2,\mathbb{C}) \to G,$$
$$d\phi_\theta\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = D_\theta, \quad \frac{1}{2}\cdot d\phi_\theta\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} = E_\theta, \quad \frac{1}{2}\cdot d\phi_\theta\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} = F_\theta$$
which respects θ:
$$\phi_\theta({}^t g^{-1}) = \theta\left(\phi_\theta(g)\right).$$
6. *With choices as in (5), the Jacobson-Morozov parabolic $Q_\theta = L_\theta U_\theta$ (defined as in Proposition 1.6(2) using D_θ) is θ-stable. The Levi decomposition*
$$Q_\theta \cap K = (L_\theta \cap K)(U_\theta \cap K)$$

restricts to a Levi decomposition

$$K^{E_\theta} = (L_\theta \cap K)^{E_\theta}(U_\theta \cap K)^{E_\theta}.$$

The first factor is K^{ϕ_θ}, a (possibly disconnected) complex reductive algebraic group, and the second factor is connected, unipotent, and normal.

7. *The orbit*

$$(L_\theta \cap K)\cdot E_\theta \simeq (L_\theta \cap K)/(L_\theta \cap K)^{E_\theta} \subset \mathfrak{s}_2$$

is open and (Zariski) dense. Furthermore

$$(Q_\theta \cap K)\cdot E_\theta \simeq (Q_\theta \cap K)/(Q_\theta \cap K)^{E_\theta} = ((L_\theta \cap K)\cdot E_\theta) + \sum_{r>2} \mathfrak{s}_r.$$

8. *After replacing (E_θ, ϕ_θ) by a conjugate $(E_{\theta,\mathbb{R}}, \phi_{\theta,\mathbb{R}})$ under K, we may assume that the map ϕ_θ also respects the real form:*

$$\phi_{\theta,\mathbb{R}}(\overline{g}) = \sigma_{\mathbb{R}}(\phi_{\theta,\mathbb{R}}(g)), \qquad F_{\theta,\mathbb{R}} = \sigma_{\mathbb{R}}(E_{\theta,\mathbb{R}}).$$

Corollary 1.9. *Suppose that $iE_{\mathbb{R}}$ and $iE'_{\mathbb{R}}$ are nilpotent elements of $i\mathfrak{g}(\mathbb{R})$; choose Lie triples $(iE_{\mathbb{R}}, D_{\mathbb{R}}, iF_{\mathbb{R}})$ and $(iE'_{\mathbb{R}}, D'_{\mathbb{R}}, iF'_{\mathbb{R}})$ as in Proposition 1.8.*

1. *The semisimple Lie algebra elements*

$$iE_{\mathbb{R}} - iF_{\mathbb{R}} \quad \text{and} \quad iE'_{\mathbb{R}} - iF'_{\mathbb{R}}$$

are conjugate by $G(\mathbb{R})$ if and only if $iE_{\mathbb{R}}$ is conjugate to $iE'_{\mathbb{R}}$ by $G(\mathbb{R})$

2. *Suppose E_θ and E'_θ are nilpotent elements of $\mathfrak{s}$; choose Lie triples $(E_\theta, D_\theta, F_\theta)$ and $(E'_\theta, D'_\theta, F'_\theta)$ as in Proposition 1.8. Then D_θ is conjugate to D'_θ by K if and only if E_θ is conjugate to E'_θ by K.*

The first assertion is not quite immediate from the proposition, and we will not use it; we include it only to show that there *is* a way of parametrizing real nilpotent classes by real semisimple classes.

Corollary 1.10. *In the setting of* (1.1) *and* (1.3), *there are bijections among the following sets:*

1. *$G(\mathbb{R})$ orbits on $\mathcal{N}^*_{i\mathbb{R}}$;*
2. *$G(\mathbb{R})$ orbits on $\mathcal{N}_{i\mathbb{R}}$;*

3. *$G(\mathbb{R})$ orbits of group homomorphisms*

$$\phi_{\mathbb{R}} \colon SL(2,\mathbb{R}) \to G(\mathbb{R});$$

4. *$K(\mathbb{R})$ orbits of group homomorphisms*

$$\phi_{\mathbb{R},\theta} \colon SL(2,\mathbb{R}) \to G(\mathbb{R})$$

   sending inverse transpose to the Cartan involution θ;
5. *$K(\mathbb{R})$ orbits of group homomorphisms*

$$\phi \colon SL(2) \to G$$

   sending inverse transpose to the Cartan involution θ, and sending complex conjugation to $\sigma_{\mathbb{R}}$;
6. *K orbits of group homomorphisms*

$$\phi \colon SL(2) \to G$$

   sending inverse transpose to the Cartan involution θ;
7. *K orbits on $\mathcal{N}_\theta$; and*
8. *K orbits on $\mathcal{N}_\theta^*$.*

The correspondences (1)$\leftrightarrow$(2) and (7)$\leftrightarrow$(8) are given by (1.4)*; (2)$\leftrightarrow$(3) by Proposition 1.8(2); and so on.*

All the maximal compact subgroups of the isotropy groups for the orbits above are naturally isomorphic, with isomorphisms defined up to inner automorphisms, to $K(\mathbb{R})^{\phi_{\mathbb{R},\theta}}$.

The bijection (1)$\leftrightarrow$(8) preserves the closure relations between orbits. Corresponding orbits $\mathcal{O}_{\mathbb{R}}$ and $\mathcal{O}_\theta$ are $K(\mathbb{R})$-equivariantly diffeomorphic.

The assertions in the last paragraph are due to Barbasch-Sepanski [5] and Vergne [19] respectively.

The bijection may also be characterized by either of the following equivalent conditions:

$$\begin{aligned} \frac{1}{2}\left(-iE_{\mathbb{R}} - iF_{\mathbb{R}} + D_{\mathbb{R}}\right) &\text{ is conjugate by } K \text{ to } E_\theta \\ iE_{\mathbb{R}} - iF_{\mathbb{R}} &\text{ is conjugate by } K \text{ to } D_\theta. \end{aligned} \tag{1.11}$$

This formulation makes clear what is slightly hidden in the formulas of Proposition 1.8(5): that the bijection *does not depend on a chosen square* *root of -1*. Changing the choice replaces $iE_{\mathbb{R}}$ by $-iE_{\mathbb{R}}$, and therefore twists the $SL(2,\mathbb{R})$ homomorphism $\phi_{\mathbb{R}}$ by inverse transpose. At the same time E_θ and F_θ are interchanged, which has the effect of twisting ϕ_θ by inverse transpose.

Definition 1.12. If $\mathcal{O}$ is a G-orbit on $\mathcal{N}^*$, then a $G(\mathbb{R})$ orbit

$$\mathcal{O}_{\mathbb{R}} \subset \mathcal{N}^*_{i\mathbb{R}} \cap \mathcal{O}$$

is called a *real form* of $\mathcal{O}$. We will call a K orbit

$$\mathcal{O}_\theta \subset \mathcal{N}^*_\theta \cap \mathcal{O}$$

a θ *form* of $\mathcal{O}$. The Kostant-Sekiguchi theorem says that there is a natural bijection between real forms and θ forms.

Definition 1.13. A *(global) geometric parameter* for (G,K) is a nilpotent K-orbit $Y \subset \mathcal{N}^*_\theta$, together with an irreducible K-equivariant vector bundle

$$\mathcal{E} \to Y.$$

Equivalently, a *(local) geometric parameter* is a K-conjugacy class of pairs

$$(\xi, (\tau, E)),$$

with $\xi \in \mathcal{N}^*_\theta$ a nilpotent element, and (τ, E) an irreducible (algebraic) representation of the isotropy group K^ξ. This bijection between local and global parameters identifies $(\xi,(\tau,E))$ with the pair

$$Y = K\cdot\xi \simeq K/K^\xi, \qquad \mathcal{E} \simeq K \times_{K^\xi} E.$$

We write $\mathcal{P}_g(G,K)$ for the collection of geometric parameters.

2. Asymptotic cones

This section is a digression, intended as another kind of motivation for the orbit method. The (very elementary) ideas play no role in the proofs of our main theorems. They appear only in the desideratum (0.1j) for deciding which representations might reasonably be attached to which coadjoint orbits. In order to provide some mathematical excuse for the material, we will include a single serious conjecture (Conjecture 2.13) about automorphic forms.

Suppose

$$V \simeq \mathbb{R}^n \tag{2.1a}$$

is a finite-dimensional real vector space. A *ray* in V is by definition a subset

$$R(v) = \mathbb{R}_{\geq 0} \cdot v \subset V \qquad (0 \neq v \in V). \tag{2.1b}$$

We write

$$\mathcal{R}(V) = \{\text{rays in } V\} \simeq S^{n-1}; \tag{2.1c}$$

an isomorphism with the $(n-1)$-sphere is induced by an isomorphism (2.1a). The resulting smooth manifold structure on $\mathcal{R}(V)$ is of course independent of the isomorphism. There is a natural fiber bundle

$$\mathcal{B}(V) = \{(v, r) \mid r \in \mathcal{R}(V),\ v \in r\} \xrightarrow{\pi} \mathcal{R}(V), \tag{2.1d}$$

the *tautological ray bundle* over $\mathcal{R}(V)$. Projection on the first factor defines a proper map

$$\mathcal{B}(V) \xrightarrow{\mu} V, \quad (v, r) \mapsto v; \tag{2.1e}$$

the map μ is an isomorphism over the preimage of $V\backslash\{0\}$ (consisting of the open rays in the bundle), and $\mu^{-1}(0) = \mathcal{R}(V)$ (the compact sphere).

Definition 2.2. In the setting of (2.1), a *cone* $C \subset V$ is any subset closed under scalar multiplication by $\mathbb{R}_{\geq 0}$. A *weak cone* is any subset closed under scalar multiplication by $\mathbb{R}_{>0}$.

Definition 2.3. In the setting of (2.1), suppose $S \subset V$ is an arbitrary subset. The *asymptotic cone of* S is

$$\mathrm{Cone}_{\mathbb{R}}(S) = \{v \in V \mid \exists \epsilon_i \to +0, s_i \in S, \lim_{i \to \infty} \epsilon_i s_i = v\}.$$

Here $\{\epsilon_i\}$ is a sequence of positive real numbers going to 0, and s_i is any sequence of elements of S.

Here are some elementary properties of the asymptotic cone.

1. The set $\mathrm{Cone}_{\mathbb{R}}(S)$ is a closed cone.
2. The set $\mathrm{Cone}_{\mathbb{R}}(S)$ is nonempty if and only if S is nonempty.
3. The set $\mathrm{Cone}_{\mathbb{R}}(S)$ is contained in $\{0\} \subset V$ if and only if S is bounded.

4. If C is a weak cone (Definition 2.2), then the asymptotic cone is the closure of C:
$$\mathrm{Cone}_{\mathbb{R}}(C) = \overline{C}.$$

The last assertion includes (0.1h) from the introduction.

Here are the same ideas in the setting of algebraic geometry. Suppose

$$V \simeq \mathbb{C}^n \tag{2.4a}$$

is a finite-dimensional complex vector space. We write

$$\mathbb{P}(V) = \{(\text{complex}) \text{ lines through the origin in } V\}. \tag{2.4b}$$

There is a natural line bundle

$$\mathcal{O}(-1)(V) = \{(v,\ell) \mid \ell \in \mathbb{P}(V), v \in \ell\} \xrightarrow{\pi} \mathbb{P}(V), \tag{2.4c}$$

the *tautological line bundle* over $\mathbb{P}(V)$. Projection on the first factor defines a proper map

$$\mathcal{O}(-1)(V) \xrightarrow{\mu} V, \quad (v,\ell) \mapsto v; \tag{2.4d}$$

the map μ is an isomorphism over the preimage of $V\backslash\{0\}$ (consisting of the bundle minus the zero section), and $\mu^{-1}(0) = \mathbb{P}(V)$.

Definition 2.5. In the setting of (2.4), a *cone* $C \subset V$ is any subset closed under scalar multiplication by $\mathbb{C}$. A *weak cone* is any subset closed under scalar multiplication by $\mathbb{C}^\times$.

Definition 2.6. In the setting of (2.4), suppose $S \subset V$ is an arbitrary subset. Define

$$I(S) = \{p \in \mathrm{Poly}(V) \mid p|_S = 0\},$$

the ideal of polynomial functions vanishing on S. This ideal is filtered by the degree filtration on polynomial functions, so we can define

$$\mathrm{gr}\, I(S) \subset \mathrm{Poly}(V)$$

a graded ideal. This is the ideal generated by the highest degree term of each nonzero polynomial vanishing on S. The *algebraic asymptotic cone of* S is

$$\mathrm{Cone}_{\mathrm{alg}}(S) = \{v \in V \mid q(v) = 0 \text{ all } q \in \mathrm{gr}\, I(S)\},$$

a Zariski-closed cone in V; or, equivalently, a closed subvariety of $\mathbb{P}(V)$.

This definition looks formally like the definition of the tangent cone to S at $\{0\}$ (see for example [11, Lecture 20]). In that definition one considers the graded ideal generated by *lowest* degree terms in the ideal of S.

Here are some elementary properties of the algebraic asymptotic cone.

1. The set $\mathrm{Cone}_{\mathrm{alg}}(S)$ is a closed cone, of dimension equal to $\dim \overline{S}$ (the Krull dimension of the Zariski closure of S).
2. The set $\mathrm{Cone}_{\mathrm{alg}}(S)$ is nonempty if and only if S is nonempty.
3. The set $\mathrm{Cone}_{\mathrm{alg}}(S)$ is contained in $\{0\} \subset V$ if and only if S is finite.
4. If C is a weak cone (Definition 2.2), then the asymptotic cone is the Zariski closure of C:
$$\mathrm{Cone}_{\mathrm{alg}}(C) = \overline{C}.$$
5. If S is a constructible algebraic set (finite union of Zariski closed intersect Zariski open) then
$$\mathrm{Cone}_{\mathrm{alg}}(S) = \{v \in V \mid \exists \epsilon_i \to 0, s_i \in S, \lim_{i\to\infty} \epsilon_i s_i = v\}.$$
Here $\{\epsilon_i\}$ is a sequence of nonzero complex numbers going to zero, and $\{s_i\}$ is any sequence of elements of S.

Finally, we note that the definition given for asymptotic cones over $\mathbb{R}$ extends to any local field. It is not quite clear what dilations ought to be allowed "in general"; we make a choice that behaves well for the coadjoint orbits we are interested in.

Suppose K is a local field of characteristic not 2, and

$$V \simeq \mathbb{K}^n \tag{2.7a}$$

is a finite-dimensional K-vector space. A *ray* in V is by definition a subset

$$R(v) = (K^\times)^2 \cdot v \subset V \qquad (0 \neq v \in V). \tag{2.7b}$$

We write

$$\mathcal{R}(V) = \{\text{rays in } V\} \to \mathbb{P}(V); \tag{2.7c}$$

the map is $\#(K^\times/(K^\times)^2)$ to one. It follows that there is a natural compact K-manifold topology on $\mathcal{R}(V)$, of dimension equal to $n-1$. There is a natural fiber bundle

$$\mathcal{B}(V) = \{(v,r) \mid r \in \mathcal{R}(V), v \in r\} \xrightarrow{\pi} \mathcal{R}(V), \quad (v,r) \mapsto r \tag{2.7d}$$

the *tautological ray bundle* over $\mathcal{R}(V)$. Projection on the first factor defines a proper map

$$\mathcal{B}(V) \xrightarrow{\mu} V, \quad (v, r) \mapsto v; \tag{2.7e}$$

the map μ is an isomorphism over the preimage of $V \backslash \{0\}$ (consisting of the open rays in the bundle), and $\mu^{-1}(0) = \mathcal{R}(V)$ (the compact space of all rays in V).

Definition 2.8. In the setting of (2.7), a *cone* $C \subset V$ is any subset closed under scalar multiplication by K^2. A *weak cone* is any subset closed under scalar multiplication by $(K^\times)^2$.

Definition 2.9. In the setting of (2.7), suppose $S \subset V$ is an arbitrary subset. The *asymptotic cone of* S is

$$\mathrm{Cone}_K(S) = \{v \in V \mid \exists \epsilon_i \to 0, s_i \in S, \lim_{i \to \infty} \epsilon_i s_i = v\}.$$

Here $\{\epsilon_i\}$ is a sequence in $(K^\times)^2$ going to 0, and s_i is any sequence of elements of S.

Here are some elementary properties of the asymptotic cone.

1. The set $\mathrm{Cone}_K(S)$ is a closed cone.
2. The set $\mathrm{Cone}_K(S)$ is nonempty if and only if S is nonempty.
3. The set $\mathrm{Cone}_K(S)$ is contained in $\{0\} \subset V$ if and only if S is bounded.
4. If C is a weak cone (Definition 2.8), then the asymptotic cone is the closure of C:
$$\mathrm{Cone}_K(C) = \overline{C}.$$

The reformulation in [6] of Howe's definition from [13] of the wavefront set of a representation (see (0.1i)) extends to groups over other local fields, by means of the "Shalika germ" expansion of characters. If G is a reductive algebraic group defined over a p-adic field K of characteristic zero, then we write

$$\begin{aligned} \mathcal{N}_K^* &= \text{nilpotent elements in } \mathfrak{g}(K)^* \\ &= \{\xi \in \mathfrak{g}(K)^* \mid t^2\xi \in \mathrm{Ad}^*(G(K))(\xi)\ (t \in K^\times)\} \end{aligned} \tag{2.10}$$

for the nilpotent cone in the dual of the Lie algebra. (That a nilpotent linear functional ξ is conjugate to $t^2\xi$ can be proved by transferring the statement to $\mathfrak{g}(K)$ as in (1.4d), and using the Jacobson-Morozov theorem to reduce to the case of $SL(2)$.) Dilation therefore defines an action of the finite group $K^\times/(K^\times)^2$ on $G(K)$ orbits in $\mathcal{N}^*_K$.

Shalika germs are defined using the Fourier transform

$$\begin{aligned} &C_c^\infty(\mathfrak{g}(K)) \to C_c^\infty(\mathfrak{g}(K)^*) \\ &\qquad \widehat{f}(\xi) = \int_{X\in\mathfrak{g}(K)} f(X)\psi(\xi(X))dX. \end{aligned} \tag{2.11}$$

Here dX is a choice of Lebesgue measure on $\mathfrak{g}(K)$: scaling the measure scales the values of the Fourier transform (and so does not change its support). Furthermore ψ is a nontrivial additive unitary character of K, which is therefore unique up to dilation by $t \in K^\times$. Dilating ψ dilates the Fourier transform as a function on $\mathfrak{g}(K)^*$, and therefore dilates the support. In particular, the support may *change* according to the $K^\times/(K^\times)^2$ action on nilpotent orbits when ψ changes.

Once the choice of the character ψ is in place, we get for every smooth admissible irreducible representation π of $G(K)$ a $G(K)$-invariant closed cone

$$\mathrm{WF}(\pi) \subset \mathcal{N}^*_K. \tag{2.12}$$

using the Shalika germ expansion of the character of π. (Our understanding is that there may be a way to make sense of this for K local of finite characteristic, but that it is not completely established.)

Conjecture 2.13. *(global coherence of WF sets) Suppose that k is a number field, and that G is a reductive algebraic group defined over k. Suppose that*

$$\pi = \otimes_v \pi_v$$

is an automorphic representation. This means (among other things) that $\{v\}$ is the set of places of k, so that each corresponding completion

$$k_v \supset k$$

is a local field, $G(k_v)$ is a reductive group over a local field as above, and π_v is a smooth irreducible admissible representation of $G(k_v)$. Accordingly for each place we get a closed $G(k_v)$-invariant cone

$$\mathrm{WF}(\pi_v) \subset \mathcal{N}^*_{k_v} \subset \mathfrak{g}(k_v)^*.$$

Suppose that all the local characters ψ_v of k_v (used in defining the Fourier transforms behind the wavefront sets) are chosen in such a way that

$$\prod_v \psi_v(x) = 1, \qquad (x \in k).$$

The conjecture is that there is a coadjoint orbit (depending on π)

$$\mathcal{O} = G(k) \cdot \xi \subset \mathfrak{g}(k)^*$$

with the property that

$$\mathrm{WF}(\pi_v) = \mathrm{Cone}_{k_v}(\mathcal{O})$$

for every place v.

It is easy to see (by considering characteristic polynomials, for example) that

$$\mathrm{Cone}_{k_v}(\mathcal{O}) \subset \mathcal{N}^*_{k_v}.$$

For similar reasons,

$$\mathrm{Cone}_{\mathrm{alg}}(G(\overline{k}) \cdot \xi) = Z \subset \mathfrak{g}(\overline{k})^*,$$

the Zariski closure of a single nilpotent coadjoint orbit Z^0 over the algebraic closure $\overline{k}$. It follows that

$$\mathrm{Cone}_{k_v}(\mathcal{O}) \subset Z(k_v),$$

the k_v-points of Z. But this local cone need not meet the open orbit Z^0. (We believe it should be possible to show that $\mathrm{Cone}_{k_v}(\mathcal{O})$ *does* meet Z^0 for all but finitely many v.)

The local cones control expansions of the local characters near the identity. It would be nice if there were some global analogue of these local expansions, involving orbits like $G(k) \cdot \xi$. But we can offer no suggestion about how to formulate any such thing.

Since all the local cones are nilpotent, it is natural to ask whether the global orbit $G(k) \cdot \xi$ in the conjecture can be taken to be nilpotent. This is *not* possible. If G is anisotropic over k (admitting no nontrivial k-split torus) then $\mathcal{N}^*_k = \{0\}$; so the conjecture would require that all the local factors π_v of any automorphic representation would have to be finite-dimensional. This does not happen (for nonabelian G).

3. Equivariant *K*-theory

In this section we recall the algebraic geometry version of equivariant K-theory, which for us will replace the $G(\mathbb{R})$-equivariant theory discussed in the introduction (for which we lack even many definitions, and certainly lack proofs of good properties).

Suppose H is a complex algebraic group. Write H^{unip} for the unipotent radical of H, and

$$H^{\mathrm{red}} \subset H, \qquad H^{\mathrm{red}} \simeq H/H^{\mathrm{unip}} \tag{3.1a}$$

for a choice of Levi subgroup. We are interested in the category

$$\mathrm{Rep}(H) = \text{finite-dimensional algebraic representations of } H. \tag{3.1b}$$

The Grothendieck group of this category is called the *representation ring of* H, and written

$$R(H) =_{\mathrm{def}} K\,\mathrm{Rep}(H). \tag{3.1c}$$

If we write

$$\begin{aligned} \widehat{H} &= \text{equivalence classes of irreducible} \\ &\qquad \text{algebraic representations of } H \\ &= \widehat{H^{\mathrm{red}}}, \end{aligned} \tag{3.1d}$$

then elementary representation theory says that

$$R(H) = K\,\mathrm{Rep}(H) \simeq \sum_{\rho\in\widehat{H}} \mathbb{Z}\rho \simeq R(H^{\mathrm{red}}). \tag{3.1e}$$

The ring structure on $R(H)$ arises from tensor product of representations. Another way to say it is using the *character* of a representation (τ, E) of H:

$$\Theta_E(h) = \mathrm{tr}\,\tau(h) \qquad (h \in H).$$

Clearly the character Θ_E is an algebraic class function on H. If (σ, S) and (κ, Q) are representations forming a short exact sequence

$$0 \to S \to E \to Q \to 0,$$

then $\Theta_E = \Theta_S + \Theta_Q$. Consequently Θ descends to a $\mathbb{Z}$-linear map

$$\Theta\colon R(H) \hookrightarrow \text{algebraic class functions on } H. \tag{3.1f}$$

The reason for the injectivity is the standard fact that characters of inequivalent irreducible representations are linearly independent functions on H. The trace of the tensor product of two linear maps is the product of the traces, so $\Theta_{E\otimes F} = \Theta_E\Theta_F$. Therefore the map of (3.1f) is also a ring homomorphism. Because irreducible representations are trivial on the unipotent radical H^{unip} of H, we get finally

$$\Theta\colon R(H) \hookrightarrow \text{algebraic class functions on } H/H^{\text{unip}} \simeq H^{\text{red}}.$$

After complexification, this map turns out to be an algebra isomorphism

$$\Theta\colon R(H) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \text{algebraic class functions on } H^{\text{red}}. \tag{3.1g}$$

Definition 3.2. Suppose X is a complex algebraic variety and suppose that H is a complex algebraic group acting on X. We are interested in the abelian category

$$\mathrm{QCoh}^H(X)$$

of H-equivariant quasicoherent sheaves on X, and particularly in the full subcategory

$$\mathrm{Coh}^H(X)$$

of coherent sheaves. We need this theory for cases in which X is *not* an affine variety. In that setting the precise definitions (found in [18, 1.2]) are a little difficult to make sense of. We will be able to work largely with the special case when X is affine; so we recall here just the more elementary description of these equivariant sheaves in the affine case.

Suppose therefore that X is a complex *affine* algebraic variety, with structure sheaf $\mathcal{O}_X$ and global ring of functions $\mathcal{O}_X(X)$. Suppose that H is an affine algebraic group acting on X. This means precisely that the algebra $\mathcal{O}_X(X)$ is equipped with an algebraic action

$$\mathrm{Ad}\colon H \to \mathrm{Aut}(\mathcal{O}_X(X))$$

by algebra automorphisms. A quasicoherent sheaf $\mathcal{M} \in \mathrm{QCoh}(X)$ is completely determined by the $\mathcal{O}_X(X)$-module

$$M = \mathcal{M}(X)$$

of global sections: any $\mathcal{O}_X(X)$-module M determines a quasicoherent sheaf

$$\mathcal{M} = \mathcal{O}_X \otimes_{\mathcal{O}_X(X)} M$$

of $\mathcal{O}_X$-modules. This makes an equivalence of categories

$$\mathrm{QCoh}(X) = \mathcal{O}_X\text{–Mod} \simeq \mathcal{O}_X(X)\text{–Mod}\,.$$

The sheaf $\mathcal{M}$ is coherent if and only if M is finitely generated:

$$\mathrm{Coh}(X) \simeq \mathcal{O}_X(X)\text{–Mod}^{\mathrm{fg}}\,.$$

Write $(\mathcal{O}_X(X), H)$–Mod for the category of $\mathcal{O}_X(X)$-modules M equipped with an algebraic action of H. This means an algebraic representation μ of H on M, such that the module action map

$$\mathcal{O}_X(X) \otimes_{\mathbb{C}} M \to M$$

intertwines $\mathrm{Ad} \otimes \mu$ with μ. We have now identified

$$\begin{aligned} \mathrm{QCoh}^H(X) &\simeq (\mathcal{O}_X(X), H)\text{–Mod} \\ \mathrm{Coh}^H(X) &\simeq (\mathcal{O}_X(X), H)\text{–Mod}^{\mathrm{fg}}, \end{aligned}$$

always for X affine.

The *equivariant K-theory of X* is the Grothendieck group $K^H(X)$ of $\mathrm{Coh}^H(X)$. (Thomason and many others prefer to write this K-theory as $G^H(X)$, reserving K for the theory with vector bundles replacing coherent sheaves. Our notation follows the text [8].)

If $\mathcal{M}$ is an H-equivariant coherent sheaf on X, we write

(3.3a) $$[\mathcal{M}] \in K^H(X)$$

for its class in K-theory. Similarly, if X is affine and M is a finitely generated H-equivariant $\mathcal{O}_X(X)$-module, we write

(3.3b) $$[M] \in K^H(X).$$

We record now some general facts about equivariant K-theory of which we will make constant use.

Suppose X is a single point. Then a coherent sheaf on X is the same thing as a finite-dimensional vector space E; an H-equivariant structure is an algebraic representation ρ of H on E. That is, there is an equivalence of categories

$$\operatorname{Coh}^H(\text{point}) \simeq \operatorname{Rep}(H) \tag{3.3c}$$

(see (3.1)). Consequently

$$K^H(\text{point}) \simeq K \operatorname{Rep}(H) \simeq \sum_{\rho \in \widehat{H}} \mathbb{Z}\rho. \tag{3.3d}$$

More generally, if (τ, V) is a finite-dimensional algebraic representation of H, then there is a natural inclusion

$$\operatorname{Rep}(H) \hookrightarrow (\operatorname{Poly}(V), H)\text{–}\operatorname{Mod}^{\text{fg}} \simeq \operatorname{Coh}^H(V), \quad E \mapsto E \otimes \operatorname{Poly}(V) \tag{3.3e}$$

(with $\operatorname{Poly}(V) = \mathcal{O}_V(V)$ the algebra of polynomial functions on V). In contrast to (3.3c), this inclusion is very far from being an equivalence of categories (unless $V = \{0\}$). Nevertheless, just as in (3.3d), it induces an isomorphism in K-theory

$$K \operatorname{Rep}(H) \simeq K^H(V); \tag{3.3f}$$

this is [18, Theorem 4.1]. (Roughly speaking, the image of the map (3.3e) consists of the *projective* $(\operatorname{Poly}(V), H)$-modules; the isomorphism in K-theory arises from the existence of projective resolutions.)

Suppose now that $H \subset G$ is an inclusion of affine algebraic groups. Then there is an equivalence of categories

$$\operatorname{Coh}^G(G/H) \simeq \operatorname{Coh}^H(\text{point}) \simeq \operatorname{Rep}(H); \tag{3.3g}$$

so the equivariant K-theory is

$$K^G(G/H) \simeq K^H(\text{point}) \simeq K \operatorname{Rep}(H). \tag{3.3h}$$

More generally, if H acts on the variety Y, then we can form a fiber product

$$X = G \times_H Y$$

and calculate

$$\begin{gathered} \operatorname{Coh}^G(G \times_H Y) \simeq \operatorname{Coh}^H(Y), \\ K^G(G \times_H Y) \simeq K^H(Y). \end{gathered} \tag{3.3i}$$

If $Y \subset X$ is a closed H-invariant subvariety, then there is a right exact sequence

$$K^H(Y) \to K^H(X) \to K^H(X - Y) \to 0. \tag{3.3j}$$

This is established without the H for example in [12, Exercise II.6.10(c)], and the argument there carries over to the equivariant case. This sequence is the end of a long exact sequence in higher equivariant K-theory ([18, Theorem 2.7]). The next term on the left is $K_1^H(X - Y)$.

Here are some of the consequences we want.

Proposition 3.4. *Suppose H acts on X with an open orbit $U \simeq H/H^u$, and $Y = X - U$ is the (closed) complement of this orbit. Then there is a right exact sequence*

$$K^H(Y) \to K^H(X) \twoheadrightarrow K^H(H/H^u) \simeq K^{H^u}(\text{point}) \simeq R(H^u).$$

Consequently the quotient

$$K^H(X)/\operatorname{Im} K^H(Y)$$

has a basis naturally indexed by the set $\widehat{H^u}$ of irreducible representations of H^u.

More generally, if U is the disjoint union of finitely many open orbits $U_j \simeq H/H^{u^j}$, then $K^H(X)/\operatorname{Im} K^H(Y)$ has a basis naturally indexed by the disjoint union of the sets $\widehat{H^{u_j}}$.

Theorem 3.5. *Suppose H acts on the affine variety X with finitely many orbits*

$$X_i = H \cdot x_i \simeq H/H_i.$$

1. *The (Zariski) closure $\overline{X_i}$ is the union of X_i and finitely many additional orbits X_j with $\dim X_j < \dim X_i$.*
2. *For each irreducible (τ, V_τ) of H_i, there is an H-equivariant coherent sheaf $\widetilde{\mathcal{V}_\tau}$ on $\overline{X_i}$ with the property that*

   $$\widetilde{\mathcal{V}_\tau}|_{X_i} = \mathcal{V}_\tau =_{\text{def}} G \times_{H_i} V_\tau.$$

   The sheaf $\widetilde{\mathcal{V}_\tau}$ may be regarded as a (coherent equivariant) sheaf on X.
3. *The equivariant K-theory $K^H(X)$ is a free $\mathbb{Z}$-module with basis consisting of the various $[\widetilde{\mathcal{V}_\tau}]$, for $X_i \subset X$ and τ an irreducible representation of H_i. Each such basis vector is uniquely defined modulo the span of the $[\widetilde{\mathcal{V}_{\tau'}}]$ supported on orbits in the boundary of X_i.*

4. If $Y \subset X$ is an H-invariant closed subvariety, then (3.3j) *is a short exact sequence*

$$0 \to K^H(Y) \to K^H(X) \to K^H(X - Y) \to 0.$$

Proof. Part (1) is a standard statement about algebraic varieties. Part (2) is established without the H for example in [12, Exercise 2.5.15], and the argument there carries over to the equivariant case.

For (3), we proceed by induction on the number of H-orbits on X. If the number is zero, then X is empty, and $K^H(X) = 0$ is indeed free. So suppose that the number of H-orbits is positive, and that (3) is already known in the case of a smaller number of orbits. Pick an orbit

$$U = H \cdot x_{i_0} \simeq H/H_{i_0} \subset X \tag{3.6a}$$

of maximal dimension. Then necessarily U is open (this follows from (1)), so $Y = X - U$ is closed. According to (3.3j), there is an exact sequence

$$K_1^H(U) \to K^H(Y) \to K^H(X) \to K^H(U) \to 0. \tag{3.6b}$$

By inductive hypothesis, $K^H(Y)$ is a free $\mathbb{Z}$-module with basis the various $[\widetilde{\mathcal{V}'_\tau}]$ living on orbits other than U. By (3.3h), $K^H(U)$ is a free $\mathbb{Z}$-module with basis the images of the various $[\widetilde{\mathcal{V}_\tau}]$ attached to U. To finish the proof of (3), we need only prove that the map from $K_1^H(U)$ is zero.

According to (3.3g), the first term of our exact sequence is the Quillen K_1 of H/H_{i_0}; that is, Quillen K_1 of a category of representations (of H_{i_0}). This category is (up to long exact sequences) a direct sum of categories of finite-dimensional complex vector spaces. By one of the fundamental facts about algebraic K-theory, it follows that $K_1^H(U) = K_1^H(H/H_{i_0})$ is a direct sum of copies of $\mathbb{C}^\times$, one for each irreducible representation of H_{i_0}.

Any group homomorphism from the divisible group $\mathbb{C}^\times$ to $\mathbb{Z}$ must be zero; so the connecting homomorphism $K_1^H(U) \to K^H(Y)$ is zero. This proves (4). □

We thank Gonçalo Tabuada for explaining to us this proof of (4).

4. Associated varieties for $(\mathfrak{g}, K)$-modules

With the structure of $\mathcal{N}^*_\theta$ from Section 1, and the generalities about equivariant K-theory from Section 3, we can now introduce the K-theory functor we will actually consider (in place of the one from (0.2) that we do not know how to define). As a replacement for the moderate growth representations of (0.2a), we will use

$$\mathcal{M}_f(\mathfrak{g}, K) = \text{category of finite length } (\mathfrak{g}, K)\text{-modules.} \tag{4.1a}$$

This category has a nice Grothendieck group $K(\mathfrak{g}, K)$, which is a free $\mathbb{Z}$-module with basis the equivalence classes of irreducible modules. Best of all, the Casselman-Wallach theorem of [25, 11.6.8] says that passage to $K(\mathbb{R})$-finite vectors is an equivalence of categories

$$\mathcal{F}_{\mathrm{mod}}(G(\mathbb{R})) \xrightarrow{\sim} \mathcal{M}_f(\mathfrak{g}, K). \tag{4.1b}$$

We write

$$[X]_{(\mathfrak{g},K)} = \text{class of } X \in K(\mathfrak{g}, K). \tag{4.1c}$$

Any such X admits a good filtration (far from unique), so that we can construct

$$\operatorname{gr} X \in \operatorname{Coh}^K(\mathcal{N}_\theta^*).$$

(The reason that the $S(\mathfrak{g})$ module $\operatorname{gr} X$ is supported on $\mathcal{N}_\theta^*$ is explained in [21, Corollary 5.13].) The class $[\operatorname{gr} X] \in K^K(\mathcal{N}_\theta^*)$ is independent of the choice of good filtration (this standard fact is proven in [21, Proposition 2.2]) so we may write it simply as $[X]_\theta$. In this way we get a well-defined homomorphism

$$\operatorname{gr}\colon K(\mathfrak{g}, K) \to K^K(\mathcal{N}_\theta^*), \qquad [X]_{(\mathfrak{g},K)} \mapsto [\operatorname{gr} X] = [X]_\theta. \tag{4.1d}$$

This is our replacement for the map (0.2c) that we do not know how to define. Here are replacements for the undefined ideas in (0.6).

Definition 4.2. Fix a *nonzero* $(\mathfrak{g}, K)$-module

$$X \in \mathcal{M}_f(\mathfrak{g}, K). \tag{4.2a}$$

Then $\operatorname{gr} X$ is a *nonzero* K-equivariant coherent sheaf on $\mathcal{N}_\theta^*$, and as such has a well-defined nonempty *support*

$$\operatorname{supp}(\operatorname{gr} X) \subset \mathcal{N}_\theta^* \tag{4.2b}$$

which is a Zariski-closed union of K-orbits. This support is also called the *associated variety* of X,

$$\begin{aligned} \mathrm{AV}(X) &=_{\mathrm{def}} \operatorname{supp}(\operatorname{gr}(X)) \\ &=_{\mathrm{def}} (\text{variety of the ideal } \operatorname{Ann}(\operatorname{gr}(X)) \subset \mathcal{N}_\theta^*. \end{aligned} \tag{4.2c}$$

(More details about the commutative algebra definition of support are recalled for example in [21, (1.2)].) Theorem 3.5 provides a formula

$$\mathrm{gr}(X) = \sum_{Z=K\cdot E_Z\subset \mathrm{AV}(X)} \sum_{\tau_Z^j\in\widehat{K^Z}} m_{\tau_Z^j}(X)[\widetilde{\mathcal{V}_{\tau_Z^j}}] \tag{4.2d}$$

If we write

$$\{Y_1, Y_2, \ldots, Y_r\}, \qquad Y_i \simeq K/K_i \tag{4.2e}$$

for the open K orbits in $\mathrm{AV}(X)$, then [21] shows that each virtual representation

$$\mu_{Y_i}(X) = \sum_j m_{\tau_{Y_i}^j(X)} \tau_{Y_i}^j \in R(K_i) \tag{4.2f}$$

is independent of the choices defining $\widetilde{\mathcal{V}_{\tau_Z^j}}$. We define the *associated cycle of* X to be

$$\mathcal{AC}(X) = \sum_i \mu_{Y_i}(X) Y_i. \tag{4.2g}$$

The *weak associated cycle of* X is

$$\mathcal{AC}_{\mathrm{weak}}(X) = \sum_i \dim \mu_{Y_i}(X) Y_i. \tag{4.2h}$$

Although we will not need it (since we are going to use supp *instead of* $\mathrm{supp}_{\mathbb{R}}$), it is perhaps reassuring to recall

Theorem 4.3. *Schmid-Vilonen* [16, Theorem 1.4] *Suppose that* (π, V) *is a nonzero* $\mathfrak{Z}(\mathfrak{g})$*-finite representation of* $G(\mathbb{R})$ *of moderate growth (see* (0.2a)*). Write* $X = V_{K(\mathbb{R})}$ *for the underlying Harish-Chandra module (see* (4.1b)*). Then*

$$\mathrm{WF}(\pi) \longleftrightarrow \mathrm{supp}(\mathrm{gr}\, X)$$

by means of the Kostant-Sekigichi identification Corollary 1.10 (of $G(\mathbb{R})$ *orbits on* $\mathcal{N}_{\mathbb{R}}^*$ *with* K *orbits on* $\mathcal{N}_\theta^*$*).*

At this point it should be possible for the reader to rewrite the introduction, replacing the equivariant K-theory for $G(\mathbb{R})$ acting on $\mathcal{N}_{\mathbb{R}}^*$ (which we do not understand) by Thomason's algebraic equivariant K-theory for K

acting on $\mathcal{N}_\theta^*$; and replacing wavefront sets and cycles by associated varieties and associated cycles. We will not do that explicitly.

Restriction to K is a fundamental tool for us, and we pause here to introduce a bit of useful formalism about that. Define

$$\mathcal{M}_f(K) = \text{category of admissible algebraic } K\text{-modules}; \tag{4.4a}$$

as usual *admissible* means that each irreducible representation of K appears with finite multiplicity. This abelian category has a nice Grothendieck group

$$K(K) = \prod_{(\rho, E_\rho)\in \widehat{K}} \mathbb{Z}\rho, \tag{4.4b}$$

the direct *product* of one copy of $\mathbb{Z}$ for each irreducible representation of K. A $(\mathfrak{g}, K)$-module X of finite length is necessarily admissible:

$$X|_K = \sum_{(\rho, E_\rho)\in \widehat{K}} m(\rho, X) E_\rho \qquad (m(\rho, X) \in \mathbb{N}). \tag{4.4c}$$

Restriction to K is therefore an exact functor

$$\mathrm{res}_K \colon \mathcal{M}_f(\mathfrak{g}, K) \to \mathcal{M}_f(K). \tag{4.4d}$$

The corresponding homomorphism of Grothendieck groups is

$$\mathrm{res}_K \colon K(\mathfrak{g}, K) \to K(K), \qquad [X]_{(\mathfrak{g},K)} \mapsto \prod_{(\rho, E_\rho)\in \widehat{K}} m(\rho, X)\rho. \tag{4.4e}$$

In exactly the same way, the fact that K is reductive implies that any irreducible representation of K must appear with finite multiplicity in global sections of an equivariant coherent sheaf on a homogeneous space for K. If K acts on an affine variety Z with finitely many orbits, restriction to K is therefore an exact functor

$$\mathrm{res}_K \colon \mathrm{Coh}^K(Z) \to \mathcal{M}_f(K). \tag{4.4f}$$

The corresponding homomorphism of Grothendieck groups is

$$\mathrm{res}_K \colon K^K(Z) \to K(K). \tag{4.4g}$$

The maps (4.4e) and (4.4g) fit into a commutative diagram

$$\begin{tikzcd} K(\mathfrak{g},K) \arrow[rr, "{[\mathrm{gr}]}"] \arrow[dr, "\mathrm{res}_K"'] & & K^K(\mathcal{N}^*_\theta) \arrow[dl, "\mathrm{res}_K"] \\ & K(K) & \end{tikzcd} \tag{4.4h}$$

5. The case of complex reductive groups

The general case of our results requires a discussion of the (rather complicated) Langlands classification of representations of real reductive groups. In order to explain our new ideas, we will therefore first consider them in the (less complicated) setting of *complex* reductive groups. Suppose therefore (still using the notation of (1.1)) that there is a complex connected reductive algebraic group G_1, and that

$$G(\mathbb{R}) \simeq G_1. \tag{5.1a}$$

Fix a compact real form

$$\sigma_1 \colon G_1 \to G_1, \qquad G_1(\mathbb{R}, \sigma_1) =_{\mathrm{def}} G_1^{\sigma_1} = K_1 \subset G_1. \tag{5.1b}$$

Then we can arrange

$$\begin{aligned} G &= G_1 \times G_1 \\ \sigma_0(g,g') &= (\sigma_1(g), \sigma_1(g')), \qquad G^{\sigma_0} = K_1 \times K_1 \\ \sigma_{\mathbb{R}}(g,g') &= (\sigma_1(g'), \sigma_1(g)), \\ G(\mathbb{R}) &= \{(g, \sigma_1(g)) \mid g \in G_1\} \simeq G_1 \\ \theta(g,g') &= (g', g) \\ K &= (G_1)_\Delta = \{(g,g) \mid g \in G_1\} \\ K(\mathbb{R}) &= (K_1)_\Delta = \{(k,k) \mid k \in K_1\} \simeq K_1. \end{aligned} \tag{5.1c}$$

(The notation K_1 may be a bit confusing because in general we write K for the *complex* group which is the complexification of the maximal compact $K(\mathbb{R}) \subset G(\mathbb{R})$; but here K_1 is a compact (real) group. We have not found a reasonable change of notation to address this issue.)

We fix also a (compact) maximal torus

$$T_1 \subset K_1; \tag{5.1d}$$

then automatically its complexification

$$H_1 = G_1^{T_1} \tag{5.1e}$$

is a (complex) maximal torus in G_1. We write

$$W_1 = N_{K_1}(T_1)/T_1 \simeq N_{G_1}(H_1)/H_1 \tag{5.1f}$$

for the Weyl group. Complexifications of these things are

$$\begin{aligned} H &= H_1 \times H_1 = \text{complexification of } H_1 \\ W &= W_1 \times W_1 = \text{Weyl group of } H \text{ in } G. \end{aligned} \tag{5.1g}$$

We fix also a Borel subgroup $B_1 \supset H_1$ of G_1. Because σ_1 preserves T_1, and K_1 is compact, the Borel subgroup $\sigma_1(B_1)$ is necessarily equal to B_1^{opp}. Therefore the complexification of B_1 is

$$B_{qs} = B_1 \times \sigma_1(B_1) = B_1 \times B_1^{\mathrm{opp}}, \tag{5.1h}$$

corresponding to the real Borel subgroup B_1 for the quasisplit $G(\mathbb{R}) = G_1$. (The subscript qs stands for "quasisplit." We are also interested in the θ-stable Borel subgroup

$$B_f = B_1 \times B_1; \tag{5.1i}$$

now the subscript f stands for "fundamental."

We turn next to a discussion of nilpotent orbits in the complex case. We write $\mathcal{N}_1^*$ for the nilpotent cone in $\mathfrak{g}_1^*$, and use other notation accordingly. Then

$$\begin{aligned} \mathcal{N}^* &= \mathcal{N}_1^* \times \mathcal{N}_1^* \\ \mathcal{N}_{i\mathbb{R}}^* &= \{(E, -\sigma_1(E)) \mid E \in \mathcal{N}_1^*\} \simeq \mathcal{N}_1^* \\ \mathcal{N}_\theta^* &= \{(E, -E)) \mid E \in \mathcal{N}_1^*\} \simeq \mathcal{N}_1^* \end{aligned} \tag{5.2a}$$

Immediately we get identifications of orbits

$$\begin{aligned} \mathcal{N}^*/G &= \mathcal{N}_1^*/G_1 \times \mathcal{N}_1^*/G_1 \\ \mathcal{N}_{i\mathbb{R}}^*/G(\mathbb{R}) &\simeq \mathcal{N}_1^*/G_1 \\ \mathcal{N}_\theta^*/K &\simeq \mathcal{N}_1^*/G_1 \end{aligned} \tag{5.2b}$$

and therefore $\mathcal{N}_{i\mathbb{R}}^*/G(\mathbb{R}) \simeq \mathcal{N}_\theta^*/K$. This last is the Kostant-Sekiguchi bijection of Corollary 1.10.

Clearly the (antiholomorphic) automorphism σ_1 of G_1 acts on the set $\mathcal{N}_1^*/G_1$ of nilpotent orbits. The Jacobson-Morozov theorem implies that this action is trivial. Here is why. The semisimple element D, whose class characterizes the orbit, belongs to $[\mathfrak{g}_1, \mathfrak{g}_1]$ and has real eigenvalues in the adjoint representation; so after conjugation we can arrange

$$D \in i\mathfrak{t}_1, \qquad \sigma_1(D) = -D.$$

The Jacobson-Morozov $SL(2)$ shows that D is conjugate to $-D$, so we have shown that σ_1 preserves the conjugacy class of D. Now apply Corollary 1.7.

It follows that $\sigma_{\mathbb{R}}$ acts on $\mathcal{N}_1^*/G_1 \times \mathcal{N}_1^*/G_1$ by interchanging the two factors. The nilpotent G-orbits preserved by this action are exactly the diagonal classes. What (5.2a) shows is that each such diagonal nilpotent class for G has a unique real form (and therefore, by Kostant-Sekiguchi, a unique θ-form).

In light of this identification of the $K = G_1$ action on $\mathcal{N}_\theta^*$ with the G_1 action on $\mathcal{N}_1^*$, we can restate Definition 1.13 (in this complex case) as follows.

> **Definition 5.3.** A *(global) geometric parameter* for a complex reductive algebraic group G_1 is a nilpotent G_1-orbit $Y \subset \mathcal{N}_1^*$, together with an irreducible G_1-equivariant vector bundle
>
> $$\mathcal{E} \to Y.$$
>
> Equivalently, a *(local) geometric parameter* is a G_1-conjugacy class of pairs
>
> $$(\xi, (\tau, E)),$$
>
> with $\xi \in \mathcal{N}_1^*$ a nilpotent element, and (τ, E) an irreducible (algebraic) representation of the isotropy group G_1^ξ. This bijection between local and global parameters identifies $(\xi, (\tau, E))$ with the pair
>
> $$Y = G_1 \cdot \xi \simeq G_1/G_1^\xi, \qquad \mathcal{E} \simeq G_1 \times_{G_1^\xi} E.$$
>
> We write $\mathcal{P}_g(G_1)$ for the collection of geometric parameters. Sometimes it will be convenient to write $\mathcal{E}(\tau)$ or $\mathcal{E}(\xi, \tau)$ to exhibit the underlying local parameter.

These geometric parameters are exactly what appears on the complicated side of the Lusztig-Bezrukavnikov bijection [7] for G_1. Lusztig's conjecture, and its proof by Bezrukavnikov, were critical to the development of the ideas in this paper. But they are not logically necessary to explain our results, so for brevity we are going to omit them.

The introduction (after reformulation in terms of K acting on $\mathcal{N}^*_\theta$) outlined a connection between the geometric parameters of Definitions 1.13 and 5.3 and the associated varieties we seek to compute. We conclude this section with an account of the Langlands classification, which describes representations of G_1 in terms that we will be able to relate to geometric parameters.

Definition 5.4. In the setting of (5.1), a *Langlands parameter* for G_1 (or for $(G,K) = (G_1 \times G_1, (G_1)_\Delta)$) is a pair of linear functionals

$$(\lambda_L, \lambda_R) \in \mathfrak{h}_1^* \times \mathfrak{h}_1^* \simeq \mathfrak{h}^*,$$

subject to the requirement that the restriction of (λ_L, λ_R) to $\mathfrak{t}_1 = (\mathfrak{h}_1)_\Delta$ is the differential of a weight:

$$\lambda_L + \lambda_R = \gamma \in X^*(H_1).$$

Two Langlands parameters are said to be *equivalent* if they are conjugate by $(W_1)_\Delta$:

$$(\lambda_L, \lambda_R) \sim (\lambda'_L, \lambda'_R) \iff (\lambda_L, \lambda_R) = (w_1 \cdot \lambda'_L, w_1 \cdot \lambda'_R)$$

for some $w_1 \in W_1$. We write $\mathcal{P}_{\mathrm{L}}(G,K)$ for the set of equivalence classes of Langlands parameters. The *discrete part* of the Langlands parameter is by definition

$$\gamma = \gamma(\lambda_L, \lambda_R) = \lambda_L + \lambda_R \in X^*(H_1).$$

Tbe *continuous part* of the parameter is

$$\nu = \nu(\lambda_L, \lambda_R) = \lambda_L - \lambda_R \in \mathfrak{h}_1^*.$$

We can recover the parameter from these two parts:

$$\lambda_L = (\gamma + \nu)/2, \qquad \lambda_R = (\gamma - \nu)/2.$$

Equivalence is easily written in terms of the discrete and continuous parameters:

$$(\lambda_L, \lambda_R) \sim (\lambda'_L, \lambda'_R) \iff (\gamma, \nu) = (w_1 \cdot \gamma', w_1 \cdot \nu') \quad (w_1 \in W_1)$$

(with obvious notation).

The Langlands classification (due in this case to Zhelobenko) attaches to each equivalence class of parameters a *standard representation* $I(\lambda_L, \lambda_R)$ (more precisely, a Harish-Chandra module for

$$(\mathfrak{g}, K) = (\mathfrak{g}_1 \times \mathfrak{g}_1, (G_1)_\Delta))$$

with the following properties.

1. There is a unique irreducible quotient $J(\lambda_L, \lambda_R)$ of $I(\lambda_L, \lambda_R)$.
2. Any irreducible Harish-Chandra module for $(\mathfrak{g}, K)$ is equivalent to some $J(\lambda_L, \lambda_R)$.
3. Two standard representations are isomorphic (equivalently, their Langlands quotients are isomorphic) if and only if their parameters are conjugate by W_1.
4. The infinitesimal character of the standard representation $I(\lambda_L, \lambda_R)$ is indexed by the $W = W_1 \times W_1$ orbit of (λ_L, λ_R).
5. The restriction of $I(\lambda_L, \lambda_R)$ to K_1 is the induced representation from T_1 to K_1 of $\gamma = \lambda_L + \lambda_R$:

$$I(\lambda_L, \lambda_R) \simeq \operatorname{Ind}_{T_1}^{K_1}(\lambda_L + \lambda_R).$$

6. The restrictions to K_1 of two standard representations are isomorphic if and only if the weights $\lambda_L + \lambda_R$ and $\lambda'_L + \lambda'_R$ are conjugate by W_1.

The representation $I(\lambda_L, \lambda_R)$ is *tempered* (an analytic condition due to Harish-Chandra, and central to Langlands' original work) if and only if the continuous parameter is purely imaginary:

$$\nu = \lambda_L - \lambda_R \in iX^*(H_1) \otimes_{\mathbb{Z}} \mathbb{R}.$$

In this case $I(\lambda_L, \lambda_R) = J(\lambda_L, \lambda_R)$.

In the general (possibly nontempered) case, the real part of ν controls the growth of matrix coefficients of $J(\lambda_L, \lambda_R)$. Tempered representations have the smallest possible growth, and larger values of $\operatorname{Re}\nu$ correspond to larger growth rates.

A K-Langlands parameter is the same thing, but with a larger equivalence relation. (In what follows, remember that K is the complexification of K_1: locally finite continuous representations of the compact group K_1 are the same as algebraic representations of the complex algebraic group K.)

Definition 5.5. A *K-Langlands parameters* for (G, K) is any of the following equivalent things.

1. A Langlands parameter (λ_L, λ_R). The equvalence relation is

$$(\lambda_L, \lambda_R) \sim_K (\lambda'_L, \lambda'_R) \iff I(\lambda'_L, \lambda'_R)|_K = I(\lambda_L, \lambda_R)|_K.$$

   The equivalence relation is throwing away the continuous parameter $\nu(\lambda_L, \lambda_R)$. Another way to state this is

$$(\lambda_L, \lambda_R) \sim_K (\lambda'_L, \lambda'_R) \iff \lambda_L + \lambda_R \in W_1 \cdot (\lambda'_L + \lambda'_R).$$

2. A tempered parameter $(\gamma/2, \gamma/2)$ (some $\gamma \in X^*(H_1)$) having real infinitesimal character (see [20, Definition 5.4.11]). The equivalence relation is

$$(\gamma/2, \gamma/2) \sim_K (\gamma'/2, \gamma'/2) \iff \gamma \in W_1 \cdot \gamma'.$$

3. A W_1 orbit of weights $\gamma \in X^*(H_1)$.
4. A dominant weight $\gamma_0 \in X^*(H_1)^+$.

The equivalence of the four conditions is standard and easy. The tempered parameter of real infinitesimal character

$$((\lambda_L + \lambda_R)/2, (\lambda_L + \lambda_R)/2)$$

is a natural representative for the K-equivalence class.

If $\gamma \in X^*(T)$ is any weight (and $\gamma_0 = w\gamma$ is its unique dominant conjugate), then the "fixed restriction to K" in (1) is

$$\mathrm{Ind}_{T_1}^{K_1}(\gamma) \simeq \mathrm{Ind}_{T_1}^{K_1}(\gamma_0).$$

Write $\mathcal{P}_{K\text{-L}}(G, K)$ for equivalence classes of K-Langlands parameters.

The conjecture of Lusztig proved by Bezrukavnikov in [7] is a *bijection* between the geometric parameters of Definition 5.3 and the K-Langlands parameters of Definition 5.5. Bezrukavnikov proceeds by using these two sets to index two bases of the same $\mathbb{Z}$-module $K^K(\mathcal{N}_\theta^*)$. He proves that his change of basis matrix is upper triangular, and in this way establishes the bijection between the index sets. He does not offer a method to *calculate* his basis indexed by geometric parameters.

Following Achar, we will use geometric parameters to index a different basis of the same vector space, and we will calculate the change of basis matrix. We are not able to prove that our basis is the same as Bezrukavnikov's.

Here are some properties of K-Langlands parameters that we will use in Section 6 to construct the basis indexed by geometric parameters.

Theorem 5.6. *Suppose* (λ_L, λ_R) *represents a* K*-Langlands parameter for the complex group* G *(Definition 5.5). Write*

$$\gamma = \lambda_L + \lambda_R \in X^*(H_1) \simeq X^*(T_1)$$

for the corresponding weight of the compact torus (5.1d).

1. *The restriction to* K_1 *of* $I(\lambda_L, \lambda_R)$ *contains the irreducible representation*

$$\mu(\lambda_L, \lambda_R) = \textit{irreducible of extremal weight } \gamma$$

with multiplicity one. The other irreducible representations of K_1 *appearing have strictly larger extremal weights.*

2. *The map*

$$\mu\colon \mathcal{P}_{K\text{-L}}(G,K) \to \widehat{K_1}, \qquad (\lambda_L, \lambda_R) \mapsto \mu(\lambda_L, \lambda_R)$$

is a bijection.

3. *The classes*

$$[\mathrm{res}_K\, I(\lambda_L, \lambda_R)] \in K(K) \qquad (\lambda_L, \lambda_R) \in \mathcal{P}_{K\text{-L}}(G,K)$$

(see (4.4e)*) are linearly independent.*

6. Representation basis for K-theory: $\mathbb{C}$ case

We continue in the setting (5.1), that is, assuming that (G,K) arises from a *complex* reductive group.

Applying Theorem 3.5 to K acting on the K-nilpotent cone gives

Corollary 6.1. *Write* $\{Y_1, \ldots, Y_r\}$ *for the orbits of* K *on* $\mathcal{N}_\theta^*$ *(*(1.3c)*). For each irreducible* K*-equivariant vector bundle* $\mathcal{E}$ *on some* Y_i*, fix a* K*-equivariant (virtual) coherent sheaf* $\widetilde{\mathcal{E}}$ *supported on the closure of* Y_i*, and restricting to* $\mathcal{E}$ *on* Y_i*. Then the classes* $[\widetilde{\mathcal{E}}]$ *are a* $\mathbb{Z}$*-basis of* $K^K(\mathcal{N}_\theta^*)$*. A little more precisely, the classes supported on any* K*-invariant closed subset* $Z \subset \mathcal{N}_\theta$ *are a basis of* $K^K(Z)$*.*

In order to describe the algorithm underlying Theorem 1.6, we need an entirely different kind of basis of $K^K(\mathcal{N}_\theta^*)$.

Definition 6.2. Suppose $\gamma \in \mathcal{P}_{K\text{-L}}(G,K)$ is a K-Langlands parameter for $(G,K) = (G_1 \times G_1, (G_1)_\Delta)$ (Definition 5.5); equivalently, a dominant weight for G_1). We attach to γ a K-equivariant coherent sheaf on the K-nilpotent cone $\mathcal{N}_\theta^*$, with well-defined image $[\gamma]_\theta \in K^K(\mathcal{N}_\theta^*)$, characterized in any of the following equivalent ways. Note first that the cotangent bundle of G_1/B_1 is

$$T^*(G_1/B_1) = G_1 \times_{B_1} (\mathfrak{g}_1/\mathfrak{b}_1)^* \xrightarrow{\pi_1} G_1/B_1. \tag{6.2a}$$

From the total space of the tangent bundle there is the *moment map*

$$T^*(G_1/B_1) \xrightarrow{\mu_1} \mathfrak{g}_1^*, \qquad \mu_1(g,\xi) = \mathrm{Ad}^*(g)(\xi) \quad (g \in G_1,\ \xi \in (\mathfrak{g}_1/\mathfrak{b}_1)^*). \tag{6.2b}$$

Of course π_1 is affine, μ_1 is proper, and both maps are G_1-equivariant; the image of μ_1 is the nilpotent cone $\mathcal{N}_1^*$.

1. Fix any Langlands parameter (λ_L, λ_R) restricting to γ; that is, elements λ_L and λ_R of $\mathfrak{h}_1^*$ such that $\gamma = \lambda_L + \lambda_R$. Fix a good filtration on the standard module $I(\lambda_L, \lambda_R)$, and define

$$[\gamma]_\theta = [\mathrm{gr}\, I(\lambda_L, \lambda_R)] = [I(\lambda_L, \lambda_R)]_\theta \tag{6.2c}$$

(see (4.1d)).

2. Extend γ to a one-dimensional (algebraic) character of $B_1 = H_1 N_1$, and let $\mathcal{L}_1$ be the corresponding G_1-equivariant (algebraic) line bundle on G_1/B_1. The pullback $\pi_1^*\mathcal{L}_1$ is a G_1-equivariant coherent sheaf on $T^*(G_1/B_1)$, so each higher direct image $R^k\mu_1^*(\pi_1^*\mathcal{L}_1)$ is (since μ_1 is proper) a G_1-equivariant coherent sheaf on $\mathcal{N}_1^*$. We define

$$[\gamma]_{1,\theta} = \sum_k (-1)^k [R^k\mu_1^*(\pi_1^*\mathcal{L}_1)] \in K^{G_1}(\mathcal{N}_1^*). \tag{6.2d}$$

Under (5.2a), $[\gamma]_{1,\theta}$ corresponds to a class $[\gamma]_\theta \in K^K(\mathcal{N}_\theta^*)$.

3. As an algebraic representation of K,

$$[\gamma]_\theta = \mathrm{Ind}_T^K \gamma = \sum_{(\rho, E_\rho) \in \widehat{K}} \dim E_\rho(\gamma)\rho; \tag{6.2e}$$

here $E_\rho(\gamma)$ denotes the γ weight space (with respect to the maximal torus T) of the K-representation E_ρ.

The equivalence of (1) and (2) is a consequence of Zuckerman's cohomological induction construction of $I(\lambda_L, \lambda_R)$ using the θ-stable Borel subgroup B_f of (5.1i). That they have the property in (3) is a standard fact about principal series representations of complex groups. That property (3) characterizes $[\gamma]_\theta$ is a consequence of Corollary 6.4 below.

Proposition 6.3. *Suppose $(Y, \mathcal{E}) \in \mathcal{P}_g(G, K)$ is a geometric parameter (Definition 5.3). Then there is an extension $\widetilde{\mathcal{E}}$ as in Corollary 6.1, and a formula in $K^K(\mathcal{N}_\theta)$*

$$[\widetilde{\mathcal{E}}] = \sum_{\gamma \in \mathcal{P}_{K\text{-L}}(G,K)} m_{\widetilde{\mathcal{E}}}(\gamma)[\gamma]_\theta.$$

Here the sum is finite, and $m_{\widetilde{\mathcal{E}}}(\gamma) \in \mathbb{Z}$.

Suppose $\widetilde{\mathcal{E}}'$ is another extension of $\mathcal{E}$ to $\overline{Y}$. Then

$$[\widetilde{\mathcal{E}}'] - [\widetilde{\mathcal{E}}] = \sum_{\substack{(Z,\mathcal{F}) \in \mathcal{P}_g(G,K) \\ Z \subset \partial Y}} n_{\mathcal{F}}[\widetilde{\mathcal{F}}].$$

Only finitely many terms appear in the sum, and $n_{\mathcal{F}} \in \mathbb{Z}$.

We will give a proof in Corollary 7.4.

Corollary 6.4. *In the setting of* (5.1)*, the classes*

$$\{[\gamma]_\theta \in K^K(\mathcal{N}_\theta^*) \mid \gamma \in \mathcal{P}_{K\text{-L}}(G, K)\}$$

are a $\mathbb{Z}$-basis of $K^K(\mathcal{N}_\theta^*)$. The restriction to K map

$$\mathrm{res}_K \colon K^K(\mathcal{N}_\theta^*) \to K(K)$$

of (4.4g) *is injective.*

Proof. That these classes span is a consequence of Corollary 6.1 and Proposition 6.3. That they are linearly independent is a consequence of Theorem 5.6(4); the argument proves injectivity of the restriction at the same time. □

7. Geometric basis for K-theory: $\mathbb{C}$ case

In this section we consider how to relate the representation-theoretic basis of Corollary 6.4 to the geometry of K-orbits on $\mathcal{N}_\theta^*$. will explain how to compute one extension of a We will proceed in the aesthetically undesirable way of using the Jacobson-Morozov theorem (and so discussing not the nilpotent elements in $\mathfrak{g}^*$ that we care about, but rather the nilpotent elements in $\mathfrak{g}$). We begin therefore with an arbitrary nilpotent element $X_1 \in \mathcal{N}_1$ (see (1.3a)). The Jacobson-Morozov theorem finds elements Y_1 and D_1 in $\mathfrak{g}_1$ so that

$$[D_1, X_1] = 2X_1, \quad [D_1, Y_1] = -2Y_1, \quad [X_1, Y_1] = D_1. \tag{7.1a}$$

We use the eigenspaces of $\mathrm{ad}(D_1)$ to define a $\mathbb{Z}$-grading of $\mathfrak{g}_1$

$$\begin{aligned} \mathfrak{g}_1[k] &=_{\mathrm{def}} \{Z \in \mathfrak{g}_1 \mid [D_1, Z] = kZ\}, \quad & \mathfrak{g}_1[\geq k] &=_{\mathrm{def}} \sum_{j \geq k} \mathfrak{g}_1[j], \\ \mathfrak{q}_1 &=_{\mathrm{def}} \mathfrak{g}_1[\geq 0], & \mathfrak{u}_1 &=_{\mathrm{def}} \mathfrak{g}_1[\geq 1]. \end{aligned} \tag{7.1b}$$

Then $\mathfrak{q}_1$ is the Lie algebra of a parabolic subgroup $Q_1 = L_1 U_1$ of G_1, with Levi factor $L_1 = G_1^{D_1}$. After conjugating X by G_1, we may assume that

$$D_1 \in \mathfrak{t}_1, \qquad T_1 \subset L_1. \tag{7.1c}$$

We will be concerned with the equivariant vector bundle

$$\mathcal{R}_1 =_{\mathrm{def}} G_1 \times_{Q_1} \mathfrak{g}_1[\geq 2] \xrightarrow{\pi} G_1/Q_1. \tag{7.1d}$$

(The reason $\mathcal{R}_1$ is of interest is that Corollary 7.4 below says that it is a G_1-equivariant resolution of singularities of the nilpotent orbit closure $\overline{G_1 \cdot X_1}$. The $\mathcal{R}$ is meant to stand for *resolution*.) According to (3.3i) and (3.3f),

$$K^{G_1}(\mathcal{R}_1) \simeq R(Q_1) \simeq R(L_1); \tag{7.1e}$$

If (σ, S) is an irreducible representation of $L_1 \simeq Q_1/U_1$, we write

$$\mathcal{S}_1(\sigma) = G_1 \times_{Q_1} S \tag{7.1f}$$

for the induced vector bundle on G_1/Q_1. The corresponding basis element of the equivariant K-theory is represented by the equivariant vector bundle

$$\mathcal{S}(\sigma) = \pi^*(\mathcal{S}_1(\sigma)) = G_1 \times_{Q_1} (\mathfrak{g}_1[\geq 2] \times S) \to \mathcal{R}_1. \tag{7.1g}$$

We are in the setting of Proposition 1.6. As a consequence of that Proposition, we have

Corollary 7.2. *Suppose we are in the setting* (7.1).

1. *The natural map*

$$\mu\colon \mathcal{R}_1 \to \mathcal{N}_1, \quad (g, Z) \mapsto \mathrm{Ad}(g)Z$$

is a proper birational map onto $\overline{G_1 \cdot X_1}$*. We may therefore* identify $G_1 \cdot X_1$ *with its preimage* U*:*

$$G_1/G_1^{X_1} \simeq G_1 \cdot X_1 \simeq U \subset \mathcal{R}_1.$$

Because $G_1 \cdot X_1$ *is open in* $\overline{G_1 \cdot X_1}$, $U = \mu^{-1}(G_1 \cdot X_1)$ *is open in* $\mathcal{R}_1$*.*

2. *The classes*

$$\{[\mathcal{S}(\sigma)] \mid \sigma \in \widehat{L_1} = \widehat{Q_1}\}$$

of (7.1g) *are a basis of the equivariant* K*-theory* $K^{G_1}(\mathcal{R}_1)$*.*

3. *Since* μ *is proper, higher direct images of coherent sheaves are always coherent. Therefore*

$$[\mu_*(\mathcal{S})] =_{\mathrm{def}} \sum_i (-1)^i [R^i \mu_* \mathcal{S}] \in K^{G_1}(\mathcal{N}_1)$$

is a well-defined virtual coherent sheaf. This defines a map in equivariant K*-theory*

$$\mu_*\colon K^{G_1}(\mathcal{R}_1) \to K^{G_1}(\mathcal{N}_1).$$

Restriction to the open set $U \simeq G_1 \cdot X_1$ *commutes with* μ_**.*

4. *Suppose* σ *is a representation of* L_1*, inducing vector bundles* $\mathcal{S}_1(\sigma)$ *on* G_1/Q_1 *and* $\mathcal{S}(\sigma)$ *on* $\mathcal{R}_1$ *as in* (7.1a)*. As a representation of* G_1*, (that is, in the Grothendieck group* $K(G_1)$*; see* (4.4a)*)*

$$\begin{aligned}
[\mu_*(\mathcal{S}(\sigma))] &= \mathrm{Ind}_{L_1}^{G_1}\left(\sigma \otimes \Big(\sum_j (-1)^j {\textstyle\bigwedge}^j \mathfrak{g}_1[1]^*\Big)\right) \\
&= \mathrm{Ind}_{T_1}^{G_1}\Bigg(\sum_\phi m_\sigma(\phi) \sum_{\substack{A \subset \Delta(\mathfrak{g}_1[1], \mathfrak{t}_1) \\ w \in W(L_1)}} (-1)^{|A| + \ell(w)} [\phi - 2\rho(A) + (\rho_{L_1} - w\rho_{L_1})]\Bigg) \\
&= \mathrm{Ind}_{T_1}^{G_1}\Bigg(\sum_\phi m_\sigma(\phi) \sum_{\substack{A \subset \Delta(\mathfrak{g}_1[1], \mathfrak{t}_1) \\ B \subset \Delta^+(\mathfrak{l}_1, \mathfrak{t}_1)}} (-1)^{|A| + |B|} [\phi - 2\rho(A) + 2\rho(B)]\Bigg).
\end{aligned}$$

Here the outer sum is over the highest weights ϕ *of the virtual representation* σ *of* L_1*, and the integers* $m_\sigma(\phi)$ *are their multiplicities. The notation* $2\rho(X)$ *stands for the sum of a set* X *of roots.*

5. *If every weight $\phi - 2\rho(A) + 2\rho(B)$ in (7) is replaced by its unique dominant conjugate, we get a computable formula*

$$[\mu_*(\mathcal{S}(\sigma))] = \sum_{\gamma \in \mathcal{P}_{K\text{-L}}(G,K)} m_\sigma(\gamma)[\gamma]_\theta.$$

Proof. For (1), that μ is proper is immediate from the fact that G_1/Q_1 is projective. That $G_1 \cdot X_1$ is open in the image follows from Proposition 1.6(6). Since μ is proper and the domain is irreducible, the image must be the closure of $G_1 \cdot X_1$. The fiber over X_1 is evidently $G^{X_1}/(G^{X_1} \cap Q_1)$, which is a single point by Proposition 1.6(4). So the map is birational.

Part(2) is (3.3f) and (3.3i).

Part (3) is standard algebraic geometry.

For (4), the Leray spectral sequences for the maps μ and π say that

$$\begin{aligned}
\sum_i (-1)^i H^0(\overline{G_1 \cdot X_1}, R^i\mu_*\mathcal{S}(\sigma)) &= \sum_i (-1)^i H^i\left(\mathcal{R}_1, \mathcal{S}(\sigma)\right) \\
&= \sum_i (-1)^i H^i\left(G_1/Q_1, G_1 \times_{Q_1} (\sigma \otimes S^\bullet(\mathfrak{g}_1[\geq 2])^*)\right) \\
&= \sum_{i,j} (-1)^{i+j} H^i\left(G_1/Q_1, G_1 \times_{Q_1} \sigma \otimes \textstyle\bigwedge^j \mathfrak{g}_1[1]^* \otimes S^\bullet(\mathfrak{g}_1[\geq 1])^*\right) \\
&= \sum_j (-1)^j \operatorname{Ind}_{L_1}^{G_1}\left(\sigma \otimes \textstyle\bigwedge^j \mathfrak{g}_1[1]^*\right).
\end{aligned} \tag{7.3a}$$

For the first equality, we use the fact that $\overline{G_1 \cdot X_1}$ is affine, so that higher cohomology vanishes. For the second, we use the fact that π is affine, so the higher direct images vanish. For the third, we use the deRham complex identity (valid in $\mathcal{M}_f(G_1)$ (see (4.4a)) for any G_1-representation V such that $S^\bullet(V)$ decomposes into irreducibles with finite multiplicities)

$$[\mathbb{C}] = \sum_j (-1)^j [S^\bullet(V) \otimes \textstyle\bigwedge^j V],$$

applied to $V = \mathfrak{g}_1[1]^*$. The fourth equality more or less identifies functions (or sections of a vector bundle) on T^*G_1/Q_1 with functions (or sections of a vector bundle) on G_1/L_1. Now (7.3a) is the first equality in (4).

The second equality in (4) uses first of all the fact that if σ_0 is an irreducible of L_1 of highest weight ϕ_0, then

$$\sigma_0 = \sum_{w \in W(L_1)} (-1)^{\ell(w)} \operatorname{Ind}_{T_1}^{L_1}(\phi_0 + \rho_{L_1} - w\rho_{L_1}). \tag{7.3b}$$

(This is a version of the Weyl character formula.) Next, it uses the fact that if E is any representation of L_1 (in this case an exterior power of $\mathfrak{g}_1[1]^*$), then

$$\operatorname{Ind}_{T_1}^{L_1}(\psi) \otimes E = \sum_{\gamma \in \Delta(E)} \operatorname{Ind}_{T_1}^{L_1}(\psi + \gamma). \tag{7.3c}$$

Here the sum runs over the weights of T_1 on E.

The third equality in (4) follows for example from the Bott-Kostant fact that (if $\mathfrak{n}_1(L_1)$ corresponds to a set of positive roots for T_1 in $\mathfrak{l}_1$) the weights of T_1 on $H^*(\mathfrak{n}_1(L_1), \mathbb{C})$ are the various $\rho_{L_1} - w\rho_{L_1}$, appearing in degree $\ell(w)$. The sum on the left side runs over the weights of cohomology (indexed by $w \in W(L_1)$), and on the right over the weights of the complex $\bigwedge^\bullet \mathfrak{n}_1(L_1)^*$ (indexed by subsets B of positive roots for L_1). $\square$

Corollary 7.4. *We continue in the setting* (7.1)*.*

1. *The restriction map in equivariant K-theory (see* (3.3j)*)*
$$R(L_1) \simeq R(Q_1) \simeq K^{G_1}(\mathcal{R}_1) \to K^{G_1}(U) \simeq R(G_1^{X_1}) = R(Q_1^{X_1})$$
sends a (virtual) representation $[\sigma]$ of Q_1 to $[\sigma|_{Q_1^{X_1}}]$.
2. *Any virtual (algebraic) representation τ of $G_1^{X_1} = Q_1^{X_1}$ can be extended to a virtual (algebraic) representation σ of Q_1. That is, the restriction map of representation rings*
$$R(L_1) \simeq R(Q_1) \twoheadrightarrow R(Q_1^{X_1}) \simeq R(L_1^{X_1})$$
is surjective.
3. *Suppose $[\tau]$ is a virtual algebraic representation of $G_1^{X_1}$, corresponding to a virtual equivariant coherent sheaf $\mathcal{T}$ on $G_1 \cdot X_1$. Choose a virtual algebraic representation $[\sigma]$ of Q_1 extending τ. Then the virtual coherent sheaf*
$$[\mu_*(\mathcal{S}(\sigma))] =_{\mathrm{def}} [\widetilde{\mathcal{T}}]$$
is a virtual extension (Corollary 6.1) of $[\mathcal{T}]$. We have a formula
$$[\widetilde{\mathcal{T}}] = \sum_{\gamma \in \mathcal{P}_{K\text{-L}}(G,K)} m_{\widetilde{\mathcal{T}}}(\gamma)[\gamma]_\theta.$$

Computability *of the extension σ in (3) is a problem in finite-dimensional representation theory of reductive algebraic groups, for which we do* not *offer a general solution.*

Proof. For (1), that the restriction in K-theory corresponds to restriction of representations is clear from the description of representatives for the classes on $\mathcal{R}_1$ in (7.1g).

For (2), restriction to an open subset in equivariant K-theory is always surjective; this is the right exactness of (3.3j). So (2) follows from (1).

Part (3) follows from the last assertion of Corollary 7.2(3), and Corollary 7.2(5). □

The last formula in Corollary 7.4 relates the geometric basis of Theorem 3.5 to the representation-theoretic basis of Corollary 6.4. The difficulty, as mentioned in the Corollary above, is that computing this formula requires (for each irreducible representation τ of $L_1^{X_1}$) a *computable* virtual representation σ of L_1 with

$$\sigma|_{L_1^X} = \tau.$$

Finding such a σ does not seem to be an intractable problem. For $GL(n)$, Achar addresses it in his thesis [1]. For other classical G_1, a fairly typical example (arising for the nilpotent element in $Sp(2n)$ corresponding to the partition 2^n of $2n$) has

$$L_1 = GL(n, \mathbb{C}), \qquad L_1^{X_1} = O(n, \mathbb{C}).$$

But we are not going to address this branching problem. Instead, we will calculate not individual basis vectors $\widetilde{\mathcal{E}}(\tau)$, but rather a basis of the *span* of all these vectors as τ varies over $\widehat{G_1^{X_1}}$ (always for a fixed X_1).

For our application to calculating associated varieties, the price is that we can calculate the components of the associated variety and their multiplicities, but not the virtual representations of isotropy groups giving rise to those multiplicities.

What we gain for this price is an algorithm, which we have implemented in the `atlas` software (see [4]).

Algorithm 7.5 (A geometric basis for equivariant K-theory)**.** We begin in the setting (7.1) with a nilpotent orbit

$$Y = G_1 \cdot X_1 \subset \mathcal{N}_1 \simeq \mathcal{N}_1^* \simeq \mathcal{N}_\theta^*. \tag{7.6}$$

(Here we use the identifications of (5.2b) and (1.4).) The goal is to produce a collection of explicit elements

(7.6a)
$$\mathcal{E}_j^{\mathrm{orbalg}}(Y) = \sum_{\gamma \in \mathcal{P}_{K\text{-}\mathrm{L}}(G,K)} m_{\mathcal{E}_j^{\mathrm{orbalg}}(Y)}(\gamma)[\gamma]_\theta \in K^K(\overline{Y}) \quad (j = 0, 1, 2 \ldots)$$

which are a *basis* of $K^K(\overline{Y})/K^K(\partial\overline{Y})$. (The superscript "orbalg" stands for "orbital algorithm." The subscript j is just an indexing parameter for the basis vectors we compute, running over

$$\{0, 1, \dots M-1\} \qquad \text{or} \qquad \mathbb{N};$$

it has no particular meaning. It replaces the parameter τ (corresponding to the coherent sheaf $\mathcal{E}$) in the basis of Theorem 3.5. The algorithm proceeds by induction on $\dim Y$; so we assume that such a basis is available for every boundary orbit $Y' \subset \partial\overline{Y}$.

Start with an arbitrary (say irreducible) representation σ of L_1, and write down the formula of Corollary 1.7(6). Replace each inducing weight $\gamma \in X^*(T_1)$ by its dominant conjugate, so that one gets an expression

$$[\mu_*\mathcal{S}(\sigma)] = \sum_{\gamma \in \mathcal{P}_{K\text{-L}}(G,K)} m_\sigma(\gamma)[\gamma]_\theta. \tag{7.6b}$$

Notice also that

$$\operatorname{rank}([\mu_*\mathcal{S}(\sigma)] = \dim(\sigma), \tag{7.6c}$$

(as a representation of L_1), and this dimension is easy to compute.

According to Corollary 7.4, the classes $\{[\mu_*\mathcal{S}(\sigma)] \mid \sigma \in \widehat{L_1}\}$, after restriction to $K^{G_1}(Y)$, are a spanning set. Furthermore the kernel of the restriction map has as basis the (already computed) set

$$\bigcup_{Z \subset \partial Y} \{\mathcal{E}_k^{\text{orbalg}}(Z) \mid k = 0, 1, 2 \dots\}. \tag{7.6d}$$

Now extracting a subset

$$\mathcal{E}_j^{\text{orbalg}}(Y) = \sum_{\sigma \in \widehat{L_1}} n_j(\sigma)[\mu_*\mathcal{S}(\sigma)] \tag{7.6e}$$

of the span of the $[\mu_*\mathcal{S}(\sigma)]$ restricting to a basis of the image of the restriction is a linear algebra problem. Because the rank (the virtual dimension of fibers over $G_1 \cdot X_1$) is additive in the Grothendieck group, we can compute each integer

$$\operatorname{rank}([\mathcal{E}_j^{\text{orbalg}}]) = \sum_{\sigma \in \widehat{L_1}} n_j(\sigma) \dim(\sigma). \tag{7.6f}$$

In this description we have swept under the rug the issue of doing finite calculations. We will now address this. Recall from (1.4c) the invariant bilinear form $\langle , \rangle$ on $\mathfrak{g}$, and from Proposition 1.5 the fact that this form defines a positive definite on any character lattice $X^*(H)$, with $H \subset G$ a maximal torus. The same proof applies to non-maximal tori; so we get a positive definite form on highest weights for any reductive subgroup of G.

Lemma 7.7. *Suppose $E \subset F$ are algebraic subgroups of G (not necessarily connected), and that τ and σ are irreducible representations of E and F respectively. Define*

$$\|\tau\| = \textit{length of a highest weight of } \tau$$

and similarly for σ. If τ appears in $\sigma|_E$, then necessarily

$$\|\tau\| \leq \|\sigma\|.$$

Because an irreducible algebraic representation must be trivial on the unipotent radical, we may assume that E and F are reductive, so that the notion of "highest weight" makes sense. The lemma reduces immediately to the case when E and F are tori, and in that case is obvious.

If now $\mathcal{E}$ is a geometric parameter corresponding to an irreducible representation τ of G_1^{ξ}, we define

$$\|\mathcal{E}\| = \|\tau\|. \tag{7.8}$$

If γ is a dominant weight thought of as a Langlands parameter, we define

$$\|\gamma\| = \text{length of } \gamma \text{ as a weight.} \tag{7.9}$$

Proposition 7.10. *Suppose we are in the setting of Corollary 7.4.*

1. *Any formula for an extension of $(Y, \mathcal{E})$ must include a term γ for which the restriction of the G_1-representation of extremal weight γ contains the $G_1^{X_1}$ representation τ defining $\mathcal{E}$; and therefore*

$$\|\gamma\| \geq \|\tau\|.$$

2. *There is a constant C depending only on G so that, there is a virtual extension of τ to L_1 in which every L_1 highest weight γ_1 appearing satisfies*

$$\|\gamma_1\| \leq \|\tau\| + C.$$

3. *There is a constant C depending only on G so that, in the formula for the virtual extension of $\mathcal{E}$ given in Corollary 7.4, every K-Langlands parameter γ appearing satisfies*

$$\|\gamma\| \leq \|\tau\| + 2C.$$

Proof. The main assertion in Part (1) is elementary, and then the last inequality is Lemma 7.7. Part (2) is elementary but tedious; we omit the argument. Part (3) is clear by inspection of Corollary 7.4(7); the constant C in this case is a bound for the sizes of the various root sums appearing. □

Here now is how to make Algorithm 7.5 into a finite calculation. We fix some bound N, and at every stage consider only the (finitely many) irreducible representations of L_1 of highest weights bounded in size by $N+C$. When this is done, all the linear algebra mentioned in the algorithm will take place in the finite-rank $\mathbb{Z}$ module spanned by K-Langlands parameters of size bounded by $N + 2C$. Instead of surjectivity for the restriction from $R(L_1)$ to $R(L_1^X)$, what we will know is that

(7.11) the image contains all irreducible representations of L_1^X of highest weight size bounded by N.

The conclusion about the algorithm is that

(7.12) proposed basis vectors with K-Langlands parameters of size bounded by N are linearly independent in $K^K(\overline{Y})/K^K(\partial\overline{Y})$.

The proposed basis vectors involving parameters of size between N and $N + 2C$ will indeed live in $K^K(\overline{Y})/K^K(\partial\overline{Y})$, but they may not be linearly independent.

8. Associated varieties for complex groups

In the setting of (5.1), suppose X is a $(\mathfrak{g}, K)$-module for the complex group G_1 (regarded as a real group). Kazhdan-Lusztig theory allows us (if X is specified as a sum of irreducibles in the Langlands classification) to find an explicit formula (in the Grothendieck group of finite length Harish-Chandra modules)

$$X = \sum_{(\lambda_L,\lambda_R)\in\mathcal{P}_{\mathrm{L}}(G,K)} m_X(\lambda_L,\lambda_R) I(\lambda_L,\lambda_R). \tag{8.13a}$$

Fix a K-invariant good filtration of the Harish-Chandra module X, so that $\operatorname{gr} X$ is a finitely generated $S(\mathfrak{g}/\mathfrak{k})$-module supported on $\mathcal{N}_\theta^*$. The class in equivariant K-theory

$$[\operatorname{gr} X] \in K^K(\mathcal{N}_\theta^*) \tag{8.13b}$$

is independent of the choice of good filtration. Because of the characterizations in Definition 6.2 of the basis $\{[\gamma]_\theta\}$ of this K-theory, we find

$$\begin{aligned}[\operatorname{gr} X] &= \sum_{\gamma \in \mathcal{P}_{K\text{-L}}(G)} \left(\sum_{\substack{(\lambda_L,\lambda_R) \in \mathcal{P}_{\mathrm{L}}(G,K) \\ (\lambda_L,\lambda_R) \sim_K \gamma}} m_X(\lambda_L,\lambda_R) \right) [\gamma]_\theta \\ &= \sum_{\gamma \in \mathcal{P}_{K\text{-L}}(G)} m_X(\gamma)[\gamma]_\theta.\end{aligned} \tag{8.13c}$$

Here the equivalence $\sim_K$ in the first inner sum is that of Definition 5.5. If we think of γ as a dominant weight, then

$$(\lambda_L,\lambda_R) \sim_K \gamma \iff \gamma \in W(K,T)\cdot(\lambda_L+\lambda_R). \tag{8.13d}$$

Recall now the classes $[\mathcal{E}_k^{\mathrm{orbalg}}(Z)]$ constructed in Algorithm 7.5. Comparing their known formulas with (8.13c), we can do an (upper triangular) change of basis calculation, and get an explicit formula

$$[\operatorname{gr} X] = \sum_{\mathcal{E}_k^{\mathrm{orbalg}}(Z)} n_X(\mathcal{E}_k^{\mathrm{orbalg}}(Z))[\mathcal{E}_k^{\mathrm{orbalg}}(Z)], \tag{8.13e}$$

with computable integers $n_X(\mathcal{E}_k^{\mathrm{orbalg}}(Z))$.

Here is how to make this calculation finite. After using Kazhdan-Lusztig theory to calculate the formula (8.13c), write N for the size of the largest highest weight appearing. (If X is irreducible, then N is just the length of the highest weight of the lowest K-type of X; no Kazhdan-Lusztig theory arises.) Then run the algorithm as described in (7.11) and (7.12), using always representations of L_1 of highest weights bounded by $N+C$.

Theorem 8.14. *Suppose X is a $(\mathfrak{g},K)$-module for the complex group G_1 (regarded as a real group). Use the notation of* (8.13).

1. *The associated variety of X (Definition 4.2) is the union of the closures of the maximal K-orbits $Z \subset \mathcal{N}_\theta^*$ with some $n_X(\mathcal{E}_k^{\mathrm{orbalg}}(Z)) \neq 0$.*

2. The multiplicity of a maximal orbit Z in the associated cycle of X is

$$\sum_{\mathcal{E}_k^{\mathrm{orbalg}}(Z)} n_X(\mathcal{E}_k^{\mathrm{orbalg}}(Z))\,\mathrm{rank}(\mathcal{E}_k^{\mathrm{orbalg}}(Z)).$$

Sketch of proof. It is an elementary matter to find a formula in equivariant K-theory (using the basis vectors of Theorem 3.5)

$$[\mathrm{gr}\, X] = \sum_{\substack{(Z,\mathcal{F})\in\mathcal{P}_g(G,K)\\ Z\subset\mathrm{supp}(\mathrm{gr}\, X)}} m_X(Z,\mathcal{F})[\widetilde{\mathcal{F}}]; \tag{8.15}$$

to see that the coefficients of the terms on maximal orbits are independent of choices; and to relate those coefficients to the multiplicities in the associated variety. We know that the basis $\{\widetilde{\mathcal{F}}\}$ can be expressed in terms of the basis $\{\mathcal{E}_k^{\mathrm{orbalg}} \mid k = 0,1,2\ldots\}$, and that the change of basis is weakly upper triangular with respect to the ordering of orbits by closure. From these facts the theorem follows. □

9. Representation basis for K-theory: $\mathbb{R}$ case

Everything has been said so as to carry over to real (linear algebraic) groups with minimal changes. We return therefore to the general setting of (1.1).

Just as in the complex case, Theorem 3.5 in the present setting gives

Corollary 9.1. *Write $\{Y_1,\ldots,Y_r\}$ for the orbits of K on $\mathcal{N}_\theta^*$ ((1.3c)). For each irreducible K-equivariant vector bundle $\mathcal{E}$ on some Y_i, fix a K-equivariant (virtual) coherent sheaf $\widetilde{\mathcal{E}}$ supported on the closure of Y_i, and restricting to $\mathcal{E}$ on Y_i. Then the classes $[\widetilde{\mathcal{E}}]$ are a basis of $K^K(\mathcal{N}_\theta^*)$. A little more precisely, the classes supported on any K-invariant closed subset $Z\subset\mathcal{N}_\theta$ are a basis of $K^K(Z)$.*

As stated at the beginning of Section 5, the distinguishing complication in the general real case is the formulation of the Langlands classification. We now begin to explain the details we need. The main point is that Harish-Chandra parametrized the discrete series of $G(\mathbb{R})$ using characters of a compact maximal torus; but this "family" of representations changes drastically as the character moves from one Weyl chamber to another. In order to have a nice family, one should keep track not only of the character, but also of the Weyl chamber in which it lies. The Weyl chamber is indexed by the positive root system Ψ in Definition 9.2 below.

Once that is done, we have the problem that each family of representations is inconveniently small: it is indexed not by *all* characters of a maximal torus, but only by appropriately *dominant* characters. We address this (following a fundamental idea of Hecht and Schmid from the 1970s) by enlarging the family to depend on arbitrary (not necessarily dominant) characters.

The resulting families of representations are convenient for our calculations, but too large to index irreducible representations. The positivity notion of *weak* in Definition 9.2 singles out those representations that have some chance to be part of the classification.

There remain two smaller issues. First, when the character is dominant but singular, it may happen that the corresponding representation of $G(\mathbb{R})$ is zero. This possibility is ruled out by the condition *nonzero* in Definition 9.2. Second (again for singular characters) it may happen that the same representation is attached to characters on two different maximal tori. In this case it turns out that (among these various realizations) there is a unique one on a most compact torus; this is the one identified by the condition *final*.

Definition 9.2. (See for example [3, Section 6] for details.) A *continued Langlands parameter* for (G, K) is a quadruple $\Gamma = (H, \gamma, \Psi)$ such that

1. H is a θ-stable maximal torus in G;
2. γ is a one-dimensional $(\mathfrak{h}, [H^\theta]^{\rho_{\mathrm{abs}}})$-module in which the kernel of the two to one covering map $[H^\theta]^{\rho_{\mathrm{abs}}} \to H^\theta$ acts nontrivially; and
3. Ψ is a system of positive imaginary (that is, θ-fixed) roots for H in G.

Two continued Langlands parameters are *equivalent* if they are conjugate by K. A continued Langlands parameter is called *weak* if in addition

4. $d\gamma \in \mathfrak{h}^*$ is weakly dominant with respect to Ψ.

The weak Langlands parameter is called *nonzero* if in addition

5. whenever $\alpha \in \Psi$ is simple and compact, $\langle d\gamma, \alpha^\vee \rangle \neq 0$.

Here $d\gamma \in \mathfrak{h}^*$ means the weight by which the Lie algebra $\mathfrak{h}$ acts in γ. The nonzero weak Langlands parameter is called *final* if in addition

6. whenever β is a real root of H such that $\langle d\gamma, \beta^\vee \rangle = 0$, then $\gamma_{\mathfrak{q}}(m_\beta) = 1$.

(Here $\gamma_{\mathfrak{q}}$ is a ρ-shift of γ defined in [3, (9.3g)].)

We write

$$\mathcal{P}_{\mathrm{L}}(G, K) = \{\text{equivalence classes of final Langlands parameters}\}.$$

The Langlands classification attaches to the equivalence class of a continued parameter Γ a *continued standard representation* $[I(\Gamma)]$ (more precisely, a virtual $(\mathfrak{g}, K)$-module in the Grothendieck group $K(\mathfrak{g}, K)$ defined in (4.1)), with the following properties.

1. The infinitesimal character of $[I(\Gamma)]$ is the $W(G, H)$ orbit of $d\gamma$.
2. The restriction of $[I(\Gamma)]$ to K depends only on
$$\Gamma_K = (H, \gamma|_{H^\theta}, \Psi).$$
More precisely,
3. the class in equivariant K-theory
$$[\operatorname{gr} I(\Gamma)] =_{\text{def}} [\Gamma]_\theta \in K^K(\mathcal{N}_\theta^*)$$
is independent of $\gamma|_{\mathfrak{h}^{-\theta}}$.
4. If Γ is weak, then $[I(\Gamma)]$ is represented by a $(\mathfrak{g}, K)$-module $I(\Gamma)$, a *weak standard representation*.
5. The weak standard representation $I(\Gamma)$ is nonzero if and only if the parameter Γ is nonzero (as defined above).
6. If Γ is weak and nonzero, then $I(\Gamma) = \oplus_{i=1}^r I(\Gamma_i)$; here $\{\Gamma_i\}$ is a computable finite set of final parameters attached to a single (more compact) θ-stable maximal torus H'.
7. If Γ is final, there is a unique irreducible quotient $J(\Gamma)$ of $I(\Gamma)$.
8. Any irreducible $(\mathfrak{g}, K)$-module is equivalent to some $J(\Gamma)$ with Γ final.
9. Two final standard representations are isomorphic (equivalently, their Langlands quotients are isomorphic) if and only if their parameters are conjugate by K.

Missing from these properties is an explicit description of the equivariant K-theory class $[\Gamma]_\theta$ like Definition 5.4(5) in the complex case. We will return to this point in Section 10.

The final standard representation $I(\Gamma)$ is *tempered* (an analytic condition due to Harish-Chandra, and central to Langlands' original work) if and only if the character γ of $H(\mathbb{R})^{\rho_{\text{abs}}}$ is unitary; equivalently, if and only if $d\gamma|_{\mathfrak{h}^{-\theta}} \in i\mathfrak{h}^*$. In this case $I(\Gamma) = J(\Gamma)$.

In the general (possibly nontempered) case, the real part of $d\gamma|_{\mathfrak{h}^{-\theta}}$ controls the growth of matrix coefficients of $J(\Gamma)$. When $\operatorname{Re} d\gamma|_{\mathfrak{h}^{-\theta}}$ is larger, the matrix coefficients grow faster.

Partly because of the notion of tempered, it is useful to define the *K-norm* of a continued parameter Γ:
$$\|\Gamma\|_K^2 =_{\text{def}} \langle d\gamma|_{\mathfrak{h}^\theta}, d\gamma|_{\mathfrak{h}^\theta} \rangle.$$

In the setting of property (6) above, $\|\Gamma\|_K = \|\Gamma_i\|_K$.

The K-norm is evidently bounded by the canonical real part of the infinitesimal character:

$$\|\Gamma\|_K^2 = \langle \operatorname{Re} d\gamma, \operatorname{Re} d\gamma\rangle - \langle \operatorname{Re} d\gamma|_{\mathfrak{h}^{-\theta}}, \operatorname{Re} d\gamma|_{\mathfrak{h}^{-\theta}}\rangle \leq \langle \operatorname{Re} d\gamma, \operatorname{Re} d\gamma\rangle,$$

with equality if and only if γ is unitary.

We pause here to mention the real groups formulation of the Langlands classification, to which we alluded in the introduction. As usual we use the notation of (1.1). There are natural bijections

$$\begin{aligned} &\{\theta\text{-stable maximal tori } H_1 \subset G\}/K\text{-conjugacy} \\ \longleftrightarrow\ &\{\theta\text{-stable real maximal tori } H_2 \subset G\}/K(\mathbb{R})\text{-conjugacy} \\ \longleftrightarrow\ &\{\text{real maximal tori } H_3 \subset G\}/G(\mathbb{R})\text{-conjugacy} \end{aligned} \tag{9.3a}$$

A θ-stable torus H_i contains an algebraic subgroup

$$H_i^\theta \qquad (i = 1, 2). \tag{9.3b}$$

A real torus H_j has a real form

$$H_j(\mathbb{R}) \qquad (j = 2, 3) \tag{9.3c}$$

which in turn has a natural maximal split subtorus

$$H_j(\mathbb{R}) \supset A_j(\mathbb{R}) \simeq (\mathbb{R}^\times)^d \qquad (j = 2, 3) \tag{9.3d}$$

It is the (topological) *identity component*

$$A_j =_{\text{def}} A_j(\mathbb{R})_0 \simeq (\mathbb{R}_+^\times)^d \qquad (j = 2, 3) \tag{9.3e}$$

that typically appears in discussions of structure theory for real reductive groups. Just as for a general reductive group, the (unique) maximal compact subgroup of $H_j(\mathbb{R})$ is the group of real points of a (unique) algebraic subgroup $T_j \subset H_j$:

$$H_j(\mathbb{R}) \supset T_j(\mathbb{R}) = \text{maximal compact subgroup.} \tag{9.3f}$$

The Cartan decomposition is the direct product decomposition

$$H_j(\mathbb{R}) = T_j(\mathbb{R}) \times A_j \qquad (j = 2, 3). \tag{9.3g}$$

Consequently the continuous characters of $H_j(\mathbb{R})$ may be described as

$$\begin{aligned} \widehat{H_j(\mathbb{R})} &\simeq \widehat{T_j(\mathbb{R})} \times \widehat{A}_j \\ &\simeq \widehat{T}_j \times \mathfrak{a}_j^*; \end{aligned} \tag{9.3h}$$

the last equality is because $T_j(\mathbb{R})$ is a compact form of the algebraic group T_j, and A_j is an abelian vector group.

Since H_2 is both real and θ-stable, we find

$$T_2 = H_2^\theta, \qquad \widehat{T_2(\mathbb{R})} \simeq \widehat{H}_2^\theta. \tag{9.3i}$$

Definition 9.4. (See for example [3, Section 6] for details.) A *continued Langlands parameter* for $G(\mathbb{R})$ is a quadruple $\Gamma = (H(\mathbb{R}), \gamma, \Psi)$ such that

1. $H(\mathbb{R})$ is a maximal torus in $G(\mathbb{R})$;
2. γ is a level one character of the ρ_{abs} double cover of $H(\mathbb{R})$, and
3. Ψ is a system of positive imaginary (that is, $\sigma_{\mathbb{R}}(\alpha) = -\alpha$) roots for H in G.

Two continued Langlands parameters are *equivalent* if they are conjugate by $G(\mathbb{R})$

This definition can be continued in a way precisely parallel to Definition 9.2, defining in the end the set of Langlands parameters

$$\mathcal{P}_{\mathrm{L}}(G(\mathbb{R})) \simeq \mathcal{P}_{\mathrm{L}}(G, K). \tag{9.5}$$

The bijection with Langlands parameters for (G, K) is an easy consequence of (9.3). This entire digression is just another instance of Harish-Chandra's idea that analytic questions about representations of $G(\mathbb{R})$ can often be phrased precisely as algebraic questions about $(\mathfrak{g}, K)$-modules.

We now return to that algebraic setting.

Definition 9.6. A *K-Langlands continued parameter* for (G, K) is a triple $\Gamma_K = (H, \gamma_K, \Psi)$ such that

1. H is a θ-stable maximal torus in G;
2. γ_K is a level one character of the ρ_{abs} double cover of H^θ; and
3. Ψ is a system of positive imaginary (that is, θ-fixed) roots for H in G.

Two continued K-Langlands parameters are *equivalent* if they are conjugate by K. A continued K-Langlands parameter is called *weak* if in addition

4. $d\gamma_K \in (\mathfrak{h}^\theta)^*$ is weakly dominant with respect to Ψ.

   The weak K-Langlands parameter is called *nonzero* if in addition

5. whenever $\alpha \in \Psi$ is simple and compact, $\langle d\gamma_K, \alpha^\vee \rangle \neq 0$.

   The nonzero K-Langlands parameter is called *final* if in addition

6. for any real root β, $\gamma_{K,\mathfrak{q}}(m_\beta) = 1$.

   The set of equivalence classes of final K-Langlands parameters is written $\mathcal{P}_{K\text{-L}}(G,K)$.

   In the complex case, we get a K-Langlands parameter from a Langlands parameter just by discarding a bit of information (the restriction to $\mathfrak{h}^{-\theta}$. In the general real case, matters are more subtle. The difference between final K-Langlands parameters and final Langlands parameters is first, that there is no character on (the split torus) $\mathfrak{h}^{-\theta}$ (or, equivalently, that $d\gamma_K$ is assumed to be zero there); and second, that the finality condition is assumed for *all* the real roots, rather than just those on which $d\gamma$ vanishes.

   We will make use of the *K-norm* of a K-Langlands parameter, defined exactly as for Langlands parameters by

$$\|\Gamma_K\|_K^2 =_{\text{def}} \langle d\gamma_K, d\gamma_K \rangle;$$

   the weight whose length we are taking belongs to $(\mathfrak{h}^\theta)^*$.

Proposition 9.7. *The set $\mathcal{P}_{K\text{-L}}(G,K)$ of equivalence classes of final K-Langlands parameters (Definition 9.6) is in one-to-one correspondence with*

1. *(final Langlands parameters for) tempered representations of real infinitesimal character (extending Γ_K by zero on $\mathfrak{h}^{-\theta}$); or*
2. *irreducible representations of K (by taking lowest K-type).*

Once we have in hand the K-Langlands parameters, there is an obvious extension of Lusztig's conjecture (what is proven by Bezrukavnikov in [7]) to real groups. But this extension is not true for $SL(2,\mathbb{R})$ (see Example 9.10 below).

We can now begin to extend to real groups the ideas in Section 6.

Proposition 9.8. *Suppose $(Y,\mathcal{E}) \in \mathcal{P}_g(G,K)$ is a geometric parameter (Definition 1.13); fix an extension $\widetilde{\mathcal{E}}$ as in Corollary 9.1. Then there is a formula in $K^K(\mathcal{N}_\theta)$*

$$[\widetilde{\mathcal{E}}] = \sum_{\Gamma_K \in \mathcal{P}_{K\text{-L}}(G,K)} m_{\widetilde{\mathcal{E}}}(\Gamma_K)[\Gamma_K]_\theta.$$

Here $m_{\widetilde{\mathcal{E}}}(\Gamma_K) \in \mathbb{Z}$*, and the sum is finite.*

Suppose $\widetilde{\mathcal{E}}'$ *is another extension of* $\mathcal{E}$ *to* $\overline{Y}$*. Then*

$$[\widetilde{\mathcal{E}}'] - [\widetilde{\mathcal{E}}] = \sum_{\substack{(Z,\mathcal{F}) \in \mathcal{P}_g(G,K) \\ Z \subset \partial Y}} n_{\mathcal{F}}[\widetilde{\mathcal{F}}].$$

Here $n_{\mathcal{F}} \in \mathbb{Z}$*, and the sum is finite.*

We will prove this in Corollary 11.3 below.

In the complex case, Bezrukavnikov's proof of the Lusztig-Bezrukavnikov conjecture guarantees the existence of an extension $\widetilde{\mathcal{E}}$ with a *single* leading term, and in this way finds a *bijection* between geometric parameters and K-Langlands parameters. In the real case there will sometimes be no reasonable way to arrange a *single* leading term, and accordingly no such bijection.Fortunately computers are better able than humans to do linear algebra with matrices that are not upper triangular.

Corollary 9.9. *In the setting of* (1.1)*, the classes*

$$\{[\Lambda_K]_\theta \in K^K(\mathcal{N}_\theta^*) \mid \Lambda_K \in \mathcal{P}_{K\text{-L}}(G,K)$$

are a $\mathbb{Z}$*-basis of* $K^K(\mathcal{N}_\theta^*)$*. The restriction to* K *map*

$$\mathrm{res}_K \colon K^K(\mathcal{N}_\theta^*) \to K(K)$$

of (4.4g) *is injective.*

Proof. That these classes span is a consequence of Theorem 3.5 and Proposition 9.8. That they are linearly independent is a consequence of Theorem 9.7; the argument proves injectivity of the restriction at the same time. □

Example 9.10. Let us take $G = SL(2,\mathbb{C})$,

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \theta(g) = DgD^{-1},$$

so that $K = H_c$ is the diagonal torus, and G is the complexification of $SU(1,1)$. We have naturally

$$K \simeq \mathbb{C}^\times, \qquad \widehat{K} \simeq \mathbb{Z}; \tag{9.10a}$$

we will write an irreducible representation of K just as an unadorned integer. The K-nilpotent cone is

$$\mathcal{N}_\theta^* \simeq \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mid ab = 0 \right\}.$$

There are two nonzero orbits of K on $\mathcal{N}_\theta^*$:

$$Y^+ = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \neq 0 \right\}, \quad Y^- = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \mid b \neq 0 \right\},$$

each isomorphic to $K/\{\pm I\}$; and the zero orbit $Y^0 \simeq K/K$. The geometric parameters are therefore

$$\mathcal{P}_g(G,K) = \{(Y^\pm, \mathcal{E}^\pm_{\mathrm{triv}}),\ (Y^\pm, \mathcal{E}^\pm_{\mathrm{sgn}}),\ (Y^0, \mathcal{E}^0_n) \mid n \in \mathbb{Z}\}. \tag{9.10b}$$

(In each case the superscript 0 or $\pm$ on the vector bundle identifies the underlying orbit.)

On the θ-stable maximal torus H_c, the Cartan involution θ acts trivially. Consequently every root is imaginary, and there are two systems of positive imaginary roots: Ψ^+ (corresponding to upper triangular matrices), and Ψ^-. Attached to each non-negative integer n there are two final K-Langlands parameters $\Gamma_K^+(n)$ (corresponding to Ψ^+, with the differential of the character identified with n) and $\Gamma_K^-(n)$. These are discrete series and limits of discrete series:

$$\begin{aligned} [\Gamma_K^+(n)]|_K &= n+1,\ n+3,\ n+5,\dots \\ [\Gamma_K^-(n)]|_K &= -n-1,\ -n-3,\ -n-5,\dots. \end{aligned} \tag{9.10c}$$

A representative of the other K-conjugacy class of θ-stable maximal torus is

$$H_s = \left\{ \begin{pmatrix} \cosh z & \sinh z \\ \sinh z & \cosh z \end{pmatrix} \right\},$$

with Lie algebra

$$\mathfrak{h}_s = \left\{ \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

Here θ acts by inversion, so there are no imaginary roots. There is exactly one final K-Langlands parameter Γ_K^0, corresponding to the spherical principal series:

$$[\Gamma_K^0]|_K = 0, \pm 2, \pm 4, \dots. \tag{9.10d}$$

so on. A reasonable partial order on these parameters is

$$\begin{aligned} \Gamma_K^0 &\prec \Gamma_K^+(1) \prec \Gamma_K^+(3) \prec \Gamma_K^+(5) \prec \cdots \\ \Gamma_K^0 &\prec \Gamma_K^-(1) \prec \Gamma_K^-(3) \prec \Gamma_K^-(5) \prec \cdots \\ \Gamma_K^+(0) &\prec \Gamma_K^+(2) \prec \Gamma_K^+(4) \prec \Gamma_K^+(6) \prec \cdots \\ \Gamma_K^-(0) &\prec \Gamma_K^-(2) \prec \Gamma_K^-(4) \prec \Gamma_K^-(6) \prec \cdots \end{aligned} \tag{9.10e}$$

Here are some reasonable choices of extensions

$$\widetilde{\mathcal{E}_n^0} = [\Gamma_K^{\mathrm{sgn}(n)}(|n|-1)]_\theta - [\Gamma_K^{\mathrm{sgn}(n)}(|n|+1)]_\theta \quad (n \neq 0), \tag{9.10f}$$

$$\widetilde{\mathcal{E}_0^0} = [\Gamma_K^0]_\theta - [\Gamma_K^+(1)]_\theta - [\Gamma_K^-(1)]_\theta, \tag{9.10g}$$

$$\widetilde{\mathcal{E}_{\mathrm{sgn}}^\pm} = [\Gamma_K^\pm(0)]_\theta; \tag{9.10h}$$

and

$$\widetilde{\mathcal{E}_{\mathrm{triv}}^\pm} = [\Gamma_K^\pm(1)]_\theta. \tag{9.10i}$$

But in the last case, there is another reasonable choice of extension:

$$\widetilde{\mathcal{E}_{\mathrm{triv}}^\pm}' = [\Gamma_K^0]_\theta - [\Gamma_K^\mp(1)]_\theta. \tag{9.10j}$$

(Here we make the natural choice of extending the structure sheaf on the open orbit $Y^\pm$ to the structure sheaf on its closure.) So here is what we have in the direction of a Lusztig-Bezrukavnikov bijection for $SL(2,\mathbb{R})$:

$$\begin{aligned} \mathcal{E}_n^0 &\longleftrightarrow \Gamma_K^{\mathrm{sgn}(n)}(|n|+1) \qquad (n \neq 0) \\ \mathcal{E}_{\mathrm{sgn}}^\pm &\longleftrightarrow \Gamma_K^\pm(0) \\ \mathcal{E}_0^0,\ \mathcal{E}_{\mathrm{triv}}^+,\ \mathcal{E}_{\mathrm{triv}}^- &\longleftrightarrow \Gamma_K^+(1),\ \Gamma_K^-(1); \end{aligned} \tag{9.10k}$$

the map from left to right is taking some kind of "leading terms" of some natural extension. One might like to include on the right in the last case the K-Langlands parameter Γ_K^0; it is not a leading term, but the result is that there is something like an "almost bijection," with the last three "smallest" geometric parameters corresponding (as a group) to the three "smallest" K-Langlands parameters.

We conclude this section by recording the (known) information we will need about continued standard parameters.

Proposition 9.11. *Use the notation of Definitions 9.2 and 9.6.*

1. *The equivalence classes (that is, orbits of K)*
$$\{[I(\Gamma) \mid \Gamma \in \mathcal{P}_{\mathrm{L}}\}$$
of final Langlands parameters are a $\mathbb{Z}$-basis of the Grothendieck group $\mathcal{M}_f(\mathfrak{g}, K)$ of finite length Harish-Chandra modules.
2. *For any final parameter Γ, Kazhdan-Lusztig theory computes*
$$[J(\Gamma)] = \sum_{\Lambda \in \mathcal{P}_{\mathrm{L}}} m_\Gamma(\Lambda)[I(\Lambda)];$$
here $m_\Gamma(\Lambda) \in \mathbb{Z}$, and the sum is finite. We have $m_\Gamma(\Gamma) = 1$, and the other nonzero terms all satisfy
$$\|\Lambda\|_K > \|\Gamma\|_K.$$
3. *For any continued parameter Γ', the unique formula*
$$[I(\Gamma')] = \sum_{\Lambda \in \mathcal{P}_{\mathrm{L}}} p_{\Gamma'}(\Lambda)[I(\Lambda]$$
can be computed using classical results of Hecht and Schmid.
4. *The equivalence classes (that is, orbits of K)*
$$\{[I(\Gamma_K)] \mid \Gamma_K \in \mathcal{P}_{K\text{-L}}\}$$
are a $\mathbb{Z}$-basis of the Grothendieck group of finite-length Harish-Chandra modules restricted to K.
5. *For any final parameter $\Gamma \in \mathcal{P}_{\mathrm{L}}$, there is an elementary computation of the unique formula*
$$[I(\Gamma)|_K] = \sum_{\Lambda_K \in \mathcal{P}_{K\text{-L}}} m_\Gamma(\Lambda_K) I(\Lambda_K).$$
All the parameters Λ_K appearing live on the same (more compact) maximal torus, satisfy
$$\|\Lambda_K\|_K = \|\Gamma\|_K,$$

and have $m_\Gamma I(\Lambda_K) = 1$*; they correspond to the lowest* K*-types of* $I(\Gamma)$ *or* $J(\Gamma)$*.*

6. *For any continued parameter* Γ'*, the unique formula*

$$[I(\Gamma')|_K] = \sum_{\Lambda_K \in \mathcal{P}_{K\text{-L}}} q_{\Gamma'}(\Lambda_K)[I(\Lambda_K)]$$

can be computed explicitly.

7. *For any final parameter* $\Gamma \in \mathcal{P}_{\mathrm{L}}$*, there is computable formula*

$$J(\Gamma)|_K = \sum_{\Lambda_K \in \mathcal{P}_{K\text{-L}}} n_{J(\Gamma)} I(\Lambda_K)|_K,$$

or equivalently

$$[\mathrm{gr}\, J(\Gamma)] = \sum_{\Lambda_K \in \mathcal{P}_{K\text{-L}}} n_{J(\Gamma)}[\Lambda_K]_\theta \in K^K(\mathcal{N}_\theta^*).$$

This proposition corresponds to the preparations made in (8.13) in the complex case.

10. Standard representations restricted to K

In this section we will recall how to compute the restrictions to K of the (continued) standard $(\mathfrak{g}, K)$-modules described in Definition 9.2. This will be critical for the description in Section 11 of how to explicitly write the formulas of Proposition 9.8.

Always we work in the setting (1.1). Suppose to begin that we have also a θ-stable parabolic subgroup with θ-stable Levi decomposition

$$Q = LU, \qquad \theta Q = Q, \qquad \theta L = L. \tag{10.1a}$$

It is not difficult to show that

$$K/(Q \cap K) \hookrightarrow G/Q \tag{10.1b}$$

is a closed embedding, so that $K/(Q \cap K)$ is projective, and therefore $Q \cap K$ is parabolic in K. (Since K may be disconnected, there is a question about the meaning of "parabolic subgroup." We will say that $P \subset K$ is *parabolic* if K/P is projective; equivalently, if $P \cap K_0 = P_0$ is parabolic in K_0, or if

P contains a (connected) Borel subgroup of K_0.) We may in particular fix a torus

$$T \subset L \cap K \tag{10.1c}$$

that is a maximal torus in K_0.

In the next proposition the disconnectedness of K complicates matters slightly, and is the reason we need not get *irreducible* representations of K from irreducibles of $L \cap K$.

Proposition 10.2 (Bott-Borel-Weil)**.** *In the setting* (10.1)*, suppose* (σ, S) *is an algebraic representation of* $L \cap K$ *(or even of* $Q \cap K$*). Then we get an equivariant algebraic vector bundle*

$$\mathcal{S} = K \times_{Q \cap K} S \to K/(Q \cap K).$$

1. *Each cohomology space* $H^i(K/(Q \cap K), \mathcal{S})$ *is a finite-dimensional algebraic representation of* K*.*
2. *The virtual representation*

$$\sum_i (-1)^i [H^i(K/(Q \cap K), \mathcal{S})] \in R(K)$$

depends only on the class $[(\sigma, S)] \in R(L \cap K)$*.*

3. *Suppose* (σ, S) *is irreducible, and that its infinitesimal character is represented by*

$$\xi_{L\cap K} \in X^*(T) - \rho_{L\cap K} \subset \mathfrak{t}^*.$$

Write

$$\xi_K = \xi_{L\cap K} - \rho(\mathfrak{u} \cap \mathfrak{k}) \in X^*(T) - \rho_K.$$

Then either

$$H^i(K/(Q \cap K), \mathcal{S}) = 0, \qquad (\textit{all } i)$$

(if ξ_K *vanishes on some coroot of* T *in* K*);* or

$$H^i(K/(Q \cap K), \mathcal{S}) = \begin{cases} \text{\textit{nonzero representation of infinitesimal character} } \xi_K & (i = i(\xi_K)) \\ 0 & (i \neq i(\xi_K)). \end{cases}$$

Our first tool for computing cohomological induction is the operation

$$\begin{aligned}(10.3)\qquad &\mathrm{Ind}_{Q\cap K}^{K}\colon R(L\cap K)\to R(K),\\ &[(\sigma,S)]\mapsto \sum_i (-1)^i[H^i(K/(Q\cap K),\mathcal{S})]\in R(K).\end{aligned}$$

One should think of the case when (σ, S) is a K-dominant irreducible representation of $L\cap K$; then $\mathrm{Ind}_{Q\cap K}^{K}(\sigma)$ lives only in the highest degree $\dim K/(Q\cap K)$, and there is essentially an irreducible representation of K of highest weight $\sigma - 2\rho(\mathfrak{u}\cap\mathfrak{k})$. (For disconnected K it may happen that there are several irreducible representations of K of highest weight $\sigma - 2\rho(\mathfrak{u}\cap\mathfrak{k})$; this is how reducible representations of K arise in Proposition 10.2.)

Proposition 10.4 (Zuckerman)**.** *In the setting* (10.1)*, there are cohomological induction functors*

$$\mathcal{R}^i\colon \mathcal{M}_f(\mathfrak{l}, L\cap K)\to \mathcal{M}_f(\mathfrak{g},K)\qquad (0\le i\le \dim \mathfrak{u}\cap\mathfrak{k})$$

(notation (4.1a)*) with the following properties.*

1. *The class*
$$[\mathcal{R}(Z)] =_{\mathrm{def}} \sum_i (-1)^i[\mathcal{R}^i(Z)]\in K(\mathfrak{g},K)$$
is well-defined, depending only on $[Z]\in K(\mathfrak{l}, L\cap K)$*.*
2. *The class*
$$[\mathrm{gr}\,\mathcal{R}(Z)] =_{\mathrm{def}} \sum_i (-1)^i[\mathrm{gr}\,\mathcal{R}^i(Z)]\in K^K(\mathcal{N}^*_\theta)$$
is well-defined, depending only on $[\mathrm{gr}\, Z]\in K^{L\cap K}(\mathcal{N}^*_{L,\theta})$*.*
3. *Suppose* $H\subset L$ *is a* θ*-stable maximal torus, and* $\Gamma_L = (H,\gamma_L,\Psi_L)$ *is a continued Langlands parameter for* $(L, L\cap K)$*. Define*
$$\gamma_G = \gamma_L\otimes \rho(\mathfrak{u})^*$$
$$\Psi_G = \Psi_L \cup \{\textit{imaginary roots of } H \textit{ in } \mathfrak{u}\},$$
so that $\Gamma_G = (H,\gamma_G,\Psi_G)$ *is a continued Langlands parameter for* (G,K)*. Then*
$$[\mathcal{R}I(\Gamma_L))] = [I(\Gamma_G)].$$

This proposition describes (or at least says that it's possible to describe) how to construct standard modules using the geometry of (10.1). We are interested in computing gr (an image in the equivariant K-theory of the nilpotent cone) of standard modules; so we need to relate that geometry to (10.1). Often the best way to think of G/Q is as a variety of parabolic subgroups:

$$G/Q \simeq \mathcal{Q} =_{\text{def}} \text{variety of parabolic subalgebras conjugate to } \mathfrak{q}. \tag{10.5a}$$

To think about nilpotent elements in $\mathfrak{g}^*$, it may be helpful to recall that in any identification $\mathfrak{g} \simeq \mathfrak{g}^*$ from an invariant bilinear form, we have

$$\mathfrak{q} \simeq (\mathfrak{g}/\mathfrak{u})^*.$$

The natural projection $\mathfrak{q} \to \mathfrak{l}$ corresponds to restriction of linear functionals

$$\pi_{\mathfrak{q}} \colon (\mathfrak{g}/\mathfrak{u})^* \to (\mathfrak{q}/\mathfrak{u})^*. \tag{10.5b}$$

An element of $\mathfrak{q}$ is nilpotent if and only if its image in $\mathfrak{l}$ is nilpotent. We therefore write

$$\begin{aligned} \mathcal{N}_{\mathfrak{l}}^* &= \text{nilpotent cone in } \mathfrak{l}^*, \\ \mathcal{N}_{\mathfrak{q}}^* &= \pi_{\mathfrak{q}}^{-1}(\mathcal{N}_{\mathfrak{l}}^*); \end{aligned} \tag{10.5c}$$

this is an "affine space bundle" over the nilpotent cone for L (roughly, a vector bundle without chosen zero section) corresponding to the vector space $(\mathfrak{g}/\mathfrak{q})^* \simeq \mathfrak{u}$. If we use the identification $\mathfrak{g}^* \simeq \mathfrak{g}$,

$$\mathcal{N}_{\mathfrak{q}}^* \simeq \mathcal{N}_{\mathfrak{l}} + \mathfrak{u}.$$

For the K-nilpotent cone,

$$\pi_{\mathfrak{q},\theta} \colon (\mathfrak{g}/(\mathfrak{u}+\mathfrak{k}))^* \to (\mathfrak{q}/(\mathfrak{u}+(\mathfrak{q}\cap\mathfrak{k})))^* \simeq (\mathfrak{l}/(\mathfrak{l}\cap\mathfrak{k}))^*. \tag{10.5d}$$

$$\begin{aligned} \mathcal{N}_{\mathfrak{l},\theta}^* &= (L\cap K)\text{-nilpotent cone in } \mathfrak{l}^* \\ &= \mathcal{N}_{\mathfrak{l}}^* \cap (\mathfrak{l}/(\mathfrak{l}\cap\mathfrak{k}))^* \\ \mathcal{N}_{\mathfrak{q},\theta}^* &= \pi_{\mathfrak{q},\theta}^{-1}(\mathcal{N}_{\mathfrak{l},\theta}^*) \\ \mathcal{N}_{\mathfrak{q},\theta}^* &\simeq \mathcal{N}_{\mathfrak{l},\theta} + (\mathfrak{u}\cap\mathfrak{s}). \end{aligned} \tag{10.5e}$$

The basic Grothendieck-Springer method to study nilpotent elements is to consider

$$\begin{aligned}\mathcal{N}^*_{\mathcal{Q}} &=_{\text{def}} \{(\xi', \mathfrak{q}') \mid \mathfrak{q}' \in \mathcal{Q},\ \xi' \in \mathcal{N}^*_{\mathfrak{q}'}\} \\ &\simeq G \times_Q \mathcal{N}^*_{\mathfrak{q}}.\end{aligned} \tag{10.5f}$$

Points here are nilpotent linear functionals ξ' on $\mathfrak{g}$ with the extra information of a chosen parabolic $\mathfrak{q}'$ (conjugate to $\mathfrak{q}$) so that ξ' vanishes on the nil radical $\mathfrak{u}'$ of $\mathfrak{q}'$. Such parabolics exist for any ξ', so the *moment map*

$$\mu_{\mathcal{Q}} \colon \mathcal{N}^*_{\mathcal{Q}} \to \mathcal{N}^*, \qquad (\xi', \mathfrak{q}') \mapsto \xi' \tag{10.5g}$$

is (projective and) surjective. In the same way, the projection

$$\pi_{\mathcal{Q}} \colon \mathcal{N}^*_{\mathcal{Q}} \to \mathcal{Q}, \qquad (\xi', \mathfrak{q}') \mapsto \mathfrak{q}' \tag{10.5h}$$

is an affine morphism (even a bundle) with fiber $\mathcal{N}^*_{\mathfrak{q}}$.

The subvariety $K/(Q \cap K)$ is

$$K/(Q \cap K) \simeq \mathcal{Q}_K =_{\text{def}} \text{variety of } \theta\text{-stable parabolic subalgebras conjugate by } K \text{ to } \mathfrak{q}, \tag{10.5i}$$

a single closed orbit of K on $\mathcal{Q}$. Over this orbit we are interested in a subbundle of $\mathcal{N}^*_{\mathcal{Q}}$

$$\begin{aligned}\mathcal{N}^*_{\mathcal{Q},\theta} &=_{\text{def}} \{(\xi', \mathfrak{q}') \mid \mathfrak{q}' \in \mathcal{Q}_K,\ \xi' \in \mathcal{N}^*_{\mathfrak{q}',\theta}\} \\ &\simeq K \times_{Q \cap K} \mathcal{N}^*_{\mathfrak{q},\theta}.\end{aligned} \tag{10.5j}$$

Points here are nilpotent linear functionals ξ' on $\mathfrak{g}/\mathfrak{k}$ with the extra information of a chosen θ-stable parabolic $\mathfrak{q}'$ (conjugate by K to $\mathfrak{q}$) so that ξ' vanishes on the nil radical $\mathfrak{u}'$ of $\mathfrak{q}'$. Such parabolics may *not* exist for some ξ', so the *moment map*

$$\mu_{\mathcal{Q},\theta} \colon \mathcal{N}^*_{\mathcal{Q},\theta} \to \mathcal{N}^*_{\theta}, \qquad (\xi', \mathfrak{q}') \mapsto \xi' \tag{10.5k}$$

is projective but not necessarily surjective. In the same way, the projection

$$\pi_{\mathcal{Q},\theta} \colon \mathcal{N}^*_{\mathcal{Q},\theta} \to \mathcal{Q}_K, \qquad (\xi', \mathfrak{q}') \mapsto \mathfrak{q}' \tag{10.5l}$$

is an affine morphism (even a bundle) with fiber $\mathcal{N}^*_{\mathfrak{q},\theta}$.

Suppose now that

$$\mathcal{E}_L \in \mathrm{Coh}^{L\cap K}(\mathcal{N}^*_{\mathfrak{l},\theta}); \tag{10.5m}$$

that is, that $\mathcal{E}_L$ is a finitely generated module for $S(\mathfrak{l})$, with $\mathfrak{l}\cap\mathfrak{k}$ and the L-invariants of positive degree acting by zero, and endowed with a compatible action of $L\cap K$. The pullback

$$\mathcal{E}_Q =_{\mathrm{def}} \pi^*_{\mathfrak{q},\theta}(\mathcal{E}_L) \in \mathrm{Coh}^{Q\cap K}(\mathcal{N}^*_{\mathfrak{q},\theta}) \tag{10.5n}$$

is obtained by first regarding $\mathcal{E}_L$ as an $S(\mathfrak{q})$ module (in which $\mathfrak{u}$ acts by zero), and then tensoring over $S(\mathfrak{q})$ with $S(\mathfrak{g})$ to extend scalars. We can define

$$\mathcal{E}_G =_{\mathrm{def}} K \times_{Q\cap K} \mathcal{E}_Q \in \mathrm{Coh}^K(\mathcal{N}^*_{\mathcal{Q},\theta}) \tag{10.5o}$$

as in (3.3i). Because $\pi_{\mathfrak{q},\theta}$ is proper, the higher direct images

$$R^i\mu_*(\mathcal{E}_G) \in \mathrm{Coh}^K(\mathcal{N}^*_\theta) \tag{10.5p}$$

are all coherent sheaves on the nilpotent cone.

Proposition 10.6 (Zuckerman's Blattner formula)**.** *In the setting* (10.1), *suppose $Z \in \mathcal{M}_f(\mathfrak{l}, L\cap K)$, with*

$$[\mathrm{gr}\, Z] \in K^{L\cap K}(\mathcal{N}^*_{\mathfrak{l},\theta})$$

is the corresponding class in equivariant K-theory. Define

$$\mathcal{R}[\mathrm{gr}\, Z] = \sum_i (-1)^i [R^i\mu_*([\mathrm{gr}\, Z]_L)] \in K^K(\mathcal{N}^*_\theta)$$

(notation as in (10.5)*). Then*

$$[\mathrm{gr}\, \mathcal{R}(Z)] = \mathcal{R}[\mathrm{gr}\, Z].$$

This is Zuckerman's proof of the Blattner formula; the representations of K appearing on the right are computable from the $L\cap K$-types of Z (by Proposition 10.2). Those on the left are the K-types of $\mathcal{R}(Z)$.

11. Geometric basis for K-theory: $\mathbb{R}$ case

In this section we will explain how to compute one extension of a geometric parameter (Proposition 9.8). As in the complex case, we will proceed in the aesthetically distasteful way of using the Jacobson-Morozov theorem (and so discussing not the nilpotent elements in $\mathfrak{g}^*$ that we care about, but rather the nilpotent elements in $\mathfrak{g}$).

We begin therefore with an arbitrary K-nilpotent element $E_\theta \in \mathcal{N}_\theta$ (see (1.4e)). The Kostant-Rallis result Proposition 1.8 finds elements $F_\theta \in \mathcal{N}_\theta$ and D_θ in $\mathfrak{k}$ so that

$$[D_\theta, E_\theta] = 2E_\theta, \quad [D_\theta, F_\theta] = -2F_\theta, \quad [E_\theta, F_\theta] = D_\theta. \tag{11.1a}$$

We use the eigenspaces of $\mathrm{ad}(D_\theta)$ to define a θ-stable parabolic subgroup $Q = LU$ of G, with Levi factor $L = G^{D_\theta}$. We will be concerned with the equivariant vector bundle

$$\mathcal{R}_\theta =_{\mathrm{def}} K \times_{Q\cap K} \mathfrak{s}[\geq 2] \xrightarrow{\pi} K/Q\cap K. \tag{11.1b}$$

(The reason $\mathcal{R}_\theta$ is of interest is that Corollary 11.2 below says that it is a K-equivariant resolution of singularities of the nilpotent orbit closure $\overline{K\cdot E_\theta}$. The $\mathcal{R}$ is meant to stand for *resolution*.) According to (3.3i) and (3.3f),

$$K^K(\mathcal{R}_\theta) \simeq R(Q\cap K) \simeq R(L\cap K); \tag{11.1c}$$

If (σ, S) is an irreducible representation of $L\cap K \simeq Q\cap K/U\cap K$, we write

$$\mathcal{S}_0(\sigma) = K \times_{Q\cap K} S \tag{11.1d}$$

for the induced vector bundle on $K/Q\cap K$. The corresponding basis element of the equivariant K-theory is represented by the equivariant vector bundle

$$\mathcal{S}(\sigma) = \pi^*(\mathcal{S}_0(\sigma)) = K \times_{Q\cap K} (\mathfrak{s}[\geq 2] \times S) \to \mathcal{R}_\theta. \tag{11.1e}$$

We are in the setting of Proposition 1.8. As a consequence of that Proposition, we have

Corollary 11.2. *Suppose we are in the setting of* (11.1).

1. *The natural map*

$$\mu\colon \mathcal{R}_\theta \to \mathcal{N}_\theta, \qquad (k, Z) \mapsto \mathrm{Ad}(k)Z$$

is a proper birational map onto $\overline{K \cdot E_\theta}$. *We may therefore* identify $K \cdot E_\theta$ *with its preimage* U*:*

$$K/K^{E_\theta} \simeq K \cdot E_\theta \simeq U \subset \mathcal{R}_\theta.$$

Because $K \cdot E_\theta$ *is open in* $\overline{K \cdot E_\theta}$, $U = \mu^{-1}(K \cdot E_\theta)$ *is open in* $\mathcal{R}_\theta$.

2. *The classes*

$$\{[\mathcal{S}(\sigma)] \mid \sigma \in \widehat{L \cap K} = \widehat{Q \cap K}\}$$

of (11.1e) *are a basis of the equivariant* K*-theory* $K^K(\mathcal{R}_\theta)$.

3. *Since* μ *is proper, higher direct images of coherent sheaves are always coherent. Therefore*

$$[\mu^*(\mathcal{S}(\sigma)] =_{\text{def}} \sum_i (-1)^i [R^i \mu^* \mathcal{S}(\sigma)] \in K^K(\mathcal{N}_\theta)$$

is a well-defined virtual coherent sheaf. This defines a map in equivariant K*-theory*

$$\mu_* \colon K^K(\mathcal{R}_\theta) \to K^K(\mathcal{N}_\theta).$$

Restriction to the open set $U \simeq K \cdot E_\theta$ *commutes with* μ_*.

Suppose σ *is a (virtual) algebraic representation of* $L \cap K$. *Write*

$$[\sigma] = \sum_i m_i(\sigma)[I(\Gamma^i_{L\cap K})]_\theta \in K^{L\cap K}(\mathcal{N}^*_{\mathfrak{l},\theta})$$

(computably, as explained in [23]*); here the* H^i *are* θ*-stable maximal tori in* L, *and*

$$\Gamma^i_{L\cap K} = (H^i, \gamma^i_{L\cap K}, \Psi^i_L) \in \mathcal{P}_{L\cap K\text{-L}}(L, L\cap K).$$

The left side $[\sigma]$ *is a finite-dimensional virtual representation of* $L \cap K$, *regarded as a class in the equivariant* K*-theory of the nilpotent cone supported at* $\{0\}$.

4. As a representation of K,

$$\begin{aligned}
[\mu^*(\mathcal{S}(\sigma))] &= \sum_j (-1)^j \mathcal{R}\left([\textstyle\bigwedge^j \mathfrak{s}[1]^* \otimes \sigma]\right) \\
&= \sum_j (-1)^j \mathcal{R}\left(\sum_i m_i [I(\Gamma^i_{L\cap K})] \otimes \textstyle\bigwedge^j \mathfrak{s}[1]^*\right) \\
&= \sum_i m_i \sum_{A\subset\Delta(\mathfrak{s}[1],H^i\cap K)} (-1)^{|A|} \mathcal{R}[\operatorname{gr} I(\Gamma^i_{L\cap K} - 2\rho(A))] \\
&= \sum_i m_i \sum_{A\subset\Delta(\mathfrak{s}[1],H^i\cap K))} (-1)^{|A|} [\operatorname{gr} I(\Gamma^i_K - 2\rho(A))].
\end{aligned}$$

Here $\Delta(\mathfrak{s}[1], H^i \cap K))$ is the set of weights of $H^i \cap K$ on $\mathfrak{s}[1]$.

5. If every continued standard representation $[I(\Gamma^i_K - 2\rho(A))]|_K$ in (6) is replaced by an integer linear combination of K-Langlands parameters in accordance with Proposition 9.11, we get a computable formula

$$[\mu^*(\mathcal{S}(\sigma))] = \sum_{\Lambda_K\in\mathcal{P}_{K\text{-L}}(G,K)} m_\sigma(\Lambda_K)[\Lambda_K]_\theta.$$

Corollary 11.3. *We continue in the setting* (11.1)*.*

1. The restriction map in equivariant K-theory

$$R(L\cap K) \simeq R(Q\cap K) \simeq K^K(\mathcal{R}_\theta) \to K^K(U) \simeq R(K^{E_\theta}) = R(Q^{E_\theta})$$

sends a (virtual) representation $[\sigma]$ of $Q\cap K$ to $[\sigma|_{(Q\cap K)^{E_\theta}}]$.

2. Any virtual (algebraic) representation τ of $K^{E_\theta} = (Q\cap K)^{E_\theta}$ can be extended to a virtual (algebraic) representation σ of $Q\cap K$. That is, the restriction map of representation rings

$$R(L\cap K) \simeq R(Q\cap K) \twoheadrightarrow R((Q\cap K)^{E_\theta}) \simeq R((L\cap K)^{E_\theta})$$

is surjective.

3. Suppose $[\tau]$ is a virtual algebraic representation of K^{E_θ}, corresponding to a virtual coherent sheaf $\mathcal{T}$ on $K\cdot E_\theta$. Choose a virtual algebraic representation σ of $Q\cap K$ extending τ. Then the virtual coherent sheaf

$$[\mu_*(\mathcal{S}(\sigma))] =_{\text{def}} [\widetilde{\mathcal{T}}]$$

is a virtual extension of $\mathcal{T}$. We have a formula

$$[\widetilde{\mathcal{T}}] = \sum_{\Lambda_K \in \mathcal{P}_{K\text{-L}}(G,K)} m_{\widetilde{\mathcal{T}}}(\Lambda_K)[\Lambda]_\theta.$$

Computability *of σ in (3) is a problem in finite-dimensional representation theory of reductive algebraic groups, for which we do* not *offer a general solution. Except for this issue, the formula in (3) is computable.*

The last formula in Corollary 11.3 relates the geometric basis of Theorem 3.5 to the representation-theoretic basis of Corollary 9.9.

Algorithm 11.4 (A geometric basis for equivariant K-theory)**.** We begin in the setting (11.1) with a nilpotent orbit

$$Y = K \cdot E_\theta \subset \mathcal{N}_\theta \simeq \mathcal{N}_\theta^*. \tag{11.5}$$

The goal is to produce a collection of explicit elements

$$\mathcal{E}_j^{\text{orbalg}}(Y) = \sum_{\Lambda_K \in \mathcal{P}_{K\text{-L}}(G,K)} m_{\mathcal{E}_j^{\text{orbalg}}(Y)}(\Lambda_K)[\Lambda_K]_\theta \in K^K(\overline{Y}) \tag{11.5a}$$

which are a *basis* of $K^K(\overline{Y})/K^K(\partial\overline{Y})$. (The superscript "orbalg" stands for "orbital algorithm." The subscript j is just an indexing parameter for the basis vectors, running over either $\{0, 1, \dots, M-1\}$ or $\mathbb{N}$.) The algorithm proceeds by induction on $\dim Y$; so we assume that such a basis is available for every boundary orbit $Y' \subset \partial\overline{Y}$.

Start with an arbitrary (say irreducible) representation σ of $L \cap K$, and write down the formula of Corollary 11.2(5):

$$[\mu_* \mathcal{S}(\sigma)] = \sum_{\Lambda_K \in \mathcal{P}_{K\text{-L}}(G,K)} m_\sigma(\Lambda_K)[\Lambda_K]_\theta. \tag{11.5b}$$

Notice also that

$$\operatorname{rank}([\mu_* \mathcal{S}(\sigma)]|_{K \cdot E_\theta}) = \dim(\sigma), \tag{11.5c}$$

which is easy to compute.

According to Corollary 11.2, the classes $\{[\mu_* \mathcal{S}(\sigma)] \mid \sigma \in \widehat{L_1}\}$, after restriction to $K^K(Y)$, are a spanning set. Furthermore the kernel of the restriction map has as basis the (already computed) set

$$\bigcup_{Z \subset \partial Y} \{\mathcal{E}_k^{\text{orbalg}}(Z)\}. \tag{11.5d}$$

Now extracting a subset

$$\mathcal{E}_j^{\mathrm{orbalg}}(Y) = \sum_{\sigma \in \widehat{L_1}} n_j(\sigma)[\mu_* \mathcal{S}(\sigma)] \tag{11.5e}$$

of the span of the $[\mu_* \mathcal{S}(\sigma)]$ restricting to a basis of the image of the restriction is a linear algebra problem. Because the rank (the virtual dimension of fibers over $K \cdot E_\theta$) is additive in the Grothendieck group, we can compute each integer

$$\mathrm{rank}([\mathcal{E}_j^{\mathrm{orbalg}}] = \sum_{\sigma \in \widehat{L_1}} n_j(\sigma) \dim(\sigma). \tag{11.5f}$$

Just as in the complex case, we have swept under the rug the issue of doing finite calculations. It can be addressed along the same lines as in the complex case; we omit the details.

12. Associated varieties for real groups

In the setting of Section 9, suppose that X is a finite length $(\mathfrak{g}, K)$-module. Kazhdan-Lusztig theory allows us (if X is specified as a sum of irreducibles in the Langlands classification) to find an explicit formula (in the Grothendieck group of finite length Harish-Chandra modules)

$$X = \sum_{\Lambda \in \mathcal{P}_{\mathrm{L}}(G,K)} m_X(\Lambda) I(\Lambda). \tag{12.6a}$$

Fix a K-invariant good filtration of the Harish-Chandra module X, so that $\mathrm{gr}\, X$ is a finitely generated $S(\mathfrak{g}/\mathfrak{k})$-module supported on $\mathcal{N}_\theta^*$. The class in equivariant K-theory

$$[\mathrm{gr}\, X] \in K^K(\mathcal{N}_\theta^*) \tag{12.6b}$$

is independent of the choice of good filtration. If we rewrite each $I(\Lambda)|_K$ in terms of K-Langlands parameters using Proposition 9.11(6), we find a computable formula

$$[\mathrm{gr}\, X] = \sum_{\Lambda_K \in \mathcal{P}_{K\text{-}\mathrm{L}}(G,K)} m_X(\Lambda_K)[\Lambda_K]_\theta. \tag{12.6c}$$

Recall now the classes $[\mathcal{E}_k^{\mathrm{orbalg}}(Z)]$ constructed in Algorithm 11.4. Comparing their known formulas with (12.6c), we can do a change of basis calculation, and get an explicit formula

$$[\mathrm{gr}\, X] = \sum_{\mathcal{E}_k^{\mathrm{orbalg}}(Z)} n_X(\mathcal{E}_k^{\mathrm{orbalg}}(Z))[\mathcal{E}_k^{\mathrm{orbalg}}(Z)], \tag{12.6d}$$

with computable integers $n_X(\mathcal{E}_k^{\mathrm{orbalg}}(Z))$.

Theorem 12.7. *Suppose X is a $(\mathfrak{g}, K)$-module. Use the notation of* (12.6)*.*

1. *The associated variety of X (Definition 4.2 is the union of the closures of the maximal K-orbits $Z \subset \mathcal{N}_\theta^*$ with some $n_X(\mathcal{E}_k^{\mathrm{orbalg}}(Z)) \neq 0$.*
2. *The multiplicity of a maximal orbit Z in the associated cycle of X is*

$$\sum_{\mathcal{E}_k^{\mathrm{orbalg}}(Z)} n_X(\mathcal{E}_k^{\mathrm{orbalg}}(Z))\,\mathrm{rank}(\mathcal{E}_k^{\mathrm{orbalg}}(Z)).$$

The proof is identical to that of Theorem 8.14 above.

References

[1] Pramod N. Achar, *On the equivariant K-theory of the nilpotent cone in the general linear group*, Represent. Theory **8** (2004), 180–211.
[2] ———, *On exotic and perverse-coherent sheaves*, Representations of reductive groups, Progr. Math., vol. 312, Birkhäuser/Springer, Cham, 2015, pp. 11–49.
[3] Jeffrey Adams, Marc van Leeuwen, Peter Trapa, and David A. Vogan Jr., *Unitary representations of real reductive groups*, available at `arXiv:1212.2192[math.RT]`.
[4] *Atlas of Lie Groups and Representations software*, 2019.
[5] Dan Barbasch and Mark R. Sepanski, *Closure ordering and the Kostant-Sekiguchi correspondence*, Proc. Amer. Math. Soc. **126** (1998), no. 1, 311–317.
[6] Dan M. Barbasch and David A. Vogan Jr., *The local structure of characters*, J. Funct. Anal. **37** (1980), 27–55.
[7] Roman Bezrukavnikov, *Quasi-exceptional sets and equivariant coherent sheaves on the nilpotent cone*, Represent. Theory **7** (2003), 1–18 (electronic).
[8] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston, Inc., Boston, MA, 1997.

[9] David H. Collingwood and William M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993.

[10] Harish-Chandra, *Representations of a semisimple Lie group on a Banach space. I*, Trans. Amer. Math. Soc. **75** (1953), 185–243.

[11] Joe Harris, *Algebraic geometry. A first course*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995. A first course; Corrected reprint of the 1992 original.

[12] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[13] R. Howe, *Wave front sets of representations of Lie groups*, Automorphic Forms, Representation Theory, and Arithmetic, 1981.

[14] Bertram Kostant and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809.

[15] George Lusztig and David A. Vogan Jr., *Singularities of closures of K-orbits on flag manifolds*, Invent. Math. **71** (1983), no. 2, 365–379.

[16] Wilfried Schmid and Kari Vilonen, *Characteristic cycles and wave front cycles of representations of reductive Lie groups*, Ann. of Math. (2) **151** (2000), no. 3, 1071–1118.

[17] Jirō Sekiguchi, *Remarks on real nilpotent orbits of a symmetric pair*, J. Math. Soc. Japan **39** (1987), no. 1, 127–138.

[18] R. W. Thomason, *Algebraic K-theory of group scheme actions*, Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), Ann. of Math. Stud., vol. 113, Princeton Univ. Press, Princeton, NJ, 1987, pp. 539–563.

[19] Michèle Vergne, *Instantons et correspondance de Kostant-Sekiguchi*, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), no. 8, 901–906.

[20] David A. Vogan Jr., *Representations of real reductive Lie groups*, Progress in Mathematics, vol. 15, Birkhäuser, Boston, Mass., 1981.

[21] ______, *Associated varieties and unipotent representations*, Harmonic analysis on reductive groups (Brunswick, ME, 1989), Progr. Math., vol. 101, Birkhäuser Boston, Boston, MA, 1991, pp. 315–388.

[22] ______, *The method of coadjoint orbits for real reductive groups*, Representation theory of Lie groups (Park City, UT, 1998), IAS/Park City Math. Ser., vol. 8, Amer. Math. Soc., Providence, RI, 2000, pp. 179–238.

[23] ______, *Branching to a maximal compact subgroup*, Harmonic Analysis, Group Representations, Automorphic Forms, and Invariant Theory: In Honour of Roger E. Howe, 2007.

[24] ______, *The size of infinite-dimensional representations*, Jpn. J. Math. **12** (2017), no. 2, 175–210.

[25] Nolan R. Wallach, *Real reductive groups. II*, Pure and Applied Mathematics, vol. 132, Academic Press, Inc., Boston, MA, 1992.

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