Dixmier Algebras, Sheets, and Representation Theory

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Dedicated to Professor Jacques Dixmier on his sixty-fifth birthday

1. Introduction.

One of the grand unifying principles of representation theory is the method of coadjoint orbits. After impressive successes in the context of nilpotent and solvable Lie groups, however, the method encountered serious obstacles in the semisimple case. Known examples (like $SL(2,\mathbb{R})$) suggested a strong connection between the structure of the unitary dual and the geometry of the orbits, but it proved very difficult to formulate any precise general conjectures that were entirely consistent with these examples.

In the late 1960's, Dixmier suggested a way to avoid some of these problems. Motivated in part by the theory of C^*-algebras, he suggested that one should temporarily set aside a direct study of unitary representations and concentrate instead on their annihilators in the universal enveloping algebra. Classification of the annihilators would be a kind of approximation to the classification of the unitary representations themselves. The hope was that this approximation would be crude enough to be tractable, and yet precise enough to provide useful insight into the unitary representations themselves. This hope has been abundantly fulfilled: the two classification problems are now inextricably intertwined, and they continue constantly to shed new light on each other.

To be more precise, suppose $G_{\mathbb{R}}$ is a connected real Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$. The set of irreducible unitary representations of $G_{\mathbb{R}}$ is written Unit $G_{\mathbb{R}}$. Write $\mathfrak{g}$ for the complexification of $\mathfrak{g}_{\mathbb{R}}$, and $U(\mathfrak{g})$ for its universal enveloping algebra. If $(\pi,\mathcal{H})$ is a unitary representation of $G_{\mathbb{R}}$, then $U(\mathfrak{g})$ acts on the dense subspace $\mathcal{H}^\infty$ of smooth vectors in $\mathcal{H}$. We define $\mathrm{Ann}(\pi)$ to be the annihilator in $U(\mathfrak{g})$ of $\mathcal{H}^\infty$. Then $\mathrm{Ann}(\pi)$ is a two-sided ideal in $U(\mathfrak{g})$. (Our ideals will always be two-sided unless the contrary is explicitly stated.)

An ideal I in any ring R with unit is called *(left) primitive* if it is the annihilator of a simple (left) R-module. (This says exactly that I is the largest two-sided ideal contained in some maximal left ideal.) A maximal ideal is always primitive, but a primitive ideal need not be maximal. The ideal I is called *prime* if whenever J and J' are ideals with $JJ' \subset I$, then either $J \subset I$ or $J' \subset I$. A primitive ideal is necessarily prime, but a prime ideal need not be primitive. We say that I is *completely prime* if the quotient ring R/I has no zero divisors. A completely prime ideal is prime, but a prime ideal need not be completely prime. We write

$$\begin{aligned} \mathrm{Spec}\ R &= \text{set of prime ideals in } R \\ \mathrm{Prim}\ R &= \text{set of primitive ideals in } R \\ \mathrm{Spec}_1 R &= \text{set of completely prime ideals in } R \\ \mathrm{Prim}_1 R &= \text{set of completely prime primitive ideals in } R\ . \end{aligned} \tag{1.1}$$

Suppose now that $(\pi,\mathcal{H})$ is an irreducible representation of G. Because the irreducibility is topological rather than algebraic, the space of smooth vectors $\mathcal{H}^\infty$ will not be a simple module for $U(\mathfrak{g})$ (unless the representation is finite-dimensional). Nevertheless, Dixmier proved

Theorem 1.2 ([7]). *Suppose π is an irreducible unitary representation of a connected Lie group $G_{\mathbb{R}}$. Then* $\mathrm{Ann}(\pi)$ *is a primitive ideal in* $U(\mathfrak{g})$.

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To explain the connection with the orbit method, suppose that $\mathcal{O}_{\mathbb{R}}$ is an orbit of $G_{\mathbb{R}}$ on $\mathfrak{g}_{\mathbb{R}}^*$. Let G be a complex connected Lie group with Lie algebra $\mathfrak{g}$, and let $\mathcal{O}_{\mathbb{C}} = G \cdot \mathcal{O}_{\mathbb{R}}$. Roughly speaking, the method of coadjoint orbits seeks to attach to $\mathcal{O}_{\mathbb{R}}$ an irreducible unitary representation $\pi(\mathcal{O}_{\mathbb{R}})$ of $G_{\mathbb{R}}$. (Actually one needs to restrict attention to certain orbits, called "admissible," and one needs some additional data beyond the orbit itself.) We will call a correspondence from orbits to representations an *orbit correspondence*, and denote it KK (for Kirillov-Kostant). Under favorable circumstances (for example if $G_{\mathbb{R}}$ is an algebraic group) it appears that almost all interesting unitary representations should appear in the image of an orbit correspondence. The problem of constructing an orbit correspondence is therefore one of the most important unsolved problems in representation theory. We consider next what Dixmier's ideas can tell us about this problem.

One of Dixmier's insights was that one should attach to the complexified orbit a primitive ideal in $U(\mathfrak{g})$. This may be formulated as

Conjecture 1.3 (Dixmier). Suppose G is a complex connected Lie group with Lie algebra $\mathfrak{g}$, and $\mathcal{O}_{\mathbb{C}}$ is an orbit of G on $\mathfrak{g}^*$. Then there is attached to $\mathcal{O}_{\mathbb{C}}$ a completely prime primitive ideal $I(\mathcal{O}_{\mathbb{C}})$ in $U(\mathfrak{g})$.

In fact the ideal should depend only on the Zariski closure of $\mathcal{O}_{\mathbb{C}}$ in $\mathfrak{g}^*$. A correspondence of the form in the conjecture is called a *Dixmier map*, and often written Dix. In its strongest original form (never formulated by Dixmier) the "Dixmier conjecture" asks that Dix should be a bijection from Zariski closures of orbits to completely prime primitive ideals. (Work of Borho and others has shown that this is not possible if the Dixmier map is to have other reasonable properties.)

If one had both an orbit correspondence and a Dixmier map, there would be (roughly speaking) a diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{R}} & \longrightarrow & \pi \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{C}} & \longrightarrow & I \end{array} \tag{1.4)(a}$$

of maps among the sets

$$\begin{array}{ccc} \mathfrak{g}_{\mathbb{R}}^*/G_{\mathbb{R}} & \xrightarrow{\text{KK}} & \text{Unit } G_{\mathbb{R}} \\ \downarrow & & \downarrow \text{Ann} \\ \mathfrak{g}^*/G & \xrightarrow{\text{Dix}} & \text{Prim } U(\mathfrak{g}) \end{array} \tag{1.4)(b}$$

A natural requirement to impose on KK and Dix is that this diagram ought to commute. Unfortunately this is not a reasonable condition. The Dixmier map is supposed to take values in $\text{Prim}_1 U(\mathfrak{g})$ (the completely prime primitive ideals); but the annihilator of a unitary representation need not be completely prime. (The simplest example is the defining representation of $SU(2)$. The corresponding quotient $U(\mathfrak{g})/\text{Ann}(\pi)$ is isomorphic to the algebra of 2 x 2 matrices, and therefore has zero divisors.)

Since the diagram (1.4) cannot commute, we look for slightly weaker requirements. The simplest one consistent with examples like $SU(2)$ is

$$\text{Ann}(\pi(\mathcal{O}_{\mathbb{R}})) \subset I(\mathcal{O}_{\mathbb{C}}) \tag{1.5}$$

Equivalently,

$$\pi(\mathcal{O}_{\mathbb{R}}) \text{ is a } U(\mathfrak{g})/I(\mathcal{O}_{\mathbb{C}})\text{-module.} \tag{1.5$'$}$$

Properly understood, (1.5)$'$ casts representation theory in an entirely new light. We want to interpret it as a program for constructing an orbit correspondence. The first step is to construct a Dixmier map: more precisely, to construct and understand the algebra $U(\mathfrak{g})/I(\mathcal{O}_{\mathbb{C}})$. A unitary representation $\pi(\mathcal{O}_{\mathbb{R}})$ attached to $\mathcal{O}_{\mathbb{R}}$ —that is, the image of the orbit correspondence — should then be constructed and understood as a module for this algebra.

From now on we will confine our attention almost exclusively to reductive groups. To a large extent the point of view just described is the one adopted by Beilinson and Bernstein in their fundamental paper [1]. (It is not difficult to find similar ideas in much earlier work — for instance in the theory of C^*-algebras, or in the work of Gelfand and Kirillov on quotient rings of enveloping algebras. What is unique to Beilinson and Bernstein is the successful application of a general structure theorem for quotients of $U(\mathfrak{g})$ to representation theory.) They showed that if I is any minimal primitive ideal in $U(\mathfrak{g})$, then $U(\mathfrak{g})/I$ is isomorphic to an algebra of differential operators on a flag variety. Consequently any irreducible $\mathfrak{g}$-module may be regarded as a module for a differential operator algebra. This perspective has proven to be tremendously illuminating in a wide range of contexts: for the classification of representations, for the construction of intertwining operators, for analysis on symmetric spaces, and for primitive ideal theory, for example.

Nevertheless, the Beilinson-Bernstein approach has some limitations. In the context of (1.5), it amounts to looking for $\pi(\mathcal{O}_{\mathbb{R}})$ as a module not for the natural algebra $U(\mathfrak{g})/I(\mathcal{O}_{\mathbb{R}})$, but rather for some much larger algebra (of which $U(\mathfrak{g})/I(\mathcal{O}_{\mathbb{R}})$ is a quotient). The modules we want are certainly present in the Beilinson-Bernstein picture, but so are many extraneous ones. Putting more precise constraints on the primitive ideal should put more precise constraints on the modules, and so (one hopes) help to suggest the definition of the orbit correspondence. For this reason, it is still worthwhile to pursue the program described after (1.5).

There is already a tremendous amount of information available about the construction of a Dixmier map. Most of it is based on the notion of "parabolic induction" in one form or another. Roughly speaking, the idea is that most coadjoint orbits for G can be constructed in a simple way from coadjoint orbits for a Levi subgroup L. Parabolic induction also provides a way to construct primitive ideals for G from those for L. If we already know something about a Dixmier map on L, then we can hope to specify part of a Dixmier map for G by requiring that induction of orbits should correspond to induction of primitive ideals under the Dixmier maps for L and G. (One can make exactly parallel remarks about representations and orbit correspondences.) In the case of $SL(n)$, Borho in [2] used exactly this idea to define a Dixmier map completely.

For reductive groups not of type A, Borho discovered a fundamental obstruction to this approach. It can happen that the same coadjoint orbit $\mathcal{O}_{\mathbf{C}}$ for G arises in two different ways by induction, and that the corresponding induced primitive ideals I_1 and I_2 are different; apparently both ought to be attached to $\mathcal{O}_{\mathbf{C}}$. One of the goals of the theory of "Dixmier algebras" initiated in [20] and [15] is to circumvent this problem. Roughly speaking, the idea is this. In Conjecture 1.3, the orbit $\mathcal{O}_{\mathbf{C}}$ is replaced by an "orbit datum," consisting of some additional algebro-geometric structure on an orbit (Definition 2.2). The primitive quotient $U(\mathfrak{g})/I$ is replaced by a "Dixmier algebra" (Definition 2.1), which is an extension ring of a quotient of $U(\mathfrak{g})$. Conjecture 1.3 is replaced by a conjectural map from orbit data to Dixmier algebras (Conjecture 2.3). (Such a map would automatically descend to a multi-valued Dixmier map in the sense of Conjecture 1.3.)

The primary purpose of this paper is to extend to orbit data and Dixmier algebras the notions of parabolic induction discussed above. This is accomplished in Proposition 3.15 and Corollary 4.17 respectively. These results suggest a way to define a Dixmier map on induced orbit data (in terms of Dixmier maps on Levi subgroups). In order to justify this definition, one would have to show that if an orbit datum is induced in two different ways, then the corresponding induced Dixmier algebras must coincide. This we have not been able to prove; it would follow from Conjectures 3.24 and 4.18. A more complete discussion of the status of Conjecture 2.3 may be found in section 5.

These ideas of course focus attention on the orbit data that are *not* induced. (In fact the primary motivation for this paper was not so much to say something about induced orbits (or primitive ideals, or Dixmier algebras) as to understand by a process of elimination those that are not.) We call these non-induced orbit data *rigid*, for reasons that will be clearer in section 3 (cf. Definition 3.22 and Proposition 3.23). An interesting point is that rigidity is a property of the full orbit datum, and not just of the underlying orbit; it may be possible to deform the orbit but not the orbit datum. In section 5 we recall from [21] a conjectural construction of Dixmier algebras attached to rigid orbit data.

The program described after (1.5) suggests that one should turn next to the description of modules for induced Dixmier algebras, seeking among these candidates for representations attached by the orbit correspondence to real forms of $\mathcal{O}_{\mathbf{C}}$. In this direction we do only a little. Induced Dixmier algebras are generalizations of Beilinson and Bernstein's twisted differential operator algebras. It should therefore be possible to analyze their modules by the kind of geometric "localization" familiar in the differential operator case. We prove here only a few of the basic facts about such a localization theory (notably Corollary 6.16 and Theorem 7.9).

Here is a more detailed outline of the contents of this paper. Section 2 recalls from [20] and [15] the definition of Dixmier algebras and orbit data, and a corresponding refinement of Conjecture 1.3. (One of McGovern's results in [15] is that the main conjecture in [20] is false; and McGovern has since found counterexamples for a revision circulated in an earlier version of this paper. Conjecture 2.3 appears to be consistent with all of his work to date.) Section 3 outlines the extension to orbit data of some of the basic structure theory for coadjoint orbits: Jordan decomposition, parabolic induction, and sheets. In the theory of sheets we find some strong (conjectural!) geometric evidence for the correctness of the general approach to the Dixmier conjecture in [20]: Conjecture 3.24 says that distinct sheets of "orbit data" should be disjoint. The failure of the corresponding fact for sheets of orbits is at the heart of the non-uniqueness problems discovered by Borho and discussed above. Section 4 presents the construction of Dixmier algebras by parabolic induction. Section 5 outlines how these ingredients should fit together to define a Dixmier map for G.

The rest of the paper is devoted to related technical results. In section 6 (following [5]), we relate induction of Dixmier algebras to ordinary induction of Harish-Chandra bimodules. (Recall that Harish-Chandra bimodules are closely related to infinite-dimensional representations of G regarded as a real Lie group. By "ordinary induction" we mean the bimodule construction corresponding to parabolic induction of group representations (in the sense of Mackey and Gelfand-Naimark).) Perhaps the most important consequence is a cohomology vanishing theorem (Corollary 6.16). This generalizes the fact that the higher cohomology of G/Q with coefficients in the sheaf of differential operators is zero. Section 7 considers the translation principle for induced Dixmier algebras and their modules.

A key tool in all the induction constructions (both for orbit data and for Dixmier algebras) is the notion of equivariant bundles on homogeneous spaces (in the algebraic category). These help to formalize the idea that G-equivariant constructions on G/H are equivalent to H-equivariant constructions at a point. A few of the basic definitions and results are summarized in an appendix for the convenience of the reader.

2. Dixmier algebras.

Suppose for the balance of this paper that G is a connected complex reductive algebraic group with Lie algebra $\mathfrak{g}$.

Definition 2.1 (cf. [15]). A *Dixmier algebra* for G is a pair (A, ϕ) satisfying the following conditions.

i) A is an algebra over $\mathbb{C}$, equipped with a locally finite algebraic action (called Ad) of G on A by algebra automorphisms.
ii) The map ϕ is an algebra homomorphism of $U(\mathfrak{g})$ into A , respecting the two adjoint actions of G. The differential of the action Ad of G on A is the difference of the left and right actions of $\mathfrak{g}$ defined by ϕ.
iii) A is a finitely generated $U(\mathfrak{g})$-module.
iv) Each irreducible G-module occurs at most finitely often in the adjoint action of G on A.

The simplest example of a Dixmier algebra is any quotient $U(\mathfrak{g})/I$ of $U(\mathfrak{g})$ by a primitive ideal.

Definition 2.2. An *orbit datum* for G is a pair (R, ψ) satisfying the following conditions.

i) R is an algebra over $\mathbb{C}$, equipped with a locally finite algebraic action (called Ad) of G on R by algebra automorphisms.
ii) The map ψ is an algebra homomorphism of $S(\mathfrak{g})$ into the center of R, respecting the two adjoint actions of G.
iii) R is a finitely generated $S(\mathfrak{g})$-module.
iv) Each irreducible G-module occurs at most finitely often in the adjoint action of G on R.

The *support* Σ of the orbit datum is the algebraic variety $\mathcal{V}(\ker\psi) \subset \mathfrak{g}^*$. (Thus Σ is the image of the *moment map* ψ^* : Spec $R \longrightarrow \mathfrak{g}^*$.) The orbit datum is called *completely prime* if R is completely prime, and *commutative* if R is commutative. It is called *geometric* if R is commutative, completely prime, and normal. It is called *pre-unipotent* if G is semisimple, and Σ is contained in the nilpotent cone. Finally, it is called *unipotent* if it is pre-unipotent and geometric.

The simplest example of an orbit datum is the quotient $S(\mathfrak{g})/J$ of $S(\mathfrak{g})$ by the ideal of functions vanishing on an orbit. In this case the support Σ is the closure of the orbit. (An example not of this kind appears as

Example 3.21 below.) For general orbit data, condition (iv) implies that Σ is a finite union of orbit closures. (To see this, consider the algebra $Z = S(\mathfrak{g})^G$ of invariants in the symmetric algebra. The maximal ideals of Z parametrize the semisimple orbits of G on $\mathfrak{g}^*$: if $\mathfrak{m}$ is such a maximal ideal, then the associated variety $\mathcal{V}(\mathfrak{m})$ consists of all elements of $\mathfrak{g}^*$ for which the semisimple part of the Jordan decomposition belongs to the corresponding orbit. Consequently each $\mathcal{V}(\mathfrak{m})$ is a finite union of coadjoint orbits; in fact Kostant's theorem on the principal nilpotent element implies that it is the closure of a single coadjoint orbit. On the other hand, the image $\psi(Z)$ is contained in the G-invariants of R, which form a finite-dimensional algebra by (2.2)(iv). It follows that the kernel of ψ contains an ideal $\mathcal{I}$ of finite codimension in Z. Consequently Σ is contained in the (finite) union of the various $\mathcal{V}(\mathfrak{m})$, with $\mathfrak{m}$ a maximal ideal in Z containing $\mathcal{I}$.) If the orbit datum is completely prime, then Σ is necessarily the closure of a single coadjoint orbit. A geometric orbit datum (R, ψ) is the same thing as a normal irreducible affine algebraic variety X (namely Spec R) equipped with a G action and an equivariant finite morphism ψ^* from X to an orbit closure in $\mathfrak{g}^*$. A unipotent orbit datum is therefore exactly a unipotent Poisson variety in the sense of [21].

Before formulating the Dixmier conjecture, we should say a little bit about filtrations. Suppose A is an algebra filtered by $\frac{1}{2}\mathbb{N}$. This means that we are given an increasing family of subspaces

$$A_0 \subset A_{\frac{1}{2}} \subset A_1 \subset \cdots$$

so that

$$\cup_j A_j = A, \qquad A_p A_q \subset A_{p+q}.$$

Then the associated graded space $\operatorname{gr} A$ is a graded algebra. (Our basic example is the standard filtration $U_n(\mathfrak{g})$; in this case $\operatorname{gr} U(\mathfrak{g}) = S(\mathfrak{g})$.) An increasing filtration on an A-module M is called *compatible* if

$$\cup_j M_j = M, \qquad A_p M_q \subset M_{p+q}.$$

In this case the associated graded space $\operatorname{gr} M$ is in a natural way a graded module for $\operatorname{gr} A$. A compatible filtration of M is called *good* if $\operatorname{gr} M$ is finitely generated as a module for $\operatorname{gr} A$.

Here is a version of the Dixmier conjecture for reductive groups. It is taken from [20], but modified in accordance with the requirements of [15].

Conjecture 2.3. Suppose G is a complex connected reductive algebraic group. Then there is a natural injection Dix from the set of geometric orbit data for G (Definition 2.2) into the set of completely prime Dixmier algebras for G. This correspondence should have the following properties. Fix an orbit datum (R, ψ), and write (A, ϕ) for the corresponding Dixmier algebra.

i) The Gelfand-Kirillov dimensions of A and R are equal.
ii) A and R are isomorphic as G-modules.
iii) A and R admit filtrations indexed by $\frac{1}{2}\mathbb{N}$, with the following properties.
   a) The filtration of A is good (and therefore by definition compatible) for A regarded as a $U(\mathfrak{g})$-module. In particular, $\phi(U_n(\mathfrak{g})) \subset A_n$.
   b) The filtration of R is good for R regarded as an $S(\mathfrak{g})$-module. In particular, $\psi(S^n(\mathfrak{g})) \subset R_n$.
   c) The associated graded algebras gr A and gr R are completely prime.
   d) There is a G-equivariant isomorphism ξ : gr $A \rightarrow$ gr R carrying gr ϕ to gr ψ.

In (a), $U_n(\mathfrak{g})$ is the nth level of the standard filtration of $U(\mathfrak{g})$; and in (b), $S^n(\mathfrak{g})$ is the nth level of the standard gradation. (Of course we could equally well use the standard filtration of $S(\mathfrak{g})$ in (b).)

Conditions (i) and (ii) are included only for expository purposes; they are consequences of (iii). That the filtration in (iii) ought to be indexed by $\frac{1}{2}\mathbb{N}$ (rather than some $\frac{1}{k}\mathbb{N}$) is suggested by [16]. The map Dix should extend to a bijection from some larger set of completely prime orbit data onto all completely prime Dixmier algebras. McGovern has pointed out that it cannot be defined on *all* completely prime orbit data, however.

Orbit data are close enough to coadjoint orbits to admit Jordan decompositions, which we now describe. Suppose (R, ψ) is a completely prime orbit datum. Fix λ in $\mathfrak{g}^*$ so that $\ker \psi$ is the ideal of functions vanishing on $\mathcal{O} = G \cdot \lambda$. Write

$$\lambda = \lambda_s + \lambda_u \tag{2.4)(a}$$

for the Jordan decomposition of λ, and

$$L = \{ g \in G \mid \mathrm{Ad}^*(g)\lambda_s = \lambda_s \} \tag{2.4)(b}$$

(a Levi subgroup of G). L is again a connected reductive algebraic group; we want to relate (R, ψ) to a completely prime orbit datum (R_L, ψ_L) for L. Define

$$\lambda_L = \lambda \mid_{\mathfrak{l}}; \tag{2.4)(c}$$

we want the support of (R_L, ψ_L) to be the closure Σ_L of $\mathcal{O}_L = L \cdot \lambda_L$. Let $\mathfrak{s}$ be the unique $\mathrm{Ad}(L)$-invariant complement for $\mathfrak{l}$ in $\mathfrak{g}$. We identify $\mathfrak{l}^*$ with the linear functionals on $\mathfrak{g}$ vanishing on $\mathfrak{s}$. By Proposition A.2(b), the natural inclusion of Σ_L in Σ induces a G-equivariant morphism

$$G \times_L \Sigma_L \to \Sigma. \tag{2.4)(d}$$

Every point $\sigma_L \in \Sigma_L$ has semisimple part λ_s. By the Jordan decomposition, the stabilizer of σ_L in G is contained in L. By (A.3), the map (2.4)(d) is one-to-one. A slightly more careful analysis, which we omit, shows that (2.4)(d) is actually an isomorphism of varieties. By Proposition A.5, the category of finitely generated modules with G-action on Σ is equivalent (by passage to geometric fibers) to the category of finitely generated modules with L-action on Σ_L. A little more explicitly, let $J_L \subset S(\mathfrak{g})$ be the defining ideal for Σ_L. Define

$$R_L = R/R\psi(J_L). \tag{2.5)(a}$$

Obviously R_L is an algebra equipped with an action of L and a map

$$\psi_L : S(\mathfrak{l}) \to R_L. \tag{2.5)(b}$$

(The map comes from ψ by restriction on the domain and passage to the quotient on the range.) The preceding discussion implies that (R_L, ψ_L) is a completely prime orbit datum for L, and that

$$R \simeq G \times_L R_L \tag{2.5)(c}$$

(cf. (A.4)). By (A.4)(c), this last formula says that R may be identified with the space of algebraic maps ρ from G to R_L, subject to the condition

$$\rho(gl) = \mathrm{Ad}(l^{-1})\rho(g) \tag{2.5)(d}$$

for g in G and l in L. The algebra structure on R is just pointwise multiplication.

Finally, one can check easily that the adjoint action of $Z(L)_0$ on R_L must be trivial. Consequently (R_L, ψ_L) gives rise to a pre-unipotent orbit datum (R_u, ψ_u) for $L/Z(L)_0$. The algebra R_u is just R_L, and the map ψ_u is the restriction of ψ_L to $[\mathfrak{l}, \mathfrak{l}]$, composed with the natural isomorphism

$$Lie(L/Z(L)_0) \simeq [\mathfrak{l}, \mathfrak{l}]. \tag{2.5)(e}$$

Conversely, one can recover ψ_L from ψ_u by the requirement

$$\psi_L(A) = \lambda_s(A) \qquad (A \in \mathfrak{z}(\mathfrak{l})). \tag{2.5)(f}$$

The following theorem summarizes this discussion.

Theorem 2.6 (Jordan decomposition for orbit data). *There is a natural bijection between the set of completely prime orbit data (R, ψ) for G (Definition 2.2) and the set of G-conjugacy classes of triples $(L, \lambda_s, (R_u, \psi_u))$. Here*

i) L is a Levi subgroup of G.

ii) $\lambda_s : \mathfrak{l} \to \mathbb{C}$ is a G-regular Lie algebra homomorphism.

iii) (R_u, ψ_u) is a completely prime pre-unipotent orbit datum for $L/Z(L)_0$.

The correspondence is specified by (2.4)-(2.5). In this bijection, R is commutative (respectively normal or geometric) if and only if R_u is.

In (ii), the "G-regular" hypothesis is just the condition (2.4)(b) above.

In light of the classification of unipotent orbit data in [21], Theorem 5.3, we get

Corollary 2.7. *The following three sets are in natural one-to-one correspondence:*

i) geometric orbit data (R, ψ) for G;

ii) G-conjugacy classes of pairs (λ, S), with λ in $\mathfrak{g}^*$ and S a subgroup of the "G-equivariant fundamental group" G^λ/G_0^λ of $G \cdot \lambda$;

iii) G-conjugacy classes of triples $(L, \lambda_s, (R_u, \psi_u))$, with L a Levi subgroup of G, λ_s a G-regular character of $\mathfrak{l}$, and (R_u, ψ_u) unipotent orbit data for $L/Z(L)_0$.

If G is simply connected, we can add

iv) finite coverings of coadjoint orbits.

Unfortunately, no analogous theorems are known for completely prime primitive Dixmier algebras. There is, however, a construction (by parabolic induction) of a Dixmier algebra associated to an orbit datum, assuming that such an algebra associated to (R_u, ψ_u) is already available. It will be described in section 4, after some geometric preliminaries in section 3.

We conclude this section with some useful formal ideas; for more background, see [17] and [8].

Definition 2.8. Suppose (A, ϕ) is a Dixmier algebra for G. The *opposite Dixmier algebra* is the pair (A^{op}, ϕ^{op}) defined as follows. The algebra A^{op} is the opposite algebra to A, with the same underlying vector space and multiplication

$$(a) \cdot_{op} (b) = ba.$$

The action of G on A^{op} is the same as on A. The map ϕ^{op} is characterized by

$$\phi^{op}(X) = \phi(-X) \qquad (X \in \mathfrak{g}).$$

A *transpose antiautomorphism* of (A, ϕ) is an isomorphism (usually written $a \mapsto {}^t a$) of A with its opposite Dixmier algebra.

Definition 2.9. Suppose (R, ψ) is an orbit datum for G with support Σ. The *opposite orbit datum* is the pair (R^{op}, ψ^{op}) defined in obvious analogy with Definition 2.8; it is an orbit datum with support $-\Sigma$. We can also define a *transpose antiautomorphism*.

The Dixmier correspondence of Conjecture 2.3 should respect passage to opposite algebras in the sense of these definitions. Now a geometric unipotent orbit datum is easily seen to be isomorphic to its opposite. The corresponding Dixmier algebras should therefore admit transpose antiautomorphisms. Such automorphisms will play a role in the theory of unipotent representations (cf. [23], Theorem 8.7(ii)).

3. Induction, sheets, and rigid orbit data.

In this section we consider parabolic induction for orbit data. Suppose L is a Levi subgroup of G. It is well known that there is a codimension-preserving map from coadjoint orbits for L to coadjoint orbits for G. This map sends a G-regular semisimple orbit $L \cdot \lambda$ to $G \cdot \lambda$, and sends $\{0\}$ to a Richardson nilpotent orbit. The purpose of this section is to extend this map to orbit data. Because the construction is not a very easy one to grasp, we will begin by recalling the theory on the level of orbits.

Definition 3.1 (cf. [14]). Suppose L is a Levi factor in G, and $\mathcal{O}_L$ is a nilpotent coadjoint orbit in $\mathfrak{l}^*$. We will construct from $\mathcal{O}_L$ a nilpotent coadjoint orbit for G. Although this orbit turns out to depend only on L, its construction requires the choice of a parabolic subgroup Q with Levi factor L. Write U for the unipotent radical of Q; then $L \simeq Q/U$. This quotient map gives rise to an injection

$$i_Q : \mathfrak{l}^* \hookrightarrow \mathfrak{q}^* \tag{3.1)(a}$$

identifying linear functionals on $\mathfrak{l}$ with linear functionals on $\mathfrak{q}$ that vanish on $\mathfrak{u}$. Define

$$\mathcal{O}_Q = i_Q(\mathcal{O}_L) \subset \mathfrak{q}^*. \tag{3.1)(b}$$

(Thus $\mathcal{O}_Q$ is isomorphic to $\mathcal{O}_L$.) Next, consider the restriction map

$$\pi_{G/Q} : \mathfrak{g}^* \twoheadrightarrow \mathfrak{q}^*. \tag{3.1)(c}$$

This exhibits $\mathfrak{g}^*$ as an affine bundle over $\mathfrak{q}^*$, with vector space $(\mathfrak{g}/\mathfrak{q})^*$. (That is, each fiber of $\pi_{G/Q}$ is a principal homogeneous space for the vector space $(\mathfrak{g}/\mathfrak{q})^*$ — a copy of the vector space with the origin forgotten.) Define

$$\tilde{\mathcal{O}}_{G/Q} = \pi_{G/Q}^{-1}(\mathcal{O}_Q) \subset \mathfrak{g}^*, \tag{3.1)(d}$$

an affine bundle over $\mathcal{O}_Q$. In particular, we have

$$\dim \tilde{\mathcal{O}}_{G/Q} = \dim \mathcal{O}_L + \dim G/Q. \tag{3.1)(e}$$

The action of Q on $\mathfrak{g}^*$ evidently preserves $\tilde{\mathcal{O}}_{G/Q}$. By Proposition A.2(b), we get a G-equivariant morphism

$$G \times_Q \tilde{\mathcal{O}}_{G/Q} \to G \cdot \tilde{\mathcal{O}}_{G/Q} \subset \mathfrak{g}^*. \tag{3.1)(f}$$

The induced bundle on the left has dimension equal to

$$\dim G/Q + \dim \tilde{\mathcal{O}}_{G/Q} = \dim \mathcal{O}_L + 2 \cdot \dim G/Q = \dim \mathcal{O}_L + dim G/L. \tag{3.1)(g}$$

It follows that any G-orbit in the induced bundle — and *a fortiori* any G-orbit in $G \cdot \tilde{\mathcal{O}}_{G/Q}$ — has dimension at most equal to $\dim \mathcal{O}_L + dim G/L$.

The *induced orbit*

$$\mathcal{O}_G = \mathrm{Ind}_{orb}(L \uparrow G)(\mathcal{O}_L)$$

is the unique nilpotent coadjoint orbit in $\mathfrak{g}^*$ satisfying any of the following equivalent conditions.

i) The orbit $\mathcal{O}_G$ is contained in $G \cdot \tilde{\mathcal{O}}_{G/Q}$; and the dimension of $\mathcal{O}_G$ is $\dim \mathcal{O}_L + \dim G/L$.
ii) There is a λ in $\mathcal{O}_G$ such that the restriction of λ to $\mathfrak{q}$ is trivial on $\mathfrak{u}$ and belongs to $\mathcal{O}_L$ on $\mathfrak{l}$; and the codimension of $\mathcal{O}_G$ in $\mathfrak{g}^*$ is equal to the codimension of $\mathcal{O}_L$ in $\mathfrak{l}^*$.
iii) Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$, and write W_L and W for the Weyl groups of L and G. Then the Springer representation of W (on W-harmonic polynomials on $\mathfrak{h}$) for $\mathcal{O}_G$ is generated by the Springer representation for $\mathcal{O}_L$.

Several remarks are in order here. First, the equivalence of (i) and (ii) is immediate from the definitions. In (iii), the Springer correspondence must be normalized to take the principal orbit to the trivial representation. We have included (iii) only because it shows that the induced orbit is independent of the choice of Q. The proof of its equivalence with (i) and (ii) ([14], Theorem 3.5) would require a substantial digression, so we omit it.

Second, there is (given Q) at most one orbit $\mathcal{O}_G$ satisfying (i) and (ii). To see this, notice that $\mathcal{O}_L$ (as a homogeneous space for a connected group) is an irreducible algebraic variety. As an affine bundle over $\mathcal{O}_L$, the variety $\tilde{\mathcal{O}}_{G/Q}$ is irreducible as well. Since G is connected, it follows that $G \cdot \tilde{\mathcal{O}}_{G/Q}$ is irreducible. We have already observed that the dimension of this set is at most $\dim \mathcal{O}_L + \dim G/L$. It can therefore contain at most one G-orbit of the desired dimension. For the proof that one exists, we refer to [14].

Third, we should check that $\mathcal{O}_G$ is nilpotent. For this it suffices to show that $\tilde{\mathcal{O}}_{G/Q}$ consists of nilpotent elements. Now it is an elementary exercise to show that a linear functional λ on a reductive Lie algebra is nilpotent if and only if there is a Borel subalgebra $\mathfrak{b}$ such that $\lambda\,|_{\mathfrak{b}} = 0$. So suppose $\lambda \in \tilde{\mathcal{O}}_{G/Q}$. Then $\lambda\,|_{\mathfrak{u}} = 0$, and $\lambda|_{\mathfrak{l}} = \lambda_L \in \mathcal{O}_L$. By hypothesis λ_L is nilpotent, so there is a Borel subalgebra $\mathfrak{b}_L$ of $\mathfrak{l}$ on which λ_L is zero. Since Q is parabolic, $\mathfrak{b} = \mathfrak{b}_L + \mathfrak{u}$ is a Borel subalgebra of $\mathfrak{g}$; and clearly $\lambda|_{\mathfrak{b}} = 0$.

A nilpotent orbit is called *rigid* if it is not induced from a proper Levi subgroup, and *induced* otherwise. It is called *Richardson* if it is induced from the zero orbit on a Levi subgroup. (Thus the zero orbit is called Richardson but not induced.)

To get a little feeling for the notion of induced orbit, we consider some examples in the classical groups. If G is $GL(n)$ or $SL(n)$, then every nilpotent orbit except 0 is induced. (To describe this explicitly, one can replace 2 by 1 everywhere in the discussion of other classical groups below.) If G_n is any other classical group of n by n matrices, then any nilpotent coadjoint orbit $\mathcal{O}$ gives rise (via Jordan blocks) to a partition

$$\pi = (p_1, ..., p_r) \tag{3.2}$$

of n. The sequence of non-negative integers p_i is (weakly) decreasing, and the sum of the p_i is n. (It is often convenient to allow some of the p_i to be zero; we identify partitions when their non-zero parts agree.) Suppose that there is a jump of 2 in this sequence; say

$$p_m - 2 \geq p_{m+1}. \qquad (3.3)(a)$$

Then $\mathcal{O}$ is induced, in the following way. G has a Levi factor

$$L = G_{n-2m} \times GL(m), \qquad (3.3)(b)$$

with the first factor a classical group of $n - 2m$ by $n - 2m$ matrices. There is a nilpotent orbit $\mathcal{O}'$ for G_{n-2m} corresponding to the partition

$$\pi' = (p_1 - 2, ..., p_m - 2, p_{m+1}, ..., p_r). \qquad (3.3)(c)$$

(In the case of $SO(2k)$ the orbit $\mathcal{O}'$ may not be uniquely determined by π'; one must make an appropriate choice of $\mathcal{O}'$.) Define $\mathcal{O}_L$ to be the orbit $(\mathcal{O}', 0)$ in L. Then

$$\mathcal{O} = \mathrm{Ind}_{orb}(L \uparrow G)(\mathcal{O}_L). \qquad (3.3)(d)$$

There are other ways for a nilpotent orbit to be induced in the classical groups, but the preceding special case captures most of the general flavor. The first example of a non-zero rigid orbit is the minimal (4-dimensional) nilpotent in $Sp(4)$; it cannot be induced because its dimension is not of the form $\dim \mathcal{O}_L + \dim G/L$ for any orbit in proper Levi subgroup L. The corresponding partition is $(2, 1, 1)$, which of course has no jumps of 2. (Viewed within the locally isomorphic group $SO(5)$, this orbit gives rise to the partition $(2, 2, 1)$, which still has no jumps.)

We now extend Definition 3.1 to include induction from non-nilpotent orbits.

Definition 3.4. Suppose L is a Levi subgroup of G, and $\mathcal{O}_L \subset \mathfrak{l}^*$ is a coadjoint orbit for L. The *induced orbit*

$$\mathcal{O}_G = \mathrm{Ind}_{orb}(L \uparrow G)(\mathcal{O}_L)$$

is defined precisely as in Definition 3.1(i) or (ii), as the unique dense orbit in $G \cdot (\pi_{G/Q}^{-1}(i_Q(\mathcal{O}_L)))$. It follows easily from the construction that the codimension of $\mathcal{O}_G$ in $\mathfrak{g}^*$ is equal to the codimension of $\mathcal{O}_L$ in $\mathfrak{l}^*$. (Again the construction depends on a choice of parabolic, but the orbit constructed does not. We will not stop to prove this independence; it follows from the special case of Definition 3.1 by Proposition 3.6.)

We have already seen one instance of this more general induction in (2.4). Here is the connection.

Lemma 3.5. *Suppose $\lambda = \lambda_s + \lambda_u$ is the Jordan decomposition of a linear functional in $\mathfrak{g}^*$. Write L for the centralizer in G of λ_s, a Levi subgroup of G, and $\lambda_L = (\lambda_L)_s + (\lambda_L)_u$ for the restriction of λ to L. Set*

$$\mathcal{O}_L = L \cdot \lambda_L = (\lambda_L)_s + L \cdot (\lambda_L)_u.$$

Then the orbit for G induced by $\mathcal{O}_L$ is

$$\mathrm{Ind}_{orb}(L \uparrow G)(\mathcal{O}_L) = G \cdot \lambda.$$

Proof. Choose a parabolic $Q = LU$ as in Definition 3.4. Define $\mathcal{O}_Q \subset \mathfrak{q}^*$ and $\tilde{\mathcal{O}}_{G/Q} \subset \mathfrak{g}^*$ as in Definition 3.1. It is immediate from the definitions that $\lambda \in \tilde{\mathcal{O}}_{G/Q}$. It remains only to check that $G \cdot \lambda$ has the correct dimension. But this follows from the fact (part of the Jordan decomposition) that the centralizer in G of λ is just the centralizer in L of λ_L. Q.E.D.

In general, the induction operation of Definition 3.4 can be expressed in two steps: the first an induction of nilpotent orbits in the sense of Definition 3.1, and the second the "Jordan decomposition" case considered in Lemma 3.5. Here is a precise statement.

Proposition 3.6. *Suppose L is a Levi subgroup of G, and $\mathcal{O}_L \subset \mathfrak{l}^*$ is a coadjoint orbit for L; write*

$$\mathcal{O}_G = \mathrm{Ind}_{orb}(L \uparrow G)(\mathcal{O}_L).$$

Then the Jordan decomposition of an element of $\mathcal{O}_G$ may be computed as follows. Fix an element λ_L of $\mathcal{O}_L$, and write its Jordan decomposition as

$$\lambda_L = \lambda_s + (\lambda_L)_u.$$

Let L' be the centralizer in G of λ_s, and put $M = L' \cap L$. Then λ_L is zero on the natural complement of $\mathfrak{m}$ in $\mathfrak{l}$, and so may be identified with an element

$$\lambda_L = \lambda_M = \lambda_s + (\lambda_M)_u.$$

of $\mathfrak{m}^*$. Similarly, λ_s may be regarded as a semisimple element of $(\mathfrak{l}')^*$ or of $\mathfrak{g}^*$.

Write

$$(\mathcal{O}_M)_u = M \cdot (\lambda_M)_u$$

$$(\mathcal{O}_{L'})_u = \mathrm{Ind}_{orb}(M \uparrow L')((\mathcal{O}_M)_u).$$

Fix a representative $(\lambda_{L'})_u \in (\mathcal{O}_{L'})_u$. Then

$$\lambda = \lambda_s + (\lambda_{L'})_u$$

is the Jordan decomposition of an element of $\mathcal{O}_G$. We have

$$\mathcal{O}_G = \mathrm{Ind}_{orb}(L' \uparrow G)(\lambda_s + (\mathcal{O}_{L'})_u).$$

Proof. Because L' fixes λ_s, the sets

$$\mathcal{O}_M = \lambda_s + (\mathcal{O}_M)_u, \qquad \mathcal{O}_{L'} = \lambda_s + (\mathcal{O}_{L'})_u$$

are coadjoint orbits for M and L' respectively. By Lemma 3.5 applied to L,

$$\mathcal{O}_L = \mathrm{Ind}_{orb}(M \uparrow L)(\mathcal{O}_M).$$

By an easy "induction by stages" fact, we deduce

$$\mathcal{O}_G = \mathrm{Ind}_{orb}(M \uparrow G)(\mathcal{O}_M).$$

Now we induce in stages first from M to L', and then to G. This gives

$$\mathcal{O}_G = \mathrm{Ind}_{orb}(L' \uparrow G)(\mathcal{O}_{L'}).$$

This is the last formula in the proposition. The description of the Jordan decomposition follows from Lemma 3.5 applied to G. Q.E.D.

We can now describe sheets of coadjoint orbits. Recall that a nilpotent coadjoint orbit is called rigid if it is not induced.

Definition 3.7 (cf. [3], [4]). Suppose L is a Levi subgroup of G, and $\mathcal{O}_u$ is a rigid nilpotent coadjoint orbit for L. The *sheet* of coadjoint orbits attached to $(L, \mathcal{O}_u)$ is the collection of orbits

$$\{\, \mathrm{Ind}_{orb}(L \uparrow G)(\lambda_s + \mathcal{O}_u) \mid \lambda_s \in (\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}])^* \,\}.$$

Two sheets are identified if they contain exactly the same orbits; by Proposition 3.6, this amounts to conjugacy of the pair $(L, \mathcal{O}_u)$ under G. A *Dixmier sheet* is one attached to the zero orbit in a Levi subgroup.

Here are some of the most important facts about sheets.

Proposition 3.8 (cf. [3]). *In the notation of Definition 3.7,*

i) Each coadjoint orbit belongs to at least one sheet.
ii) Each sheet contains exactly one nilpotent orbit (namely $\mathrm{Ind}_{orb}(L \uparrow G)(\mathcal{O}_u)$*).*
iii) All of the orbits in a sheet have the same dimension (namely $\dim \mathcal{O}_u + \dim G/L$*).*

Proof. By induction by stages, every nilpotent orbit is induced from a rigid nilpotent orbit. Now (i) follows from Proposition 3.6. Part(ii) is also clear from Proposition 3.6, which says that induction preserves "semisimple part." Part (iii) follows from Definition 3.4. (In fact the original definition of a sheet is as a component of the variety of coadjoint orbits of a fixed dimension; from this point of view the description in Definition 3.7 is something to be proved.) Q.E.D.

In the case of $SL(n)$, distinct sheets are actually disjoint. This partition of the orbits is the key to the definition of a Dixmier map in that case (cf. [2]). Even for the other classical groups, however, distinct sheets can overlap: for $Sp(4)$, the (Dixmier) sheets corresponding to $(GL(2), \{0\})$ and $(GL(1) \times Sp(2), \{0\})$ share the same nilpotent orbit. As was explained in the introduction, this failure of disjointness is the main reason that one cannot have a nice bijection between orbits and completely prime primitive ideals for the other groups.

We turn now to the problem of inducing orbit data. Each step in the geometric construction of Definition 3.1 (or Definition 3.4) has an algebraic analogue, provided by the standard dictionary between algebraic geometry and commutative algebra.

Definition 3.9. Fix a pair $(L, (R_L, \psi_L))$, where

L is a Levi subgroup of G, and
(R_L, ψ_L) is an orbit datum for L.

(Definition 2.2). Notice that these hypotheses are much weaker than those of Theorem 2.6: the main difference is that R_L (or rather its restriction to the commutator subgroup) is not required to be pre-unipotent. We will construct from these data (and a parabolic subgroup) an orbit datum for G. Define

$$\Sigma_L = \mathrm{Supp}\ R_L, \tag{3.10)(a}$$

the image of the morphism ψ_L^* from Spec R_L to $\mathfrak{l}^*$. If R_L is completely prime, then

$$\Sigma_L = \overline{\mathcal{O}}_L, \tag{3.10)(b}$$

the closure of a single coadjoint orbit for L. If R_L is geometric, we write

$$X_L = \text{Spec } R_L \tag{3.10)(c}$$

for the corresponding variety (a ramified finite cover of Σ_L). Our orbit datum for G is going to be supported on the closure of

$$\mathcal{O}_G = \text{Ind}_{orb}(L \uparrow G)(\mathcal{O}_L). \tag{3.10)(d}$$

Now fix a parabolic subgroup $Q = LU$ of G with Levi factor L. (Recall that L is most naturally regarded as the quotient of Q by its unipotent radical, rather than as a subgroup. The reader may observe that it is this structure that we will actually use.) We will use the notation of Definitions 3.1 and 3.4. Define

$$R_Q = R_L, \tag{3.11)(a}$$

and let Q act on R_Q by making U act trivially. Define a map ψ_Q from $S(\mathfrak{q})$ to R_Q by

$$\psi_Q(A) = \begin{cases} \psi_L(A) & (A \in \mathfrak{l}) \\ 0 & (A \in \mathfrak{u}). \end{cases} \tag{3.11)(b}$$

The image of ψ_Q^* is

$$\Sigma_Q = i_Q(\Sigma_L) \simeq \Sigma_L; \tag{3.11)(c}$$

here $\mathfrak{l}^*$ is identified with $(\mathfrak{q}/\mathfrak{u})^* \subset \mathfrak{q}^*$. If R_L is completely prime, then Σ_Q is the closure of $\mathcal{O}_Q \subset \mathfrak{q}^*$. If R_L is geometric, then $X_Q = \text{Spec } R_Q$ is a ramified finite cover of Σ_Q as before.

The algebraic version of (3.1)(d) is the definition

$$R_{G/Q} = S(\mathfrak{g}) \otimes_{S(\mathfrak{q})} R_Q. \tag{3.12)(a}$$

This algebra carries a natural action Ad of Q by algebraic automorphisms. It is the tensor product of the adjoint action on $S(\mathfrak{g})$ with the action on R_Q. There is an Ad(Q)-equivariant homomorphism

$$\psi_{G/Q} : S(\mathfrak{g}) \to R_{G/Q}, \qquad p \mapsto p \otimes 1. \tag{3.12)(b}$$

The image of $\psi_{G/Q}^*$ is

$$\begin{aligned} \Sigma_{G/Q} &= \pi_{G/Q}^{-1}(\Sigma_Q) \\ &= \{ \lambda \in \mathfrak{g}^* \mid \lambda|_{\mathfrak{q}} \in \Sigma_Q \}. \end{aligned} \tag{3.12)(c}$$

This exhibits $\Sigma_{G/Q}$ as an affine bundle over Σ_Q with fiber $(\mathfrak{g}/\mathfrak{q})^*$. If R_L is geometric, we write $X_{G/Q} =$ Spec $R_{G/Q}$; again this is an affine bundle over X_Q. Suppose that R_L is completely prime, so that Σ_Q and therefore $\Sigma_{G/Q}$ are irreducible. The existence of the induced orbit $\mathcal{O}_G$ is equivalent (by (A.3) and the definitions) to the fact that Q has a (unique) open orbit $\mathcal{O}_{G/Q}$ on $\Sigma_{G/Q}$.

We define a sheaf of algebras on G/Q by

$$\mathcal{R}_G = G \times_Q R_{G/Q}. \tag{3.13)(a}$$

If V is an open set in G, then the space of sections of $\mathcal{R}_G$ over VQ is the space of algebraic maps ρ from VQ to $R_{G/Q}$, satisfying

$$\rho(gq) = \text{Ad}(q^{-1})\rho(g) \tag{3.13)(b}$$

(cf. (2.5)(d)). (It might be slightly more natural first to identify $R_{G/Q}$ with a sheaf of algebras $\mathcal{R}_{G/Q}$ over $\Sigma_{G/Q}$, and then to define

$$\mathcal{R}'_G = G \times_Q \mathcal{R}_{G/Q} \tag{3.13)(a)'}$$

as a sheaf of algebras over $G \times_Q \Sigma_{G/Q}$ (cf. (A.4)). Because $\Sigma_{G/Q}$ is affine, the two approaches are interchangeable; the one we have chosen seems slightly simpler.) The group G acts on the sheaf $\mathcal{R}_G$; an element g defines an algebra homomorphism from the sections over VQ to the sections over $(g^{-1})VQ$ by right translation. The *induced orbit datum*

$$(R_G, \psi_G) = \mathrm{Ind}_{orb}(Q \uparrow G)(R_L, \psi_L) \tag{3.14)(a}$$

is defined by

$$R_G = \text{global sections of } \mathcal{R}_G; \tag{3.14)(b}$$

these are just the algebraic maps from G to $R_{G/Q}$ satisfying (3.13)(b) (cf. (A.4)(c)). The algebra structure is pointwise multiplication, and the action of G is $(g \cdot \rho)(x) = \rho(g^{-1}x)$. Finally,

$$\psi_G : S(\mathfrak{g}) \to R_G, \qquad (\psi_G(p))(x) = (\mathrm{Ad}(x^{-1})p) \otimes 1 \in R_{G/Q}. \tag{3.14)(c}$$

Proposition 3.15. *Suppose $Q = LU$ is a parabolic subgroup of G, and (R_L, ψ_L) is an orbit datum for L. Then the pair*

$$(R_G, \psi_G) = \mathrm{Ind}_{orb}(Q \uparrow G)(R_L, \psi_L)$$

(Defintion 3.9) is an orbit datum for G. It is completely prime (respectively geometric) if (R_L, ψ_L) is. The support of R_G is

$$\Sigma_G = Ad^*(G) \cdot (\Sigma_{G/Q}) = Ad^*(G) \cdot (\pi_{G/Q}^{-1}(i_Q(\Sigma_L)))$$

Proof. Conditions (i) and (ii) in Definition 2.2 follow easily from the definitions. We consider (iii). By hypothesis, R_L is a finitely generated $S(\mathfrak{l})$-module supported on Σ_L. It follows easily from the definitions that $R_{G/Q}$ is a finitely generated $S(\mathfrak{g})$-module supported on Σ_Q. We may therefore identify it with a coherent sheaf $\mathcal{R}_{G/Q}$ on $\Sigma_{G/Q}$. By Proposition A.5,

$$\mathcal{R}'_G = G \times_Q \mathcal{R}_{G/Q}$$

is a coherent sheaf on $G \times_Q \Sigma_{G/Q}$. By Proposition A.2(c), the natural map

$$G \times_Q \Sigma_{G/Q} \to G \cdot \Sigma_{G/Q} \tag{3.16}$$

is proper. (In fact the proof shows that the morphism is projective.) By [9], Corollary II.5.20, the space of global sections of $\mathcal{R}'_G$ is a finitely generated $S(\mathfrak{g})$-module supported on $G \cdot \Sigma_{G/Q}$. By the remark at (3.13)(a)', this space of global sections is precisely R_G.

We know that Σ_L is a finite union of orbits of L (see the remarks after Definition 2.2). The theory of induced orbits recalled at the beginning of this section therefore implies that $G \cdot \Sigma_{G/Q}$ is a finite union of orbits of G. Condition (iv) in Definition 2.2 follows.

That the support of R_G is all of $G \cdot \Sigma_{G/Q}$ (rather than some proper subvariety) follows from the fact that the map (3.16) is generically finite; this in turn is a consequence of the the theory of induced orbits (cf. (3.1) and (A.3)).

That R_G is completely prime (or commutative) whenever R_L is follows from the definitions. Suppose that R_L is geometric; write

$$X_L = \text{Spec } R_L, \qquad X_{G/Q} = \text{Spec } R_{G/Q} \tag{3.17)(a}$$

as in Definition 3.9. Then the sheaf of algebras $\mathcal{R}_G$ is just the structure sheaf of the bundle

$$\mathcal{X}_G = G \times_Q X_{G/Q}, \tag{3.17)(b}$$

over G/Q. Since $X_{G/Q}$ is normal, so is $\mathcal{X}_G$. It follows that R_G, as the algebra of global functions on $\mathcal{X}_G$, is normal as well. Consequently $X_G = \text{Spec } R_G$ is normal, so the orbit datum (R_G, ψ_G) is geometric. Q.E.D.

Just as in the case of coadjoint orbits (Proposition 3.6), induction of orbit data is closely related to the Jordan decomposition.

Proposition 3.18. *Suppose $Q = LU$ is a parabolic subgroup of G, and (R_L, ψ_L) is a completely prime orbit datum for L. Suppose that this orbit datum corresponds in the Jordan decomposition for L (Theorem 2.6) to $(M, \lambda_s, (R_u, \psi_u))$. Let L' be the centralizer in G of λ_s. Then $Q' = Q \cap L' = MU'$ is a parabolic subgroup of L', and*

$$(R'_u, \psi'_u) = \mathrm{Ind}_{orb}(Q' \uparrow L')(R_u, \psi_u)$$

is a completely prime pre-unipotent orbit datum for $L'/Z(L')_0$. The induced orbit datum

$$(R_G, \psi_G) = \mathrm{Ind}_{orb}(Q \uparrow G)(R_L, \psi_L)$$

has Jordan decomposition $(L', \lambda_s, (R'_u, \psi'_u))$.

The proof is parallel to that of Proposition 3.6, and we omit it. (In the statement, we have been a little sloppy about identifying orbit data for (for example) L' and $L'/Z(L')_0$.)

We know that the support of an induced orbit datum does not depend on the choice of parabolic Q; and we know (Corollary 2.7, for example) that an orbit datum is nearly determined by its support. This suggests

Conjecture 3.19. In the setting of Definition 3.9, the orbit datum $\mathrm{Ind}_{orb}(Q \uparrow G)(R_L, \psi_L)$ is independent of the choice of parabolic subgroup with Levi factor L.

An interesting special case (both of the construction and of the conjecture) is when R_L is $\mathbb{C}$ and ψ_L is zero on $\mathfrak{l}$. Then $R_{G/Q}$ is the algebra $S(\mathfrak{g}/\mathfrak{q})$ of functions on the cotangent space at eQ to G/Q, and $\mathcal{X}_G$ is just $T^*(G/Q)$. The algebra R_G is the algebra of regular functions on $T^*(G/Q)$. If Q' is another parabolic subgroup with Levi factor L, then $T^*(G/Q)$ need not be isomorphic to $T^*(G/Q')$. The conjecture asks for an isomorphism between the algebras of global regular functions on these (non-affine) varieties (respecting the G actions and the maps ψ_G). Such an isomorphism does exist, but I do not know any very satisfactory proof of the fact. (Because of normality, it suffices to show that the unique dense G-orbits in the two cotangent bundles are isomorphic as homogeneous spaces. Fix elements

$$\lambda \in (\mathfrak{g}/\mathfrak{q})^* \simeq T^*_{eQ}(G/Q),$$

$$\lambda' \in (\mathfrak{g}/\mathfrak{q}')^* \simeq T^*_{eQ'}(G/Q')$$

representing these orbits. Then λ and λ' are conjugate as elements of $\mathfrak{g}^*$, so the isotropy groups $G(\lambda)$ and $G(\lambda')$ are conjugate. The identity components of these groups are contained in Q and Q' respectively. The desired isomorphism is equivalent to the fact that $G(\lambda) \cap Q$ is conjugate to $G(\lambda') \cap Q'$. This can be verified by more or less explicit computation on a case-by-case basis.) The simplest non-trivial example is for $G = GL(3)$ and $L = GL(2) \times GL(1)$. We can take G/Q to be the variety of lines in $\mathbb{C}^3$, and G/Q' to be the variety of planes in $\mathbb{C}^3$. Of course these varieties are isomorphic, but the isomorphism cannot be made to respect the G actions. Nevertheless the algebras of regular functions on the respective tangent bundles *are* isomorphic in a G-invariant way: they are both isomorphic to the algebra of regular functions on the corresponding (sub-regular) nilpotent orbit in $\mathfrak{g}^*$.

Because of Proposition 3.18, a completely prime orbit datum for a semsimple group that is *not* pre-unipotent must be induced from a proper parabolic subgroup. It therefore makes sense to concentrate on pre-unipotent orbit data.

Definition 3.20. Suppose (R, ψ) is a completely prime pre-unipotent orbit datum for the semisimple group G. We say that (R, ψ) is *rigid* if it is not induced (in the sense of Definition 3.9) from a completely prime orbit datum on a proper Levi subgroup. Otherwise we say that (R, ψ) is *induced.*

This notion of rigidity is much broader than that for orbits (defined after Definition 3.1). That is, an orbit datum will be rigid if its support is rigid; but it may be rigid even if its support is not. If G is $GL(n)$, every non-zero completely prime orbit datum is induced; but already for $SL(2)$ this is not true.

Example 3.21. Suppose G is $SL(2)$. We define a unipotent orbit datum (R, ψ) for G as follows. The algebra R is $\mathbb{C}[p, q]$, with the action of G induced by linear change of variables. We define ψ from $S(\mathfrak{g})$ to R on the standard basis (H, E, F) of $\mathfrak{g}$ by

$$\psi(H) = 2pq, \qquad \psi(E) = p^2, \qquad \psi(F) = -q^2.$$

Obviously R is finitely generated as an $S(\mathfrak{g})$ module (by 1, p, and q). It is easy to check that ψ respects the action of G. Every irreducible representation of G appears in R exactly once. The image of ψ^* is the cone $H^2 + 4EF = 0$, which is the nilpotent cone in $\mathfrak{g}^*$. It follows that (R, ψ) is a unipotent orbit datum. One can show that it is rigid for example by computing all the induced orbit data; this is not difficult since there is only one proper Levi subgroup. In the parametrization of Corollary 2.7, R corresponds to the double cover of the principal nilpotent orbit.

Iwan Pranata has observed that the preceding example can be modified to produce many non-geometric completely prime commutative orbit data. To do that, fix a non-negative integer k, and define R_k to be the subring of R generated by the image of ψ and the polynomials of degree $2k + 1$. (Pranata also found the Dixmier algebras associated to these orbit data.) Together with $(\mathbb{C}, 0)$, the various (R_k, ψ) exhaust the rigid completely prime orbit data for $SL(2)$.

Here is the analogue of Definition 3.7.

Definition 3.22. Suppose $Q = LU$ is a parabolic subgroup of G, and (R_u, ψ_u) is a rigid completely prime pre-unipotent coadjoint orbit datum for $L/Z(L)_0$. To every character λ_s of $\mathfrak{l}$ we can associate a completely prime orbit datum $(R_L, \psi_L(\lambda_s))$ for L as in (2.5)(e) and (f):

$$R_L = R_u, \qquad \psi_L(A) = \psi_u(A + \mathfrak{z}(\mathfrak{l})) + \lambda_s(A) \quad (A \in \mathfrak{l}).$$

The *sheet* of completely prime orbit data attached to $(L, (R_u, \psi_u))$ is

$$\{ \operatorname{Ind}_{orb}(Q \uparrow G)(R_L, \psi_L(\lambda_s)) \mid \lambda_s \in (\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}])^* \}$$

(Definition 3.9). It is called a *Dixmier sheet* if $R_u = \mathbb{C}$. Two sheets are identified if they contain exactly the same orbit data; according to Conjecture 3.19, this amounts to conjugacy of the pair $(L, (R_u, \psi_u))$ under G. (By considering regular λ_s in Proposition 3.18, one sees that this conjugacy is a *necessary* condition for the sheets to coincide.)

We have at once an analogue of Proposition 3.8.

Proposition 3.23. *In the notation of Definitions 3.9 and 3.22,*

i) Each completely prime coadjoint orbit datum belongs to at least one sheet.

ii) If G is semisimple, each sheet contains exactly one preunipotent orbit datum (corresponding to $\lambda_s = 0$).

iii) The supports of the orbit data in a single sheet are contained in a single sheet of coadjoint orbits.

This follows from Proposition 3.18. In (iii), the supports may not constitute an entire sheet of coadjoint orbits. This happens exactly when the support of the rigid orbit datum R_u is not a rigid coadjoint orbit.

Suppose that $G = Sp(4)$, $L = GL(2)$, and $L' = GL(1) \times Sp(2)$. Fix parabolic subgroups Q and Q' with Levi factors L and L'. As was pointed out after Proposition 3.8, the Dixmier sheets of coadjoint orbits attached to L and L' share the same nilpotent orbit. Let us now consider the corresponding sheets of orbit data. The unipotent orbit data for these sheets are the rings of regular functions on $T^*(G/Q)$ and $T^*(G/Q')$ respectively. They are *not* isomorphic as Dixmier algebras: the second contains the five-dimensional representation of G, and the first does not. It follows that these two sheets of orbit data are disjoint. (The non-unipotent orbit data are distinguished by their supports.) Many similar examples suggest

Conjecture 3.24. Two sheets of orbit data are disjoint or they coincide. The first possibility occurs exactly when the pairs $(L, (R_u, \psi_u))$ defining the sheets are conjugate by G.

Conjecture 3.19 is more or less subsumed in this formulation. In the case of geometric orbit data, Conjecture 3.24 amounts to an assertion about the behavior of fundamental groups under induction (compare Corollary 2.7 and the example after Conjecture 3.19). It could in principle be verified by a finite calculation for each group. For the classical groups this should not be too difficult to carry out, but I have not done so. For non-geometric sheets (in particular when the normality hypothesis is dropped) the number of essentially different cases becomes infinite, and some more conceptual approach is probably necessary.

4. Parabolic induction of Dixmier algebras.

The main purpose of this section is to give a construction of Dixmier algebras parallel to the construction of orbit data described in section 3. Fix a pair $(Q, (A_L, \phi_L))$, where

$$Q \text{ is a parabolic subgroup of } G, \text{ and } (A_L, \phi_L) \text{ is a Dixmier algebra for } L. \tag{4.1}$$

We want to construct an "induced Dixmier algebra" (A_G, ϕ_G) for G. Of course we will imitate and extend the construction of induced primitive ideals (cf. [6]).

Before we embark on the rather convoluted construction of A_G, it is worth pausing to explain why the problem is not trivial. Suppose $V_{\mathfrak{l}}$ is a faithful module for A_L. Let $V_{\mathfrak{g}} = U(\mathfrak{g}) \otimes_{\mathfrak{q}} V_{\mathfrak{l}}$ be a parabolically induced module for $\mathfrak{g}$. (We are neglecting "ρ-shifts" — that is, the twist by the character δ in (4.2)(b) — for this discussion.) The ideal I in $U(\mathfrak{g})$ induced by the annihilator of $V_{\mathfrak{l}}$ is by definition the annihilator of $V_{\mathfrak{g}}$; so the quotient ring $U(\mathfrak{g})/I$ is naturally embedded in End $V_{\mathfrak{g}}$. The Dixmier algebra A_G is supposed to be some sort of nice ring extension of $U(\mathfrak{g})/I$, so it is natural to look for this extension in End $V_{\mathfrak{g}}$. Now it is an important theme of Joseph's work that the algebra of G-finite endomorphisms of $V_{\mathfrak{g}}$ is a natural and well-behaved ring; so it is an obvious candidate for A_G. To see that it is not the right one, consider the case $L = G$. In that case we had better have $A_L = A_G$ (if our notion of induction is to be related to induction of orbit data). Our "obvious candidate" for A_G will have this property only if the Dixmier algebra A_L is already the *full* algebra of L-finite endomorphisms of $V_{\mathfrak{l}}$. So this approach from the outset runs into a rather difficult technical problem: is every Dixmier algebra for L the full algebra of L-finite endomorphisms of some $\mathfrak{l}$-module? (The answer in this generality is "no;" if A_L is required to be reasonable (say prime, for example) the answer may be "yes," but I have no idea how to prove it.)

What we actually do is this. We show that a large class of endomorphisms of $V_{\mathfrak{g}}$ (including all the G-finite ones) can be described essentially as matrices with entries in the endomorphism algebra of $V_{\mathfrak{l}}$ (cf. Definition 4.6 and (4.9)). We can then define A_G to consist of those G-finite endomorphisms of $V_{\mathfrak{g}}$ whose "matrix entries" belong to A_L (Definition 4.7). This definition is not so difficult. What is slightly less easy is proving that A_G satisfies the finiteness requirements in the definition of Dixmier algebras, and relating the definition to induction of orbit data. To do that, we have to give a rather different description of A_G (Corollary 4.16). (We cannot easily use Corollary 4.16 as the definition of A_G, because it is difficult to see the algebra structure from this point of view.)

Especially for the material on the symbol calculus, the reader should keep in mind the case $A_L = \mathbb{C}$; in that case A_G is the algebra of (holomorphic linear) differential operators on G/Q. (Recall that we are still neglecting ρ shifts!) That case has been thoroughly analyzed in [Borho-Brylinski], and most of the serious ideas we use may be found there.

Suppose then that

$$V_{\mathfrak{l}} \text{ is a faithful module for } A_L. \tag{4.2)(a}$$

(By Proposition 6.0 of [15], any prime Dixmier algebra is primitive; if A_L is prime, we could therefore even choose $V_{\mathfrak{l}}$ to be simple as an A_L-module. This makes no difference in the construction, however.) Let 2δ denote the character of L on the top exterior power of $\mathfrak{g}/\mathfrak{q}$. Then δ is a well-defined character of $\mathfrak{l}$. Twisting the map ϕ_L by δ we get a new Dixmier algebra (A'_L, ϕ'_L): the algebra A'_L is equal to A_L, and

$$\phi'_L(X) = \phi_L(X) + \delta(X) \qquad (X \in \mathfrak{l}). \tag{4.2)(b}$$

This algebra has a faithful module

$$V'_{\mathfrak{l}} = V_{\mathfrak{l}} \otimes \mathbb{C}_\delta. \tag{4.2)(c}$$

Make Q act on A'_L by making U act trivially; to emphasize this structure, we call the algebra A_Q, and write

$$\mathrm{Ad} : Q \to \mathrm{Aut}(A_Q). \tag{4.2)(d}$$

Extend ϕ'_L to

$$\phi_Q : U(\mathfrak{q}) \to A_Q \tag{4.2)(e}$$

by sending $\mathfrak{u}$ to 0. When we regard $V'_{\mathfrak{l}}$ as a module for A_Q, we call it $V_{\mathfrak{q}}$. Write $\mathfrak{u}^-$ for the nil radical of the parabolic subalgebra of $\mathfrak{g}$ opposite to Q. Now define

$$V_{\mathfrak{g}} = U(\mathfrak{g}) \otimes_{\mathfrak{q}} V_{\mathfrak{q}} \simeq U(\mathfrak{u}^-) \otimes_{\mathbf{C}} V_{\mathfrak{q}}. \tag{4.2)(f}$$

We are going to construct A_G as an algebra of endomorphisms of $V_{\mathfrak{g}}$. We begin by constructing a larger algebra. To motivate the definition of this algebra, we will study the adjoint action of $\mathfrak{z}(\mathfrak{l})$ on the endomorphisms of $V_{\mathfrak{g}}$. Our immediate goal is Lemma 4.5 below.

Because A_L is a Dixmier algebra, the kernel of ϕ_L must contain an ideal of finite codimension in $U(\mathfrak{z}(\mathfrak{l}))$. To simplify the notation, we assume that this ideal is maximal (as it must be if A_L is prime); the reader can easily modify the constructions that follow to cover the general case. Then $\mathfrak{z}(\mathfrak{l})$ acts by a character λ on $V_{\mathfrak{q}}$. If α is any character of $\mathfrak{z}(\mathfrak{l})$, write $U(\mathfrak{u}^-)_\alpha$ for the (finite-dimensional) subspace of $U(\mathfrak{u}^-)$ on which $\mathfrak{z}(\mathfrak{l})$ acts by α. Then $V_{\mathfrak{g}}$ is the direct sum of its weight spaces

$$(V_{\mathfrak{g}})_\beta = U(\mathfrak{u}^-)_{\beta-\lambda} \otimes V_{\mathfrak{q}}. \tag{4.3}$$

Next, recall that the algebra End $V_{\mathfrak{g}}$ is a bimodule for $U(\mathfrak{g})$. The left action of an element u on an endomorphism T is on the range of T, and the right action is on the domain. Write ad for the corresponding diagonal action of $\mathfrak{g}$: if T is an endomorphism, X is in $\mathfrak{g}$, and v is in $V_{\mathfrak{g}}$, then

$$(\mathrm{ad}(X)T)(v) = X \cdot (Tv) - T(X \cdot v). \tag{4.4}$$

An endomorphism T has weight γ for $\mathrm{ad}(\mathfrak{z}(\mathfrak{l}))$ if and only if it carries $(V_{\mathfrak{g}})_\beta$ to $(V_{\mathfrak{g}})_{\beta+\gamma}$ for every β. Using (4.3), one deduces immediately

Lemma 4.5. *Suppose T is an $ad(\mathfrak{z}(\mathfrak{l}))$-finite endomorphism of $V_{\mathfrak{g}}$. Then for every u in $U(\mathfrak{u}^-)$ there are finitely many elements $u_1, \ldots, u_n$ in $U(\mathfrak{u}^-)$, and endomorphisms $E_1, \ldots, E_n$ of $V_{\mathfrak{q}}$, so that for each v in $V_{\mathfrak{q}}$,*

$$T(u \otimes v) = \sum u_i \otimes E_i v.$$

Here is a first approximation to the induced Dixmier algebra.

Definition 4.6. Suppose we are in the setting (4.2). An endomorphism T of $V_{\mathfrak{g}}$ is said to be *of type A_L* if and only if for every u in $U(\mathfrak{u}^-)$ there are finitely many elements $u_1, \ldots, u_n$ in $U(\mathfrak{u}^-)$, and endomorphisms $E_1, \ldots, E_n$ in A_Q, so that for each v in $V_{\mathfrak{q}}$,

$$T(u \otimes v) = \sum u_i \otimes E_i v. \tag{4.6)(a}$$

Using the defining relations for $V_{\mathfrak{g}}$, one checks that it is equivalent to require this condition for every u in $U(\mathfrak{g})$, or to require only that u_i belong to $U(\mathfrak{g})$. Write

$$A_{\mathfrak{g}} = \{\, T \in \mathrm{End}(V_{\mathfrak{g}}) \mid T \text{ is of type } A_L \,\}$$

It is easy to check that $A_{\mathfrak{g}}$ is an algebra.

If w is in $U(\mathfrak{g})$, define $\phi_{\mathfrak{g}}(w)$ to be the endomorphism of $V_{\mathfrak{g}}$ defined by the action of w. This is always of type A_L, so we get

$$\phi_{\mathfrak{g}} : U(\mathfrak{g}) \to A_{\mathfrak{g}}, \tag{4.6)(b}$$

a homomorphism of algebras.

In (4.9) below we will give another description of $A_{\mathfrak{g}}$ from which one can see that it depends only on the data (4.1) (and not on the choice of $V_{\mathfrak{l}}$).

Because of (4.4) and the existence of the map $\phi_{\mathfrak{g}}$, we see that the algebra $A_{\mathfrak{g}}$ is $\mathrm{ad}(\mathfrak{g})$-stable. The reason $A_{\mathfrak{g}}$ is not a Dixmier algebra is that this Lie algebra adjoint action does not exponentiate to an algebraic action of G. We remedy this in the simplest possible way.

Definition 4.7. In the setting of Definition 4.6, define

$$\begin{aligned} A_G &= \mathrm{Ad}(G)\text{-finite subalgebra of } A_{\mathfrak{g}} \\ &= \mathrm{Ad}(G)\text{-finite endomorphisms of } V_{\mathfrak{g}} \text{ of type } A_L \end{aligned} \tag{4.7)(a}$$

To understand this definition, recall that an element T of $A_{\mathfrak{g}}$ is called $\mathrm{Ad}(G)$*-finite* if it belongs to a finite-dimensional $\mathrm{ad}(\mathfrak{g})$-invariant subspace F of $A_{\mathfrak{g}}$, on which there exists an algebraic action Ad_F of G with differential ad. The elements in the image of $\phi_{\mathfrak{g}}$ certainly have this property: one can take for F the image of some level $U_n(\mathfrak{g})$ of the standard filtration of $U(\mathfrak{g})$. Restricting the range of $\phi_{\mathfrak{g}}$ therefore defines

$$\phi_G : U(\mathfrak{g}) \to A_G, \tag{4.7)(b}$$

a homomorphism of algebras. It is easy to check that A_G is a subalgebra of $A_{\mathfrak{g}}$ containing the identity element. Since G is connected, the actions Ad_F are uniquely determined, and can be assembled into an algebraic action

$$\mathrm{Ad} : G \to \mathrm{Aut}(A_G). \tag{4.7)(c}$$

The action is by algebra automorphisms since $\mathrm{ad}(\mathfrak{g})$ acts by derivations. We define the *induced Dixmier algebra* to be

$$(A_G, \phi_G) = \mathrm{Ind}_{Dix}(Q \uparrow G)(A_L, \phi_L). \tag{4.7)(c}$$

To show that A_G is a Dixmier algebra, we only need to check the finiteness conditions (iii) and (iv) of Definition 2.1. To do that, and to get a clearer picture of the structure of A_G, we need first to study $A_{\mathfrak{g}}$ much more carefully.

Lemma 4.8. *The set $A_{\mathfrak{g}}$ is an algebra of endomorphisms of $V_{\mathfrak{g}}$. If A_L consists of all L-finite endomorphisms of $V_{\mathfrak{l}}$, then $A_{\mathfrak{g}}$ contains all the L-finite endomorphisms of $V_{\mathfrak{g}}$.*

This is straightforward.

Here is another description of $A_{\mathfrak{g}}$. Fix characters β and γ of $\mathfrak{z}(\mathfrak{l})$. Define $(\mathrm{End}\ V_{\mathfrak{g}})_{\beta\gamma}$ to consist of those endomorphisms transforming by the character β by the left action of $\mathfrak{z}(\mathfrak{l})$, and by γ under the right action. These are precisely the endomorphisms vanishing on all the weight spaces of $V_{\mathfrak{g}}$ except the one for γ, and whose image is contained in the β weight space:

$$\begin{aligned} (\mathrm{End}\ V_{\mathfrak{g}})_{\beta\gamma} &\simeq \mathrm{Hom}((V_{\mathfrak{g}})_{\gamma}, (V_{\mathfrak{g}})_{\beta}) \\ &\simeq \mathrm{Hom}(U(\mathfrak{u}^-)_{\gamma-\lambda} \otimes V_{\mathfrak{q}}, U(\mathfrak{u}^-)_{\beta-\lambda} \otimes V_{\mathfrak{q}}) \\ &\simeq \mathrm{Hom}(U(\mathfrak{u}^-)_{\gamma-\lambda}, U(\mathfrak{u}^-)_{\beta-\lambda}) \otimes \mathrm{End}\ V_{\mathfrak{q}}. \end{aligned} \tag{4.9)(a}$$

The corresponding double weight space for $A_{\mathfrak{g}}$ is

$$(A_{\mathfrak{g}})_{\beta\gamma} \simeq \mathrm{Hom}(U(\mathfrak{u}^-)_{\gamma-\lambda}, U(\mathfrak{u}^-)_{\beta-\lambda}) \otimes A_Q. \tag{4.9)(b}$$

The algebra $A_{\mathfrak{g}}$ is built from these pieces in a slightly subtle way:

$$A_{\mathfrak{g}} \simeq \prod_{\gamma} \left(\sum_{\beta} (A_{\mathfrak{g}})_{\beta\gamma} \right). \qquad (4.9)(c)$$

The reason we insist on a direct product outside is to ensure that the identity operator belongs to $A_{\mathfrak{g}}$. The direct sum inside ensures that the resulting algebra acts on $V_{\mathfrak{g}}$.

In the description (4.9) of $A_{\mathfrak{g}}$, the algebra structure is the obvious one (induced by the algebra structure on A_Q and the "composition maps"

$$\mathrm{Hom}(U(\mathfrak{u}^-)_\gamma, U(\mathfrak{u}^-)_\beta) \times \mathrm{Hom}(U(\mathfrak{u}^-)_\delta, U(\mathfrak{u}^-)_\gamma) \to \mathrm{Hom}(U(\mathfrak{u}^-)_\delta, U(\mathfrak{u}^-)_\beta).$$

The structure that is subtle in this picture is the homomorphism $\phi_{\mathfrak{g}}$ of (4.7)(b). At any rate, (4.9) shows that $A_{\mathfrak{g}}$ depends only on A_L (and Q), not on the choice of module $V_{\mathfrak{l}}$.

We need two more descriptions of $A_{\mathfrak{g}}$. For each of them, it is convenient to introduce an auxiliary space.

Definition 4.10. In the setting of (4.2), define

$$A_{G/Q} = U(\mathfrak{g}) \otimes_{\mathfrak{q}} A_Q. \qquad (4.10)(a)$$

This space is clearly analogous to the algebra $R_{G/Q}$ of (3.12), but I do not know any very simple motivation for its introduction. Clearly $A_{G/Q}$ carries a left action of $U(\mathfrak{g})$ and a commuting right action of $U(\mathfrak{q})$. There is also an algebraic action

$$\mathrm{Ad} : Q \to End(A_{G/Q}), \qquad (4.10)(b)$$

which is the tensor product of the adjoint actions on $U(\mathfrak{g})$ and A_Q. The differential of Ad is the difference of the left and right actions of $\mathfrak{q}$.

The first of our final descriptions of $A_{\mathfrak{g}}$ is

Lemma 4.11. *In the setting of Definition 4.10, define*

$$A^1_{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{q}(right,right)}(U(\mathfrak{g}), A_{G/Q}). \qquad (i)$$

(Here $\mathfrak{q}$ acts on the first $U(\mathfrak{g})$ and on $A_{G/Q}$ on the right to define the $\mathrm{Hom}_{\mathfrak{q}}$.*) Then there is an isomorphism*

$$\alpha^1 : A_{\mathfrak{g}} \to A^1_{\mathfrak{g}}. \qquad (ii)$$

In this isomorphism, the left action of $U(\mathfrak{g})$ on $A_{\mathfrak{g}}$ corresponds to the left action on $A_{G/Q}$; and the right action of $U(\mathfrak{g})$ on $A_{\mathfrak{g}}$ corresponds to the left action on the domain $U(\mathfrak{g})$ in the Hom. *Explicitly,*

$$\alpha^1(xTy)(u) = x\left(\alpha^1(T)(yu)\right) \qquad (x, y, u \in U(\mathfrak{g}),\ T \in A_{\mathfrak{g}}. \qquad (iii)$$

For $x \in U(\mathfrak{g})$, the element $\phi_{\mathfrak{g}}(x)$ of (4.6)(b) satisfies

$$\alpha^1(\phi_{\mathfrak{g}}(x))(u) = xu \otimes 1. \qquad (iv)$$

We use the cumbersome subscript in (i) to distinguish this realization from that of Lemma 4.14 below.

Proof. Fix $T \in A_{\mathfrak{g}}$; we must define $\alpha^1(T)$ as a map from $U(\mathfrak{g})$ to $A_{G/Q} = U(\mathfrak{g}) \otimes A_Q$. So fix u in $U(\mathfrak{g})$, and write

$$T(u \otimes v) = \sum u_i \otimes E_i v \qquad (4.12)(a)$$

(with $u_i \in U(\mathfrak{g})$ and $E_i \in A_Q$) as in Definition 4.6. Then put

$$\alpha^1(T)(u) = \sum u_i \otimes E_i. \qquad (4.12)(b)$$

We leave to the reader the straightforward verification that $\alpha^1(T)$ is a well-defined element of $A^1_{\mathfrak{g}}$, and that α^1 is an isomorphism. The assertion (iii) follows from (4.12). For (iv), one can use (iii) and the fact that $\phi_{\mathfrak{g}}(x) = x \cdot 1_{A_{\mathfrak{g}}}$ (the left action of x on the identity element of $A_{\mathfrak{g}}$). Q.E.D.

Using (4.12), we can compute the algebra structure on $A^1_{\mathfrak{g}}$ induced by the isomorphism α^1. Suppose S' and S are in $A^1_{\mathfrak{g}}$. To evaluate $S'S$ at an element u of $U(\mathfrak{g})$, first write

$$S(u) = \sum u_i \otimes E_i. \tag{4.13)(a}$$

Next, write

$$S'(u_i) = \sum_j u_{ij} \otimes E_{ij}. \tag{4.13)(b}$$

Then

$$(S'S)(u) = \sum_{i,j} u_{ij} \otimes E_{ij}E_i. \tag{4.13)(c}$$

We leave the straightforward verification of this to the reader.

The last description of $A_{\mathfrak{g}}$ is the least transparent of all, but it will be crucial to the symbol calculus.

Lemma 4.14. *In the setting of Lemma 4.11, define*

$$A^2_{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{q}(left,ad)}(U(\mathfrak{g}), A_{G/Q}) \tag{i}$$

to be the space of maps Σ from $U(\mathfrak{g})$ to $A_{G/Q}$ with the property that

$$\Sigma(Xu) = ad(X)\Sigma(u) \qquad (X \in \mathfrak{q}, u \in U(\mathfrak{g})). \tag{ii}$$

Then there is an isomorphism

$$\alpha^2 : A_{\mathfrak{g}} \to A^2_{\mathfrak{g}}. \tag{iii}$$

Under the isomorphism α^2, the adjoint action of $U(\mathfrak{g})$ on $A_{\mathfrak{g}}$ corresponds to the right action on the domain $U(\mathfrak{g})$ term in $A^2_{\mathfrak{g}}$:

$$\alpha^2(\mathrm{ad}(x)T)(u) = \alpha^2(T)(ux). \tag{iv}$$

The map $\phi_{\mathfrak{g}}$ is computed in this picture by

$$\alpha^2(\phi_{\mathfrak{g}}(x))(u) = ad(u)x \otimes 1. \tag{v}$$

Before proving this lemma, we deduce the consequences we want.

Definition 4.15. The *complete symbol sheaf* (for A_L and Q) is the quasicoherent sheaf

$$\mathcal{A}_G = G \times_Q A_{G/Q}$$

on G/Q. Here we use the action Ad of Q on $A_{G/Q}$.

It should be fairly easy to make $\mathcal{A}_G$ into a sheaf of algebras on G/Q, but I have not done this.

Corollary 4.16. *The algebra $A_{\mathfrak{g}}$ may be identified with the space of formal power series sections of $\mathcal{A}_G$ at the identity coset eQ. The subalgebra A_G then corresponds to the global sections of $\mathcal{A}_G$; the left action of G on sections corresponds to the adjoint action on A_G. In particular,*

$$\begin{aligned} A_G &\simeq \mathrm{Ind}_{alg}(Q \uparrow G)(A_{G/Q}) \\ &= \mathrm{Ind}_{alg}(Q \uparrow G)(U(\mathfrak{g}) \otimes_{\mathfrak{q}} A_Q). \end{aligned}$$

Here we use induction of algebraic representations (cf. (A.8)); Q acts on the inducing representation by $\mathrm{Ad} \otimes \mathrm{Ad}$.

Sketch of proof. This is a consequence of Lemma 4.14 and general facts about homogeneous vector bundles. A formal power series section Σ of the "vector bundle" $G \times_Q A_{G/Q}$ is specified by a map from $U(\mathfrak{g})$ to $A_{G/Q}$, sending u to the value at eQ of the derivative $\partial_u(\Sigma)$. Since Σ must transform (infinitesimally) under Q in a certain way, this map must respect the action of $\mathfrak{q}$; that is, it must belong to the space $A^2_{\mathfrak{g}}$ of Lemma 4.14. This gives the first assertion. For the rest, one needs to know that a G-finite formal power series section must come from a globally defined section; this is easy. Q.E.D.

Corollary 4.17. *In the setting of Definition 4.7, the pair*

$$(A_G, \phi_G) = \mathrm{Ind}_{Dix}(Q \uparrow G)(A_L, \phi_L)$$

is a Dixmier algebra. The kernel of ϕ_G in $U(\mathfrak{g})$ is the ideal induced (via $\mathfrak{q}$) from the kernel of ϕ_L in $U(\mathfrak{l})$.

Proof. Since A_L is a Dixmier algebra for L, the kernel of ϕ_L contains an ideal of finite codimension in the center of $U(\mathfrak{l})$. Since A_G is an algebra of endomorphisms of $V_{\mathfrak{g}}$, it follows from the theory of the Harish-Chandra homomorphism that the kernel of ϕ_G contains an ideal of finite codimension in the center of $U(\mathfrak{g})$. By the theory of Harish-Chandra modules for G, condition (iii) in the definition of Dixmier algebra (Definition 2.1) is a consequence of condition (iv). But condition (iv) is an elementary consequence of the description of A_G as an induced module in Corollary 4.16; we leave the calculation to the reader.

For the statement about induced ideals, notice first that the annihilator of $V_{\mathfrak{l}}$ in $U(\mathfrak{l})$ is the kernel of ϕ_L (since $V_{\mathfrak{l}}$ is assumed to be faithful for A_L). By definition the induced ideal is therefore the annihilator in $U(\mathfrak{g})$ of $V_{\mathfrak{g}}$. Since A_G is an algebra of endomorphisms of $V_{\mathfrak{g}}$, the claim follows. Q.E.D.

Conjecture 4.18. In the setting of Definition 4.7, the induced Dixmier algebra is independent of the choice of Q.

Of course this conjecture is analogous to Conjecture 3.19. The corresponding assertion for induced ideals is true (cf. [8]). Beyond this, there is very little evidence. It may be that one has to impose some extra condition on A_L, such as complete primality.

Proof of Lemma 4.14. Rather than passing directly from $A_{\mathfrak{g}}$ to $A^2_{\mathfrak{g}}$, we will construct an isomorphism

$$\alpha^{12} : A^1_{\mathfrak{g}} \to A^2_{\mathfrak{g}}. \tag{4.19)(a}$$

In light of Lemma 4.11, it will suffice to show that α^{12} has nice properties; then the isomorphism

$$\alpha^1 = \alpha^{12} \circ \alpha^1 \tag{4.19)(b}$$

will satisfy the requirements of Lemma 4.14. So fix $S \in A^1_{\mathfrak{g}}$, and define

$$\alpha^{12}(S)(u) = (\mathrm{ad}(u)S)(1). \tag{4.19)(c}$$

That the map $\alpha^{12}(S)$ satisfies condition (ii) in Lemma 4.14, and that

$$\alpha^{12}(S)(uv) = \alpha^{12}(\mathrm{ad}(v)S)(u) \tag{4.19)(d}$$

(which implies (4.14)(iv)) are easy calculations. The formula (4.14)(v) follows from (4.11)(iv) and the fact that the map $\phi_{\mathfrak{g}}$ is ad-equivariant.

The main point is therefore to show that the map α^{12} is a linear isomorphism. To see that, we will write an explicit inverse. Write $u \to {}^t u$ for the transpose antiautomorphism of $U(\mathfrak{g})$; we have ${}^t X = -X$ for X in $\mathfrak{g}$. Write

$$h : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

for the Hopf map; this is the algebra homomorphism sending X in $\mathfrak{g}$ to $X \otimes 1 + 1 \otimes X$. If M is a bimodule for $U(\mathfrak{g})$, then the adjoint action of $U(\mathfrak{g})$ may be computed as follows. Fix u in $U(\mathfrak{g})$, and write

$$h(u) = \sum u_i \otimes v_i. \tag{4.20)(a}$$

Then

$$\mathrm{ad}(u)m = \sum (u_i)m({}^t v_i). \tag{4.20)(b}$$

We have described the left and right actions of $U(\mathfrak{g})$ in $A^1_{\mathfrak{g}}$ explicitly; we deduce that for $S \in A^1_{\mathfrak{g}}$,

$$(\mathrm{ad}(u)S)(x) = \sum u_i[S({}^t v_i x)]. \tag{4.20)(c}$$

Consequently

$$\alpha^{12}(S)(u) = \sum u_i(S({}^t v_i)). \tag{4.20)(d}$$

Now suppose that Σ is an element of $A^2_{\mathfrak{g}}$. Define a map $\alpha^{21}(\Sigma)$ from $U(\mathfrak{g})$ to $A_{G/Q}$ by

$$\alpha^{21}(\Sigma)(u) = \sum u_i(\Sigma({}^t v_i)). \tag{4.21)(a}$$

One way to see that α^{21} is well-defined is to regard $A^2_{\mathfrak{g}}$ as a subspace of $\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), A_{G/Q})$. This larger space carries commuting left and right $U(\mathfrak{g})$ actions, from the left actions on $A_{G/Q}$ and $U(\mathfrak{g})$ respectively. Consequently there is an adjoint action ad; and

$$\alpha^{21}(\Sigma)(u) = (ad(u)\Sigma)(1). \tag{4.21)(b}$$

To check that $\alpha^{21}(\Sigma)$ belongs to $A^1_{\mathfrak{g}}$, fix u in $U(\mathfrak{g})$ and X in $\mathfrak{q}$; we must show that

$$\alpha^{21}(\Sigma)(uX) = (\alpha^{21}(\Sigma)(u))X.$$

Now $h(uX) = \sum(u_i X \otimes v_i + u_i \otimes v_i X)$, so

$$\alpha^{21}(\Sigma)(uX) = \sum(u_i X(\Sigma({}^t v_i)) + u_i(\Sigma(-X {}^t v_i))).$$

By (4.14)(ii), the second terms can be rewritten to give

$$\begin{aligned}\alpha^{21}(\Sigma)(uX) &= \sum(u_i X(\Sigma({}^t v_i)) - u_i(ad(X)\Sigma({}^t v_i))) \\ &= \sum(u_i X(\Sigma({}^t v_i)) - u_i(Xs({}^t v_i) - \Sigma({}^t v_i)X)).\end{aligned}$$

The first two terms in each summand cancel, leaving

$$\begin{aligned}\alpha^{21}(\Sigma)(uX) &= \sum(u_i s({}^t v_i)X) \\ &= (\alpha^{21}(\Sigma)(u))X,\end{aligned}$$

as we wished to show.

Finally, we must show that the correspondences defined by (4.20) and (4.21) are mutual inverses. Fix S in $A^1_{\mathfrak{g}}$; we will show that

$$\alpha^{21}(\alpha^{12}(S)) = S. \tag{$*$}$$

(The proof that $\alpha^{12}(\alpha^{21}(\Sigma)) = \Sigma$ is identical.) To do this, fix $X_1, \ldots, X_n$ in $U(\mathfrak{g})$; we will compute the left side of $(*)$ at $u = X_1 \ldots X_n$. If A is any subset of $\{1, \ldots, n\}$, write X_A for the corresponding product (with the indices arranged in increasing order). The complement of A is written A^c. Then the Hopf map is given by

$$h(u) = \sum_{A \subset \{1,\ldots,n\}} X_A \otimes X_{A^c}.$$

Therefore

$$\begin{aligned}\alpha^{21}(\alpha^{12}(S))(u) &= \sum_A X_A(\alpha^{12}(S)({}^tX_{A^c}))\\ &= \sum_A X_A\left(\sum_{B\subset A^c} {}^tX_B S(X_{A^c-B})\right)\\ &= \sum_{\substack{A\\ B\subset A^c}} X_A{}^tX_B S(X_{A^c-B}).\end{aligned}$$

This last expression may be regarded as a sum over partitions of $\{1,\ldots,n\}$ into three disjoint subsets A, B, C; it is

$$\begin{aligned}&\sum_{A,B,C} X_A{}^tX_B S(X_C)\\ &=\sum_C(\mathrm{ad}(X_{C^c})\cdot 1)S(X_C).\end{aligned}$$

Of course $\mathrm{ad}(X_{C^c})$ acts by zero on 1 unless C^c is empty. The only non-zero term is therefore

$$(ad(1)\cdot 1)S(X_{\{1,\ldots,n\}}),$$

which is $S(u)$ as we wished to show. Q.E.D.

We conclude this section with a symbol calculus for the induced Dixmier algebra.

Definition 4.22. Suppose we are in the setting of Definition 4.10. The standard filtration of $U(\mathfrak{g})$ induces an $\mathrm{Ad}(Q)$-stable filtration

$$A_{G/Q,n} = \text{ image of } U_n(\mathfrak{g})\otimes A_Q \tag{4.22)(a}$$

of $A_{G/Q}$. The left action of $U(\mathfrak{g})$ is compatible with this filtration in the usual sense (see the discussion preceding Conjecture 2.3). Define $S_{G/Q}$ to be the associated graded object:

$$S^n_{G/Q} = S^n(\mathfrak{g}/\mathfrak{q})\otimes A_Q. \tag{4.22)(b}$$

This is a graded algebra with a graded algebraic action (still called Ad) of Q by automorphisms. (In fact it is the pushforward to a point of a constant sheaf of $\mathcal{O}$-algebras on $(\mathfrak{g}/\mathfrak{q})^*$.) The *principal symbol sheaf* is the graded sheaf of algebras on G/Q

$$\mathcal{S}_G = G\times_Q S_{G/Q} \tag{4.22)(c}$$

(cf. (3.13)). (This is the pushforward to G/Q of a sheaf of $\mathcal{O}$-algebras on $T^*(G/Q)$.) The space of global sections of $\mathcal{S}_G$ is a graded algebra S_G that we call the *principal symbol algebra*; explicitly,

$$\begin{aligned}S_G &= \mathrm{Ind}_{alg}(Q\uparrow G)(S(\mathfrak{g}/\mathfrak{q})\otimes A_Q)\\ &= \text{ functions on } G \text{ with values in } S(\mathfrak{g}/\mathfrak{q})\otimes A_Q,\\ &\qquad \text{transforming by Ad under } Q.\end{aligned} \tag{4.22)(d}$$

An immediate consequence is that if A_Q (or, equivalently, A_L) is completely prime, then S_G is as well. We write Ad for the action of G on S_G by left translation of sections.

Now define the *weak filtration* on $A_{\mathfrak{g}}$ by

$$A^1_{\mathfrak{g},p,wk} = \{\, S\in A^1_{\mathfrak{g}} \mid S(U_n(\mathfrak{g}))\subset (A_{G/Q})_{p+n}\}. \tag{4.23)(a}$$

(Here and below we will use the isomorphisms α^1 and α^2 to transfer filtrations among $A_{\mathfrak{g}}$, $A^1_{\mathfrak{g}}$, and $A^2_{\mathfrak{g}}$.) This makes $A^1_{\mathfrak{g}}$ a filtered algebra (as one checks using (4.13(c)), but not every element of $A^1_{\mathfrak{g}}$ belongs to some $A^1_{\mathfrak{g},p}$. It follows from (4.11)(iv) that

$$\phi_{\mathfrak{g}}(U_n(\mathfrak{g}))\subset (A_{\mathfrak{g}})_{n,wk}. \tag{4.23)(b}$$

Using (4.20) and (4.21), one can describe the weak filtration directly on $A^2_{\mathfrak{g}}$ as well: it is

$$A^2_{\mathfrak{g},p,wk} = \{\, \Sigma \in A^2_{\mathfrak{g}} \mid \Sigma(U_n(\mathfrak{g})) \subset (A_{G/Q})_{p+n} \}. \tag{4.23)(c}$$

The difficulty with the weak filtration is that it is not ad($\mathfrak{g}$)-invariant. Given any filtered algebra A and a family $\mathcal{D}$ of derivations of A, we can define a smaller filtration of A by

$$A_{n,sm} = \{\, a \in A \mid D_1 \dots D_k(a) \in A_n, \text{ all } \{D_i\} \subset \mathcal{D} \,\}.$$

This gives A a new structure of filtered algebra, this one preserved by $\mathcal{D}$. In our case, we set

$$A_{\mathfrak{g},p} = \{\, T \in A_{\mathfrak{g}} \mid ad(u)T \in A_{\mathfrak{g},p,wk}, \text{ all } u \in U(\mathfrak{g}) \,\}. \tag{4.24)(a}$$

This is not very easy to understand directly; but Lemma 4.14(iv) and (4.23)(c) show immediately that the corresponding filtration of $A^2_{\mathfrak{g}}$ is

$$A^2_{\mathfrak{g},p} = \{\, \Sigma \in A^2_{\mathfrak{g}} \mid \Sigma(U_n(\mathfrak{g})U(\mathfrak{g})) \subset (A_{G/Q})_{p+n}, \text{ all } n \,\}. \tag{4.24)(a}$$

Clearly this is

$$A^2_{\mathfrak{g},p} = \{\, \Sigma \mid \Sigma(U(\mathfrak{g})) \subset (A_{G/Q})_p \,\}. \tag{4.24)(b}$$

Using Corollary 4.16, we deduce a description of the restriction of this filtration to A_G:

$$A_{G,p} = \mathrm{Ind}_{alg}(Q \uparrow G)(A_{G/Q,p}). \tag{4.24)(c}$$

Regarded as an algebraic map from G to $A_{G/Q}$, every element of A_G has its image contained in some finite-dimensional subspace. Consequently

$$\bigcup_p A_{G,p} = A_G. \tag{4.24)(d}$$

On the level of sheaves, Definition 4.22 gives rise to an exact sequence

$$0 \to \mathcal{A}_{G,p-1} \to \mathcal{A}_{G,p} \to \mathcal{S}^p_G \to 0. \tag{4.25)(a}$$

(Here we have used and extended slightly the notation of Definition 4.15 and (4.22)(c).) Taking global sections gives an exact sequence

$$0 \to A_{G,p-1} \to A_{G,p} \to S^p_G. \tag{4.25)(b}$$

The last map in this sequence is called a *principal symbol map*, and is denoted π_p. Because of (4.24)(c), these maps can be assembled to a graded injection

$$\pi : \mathrm{gr}\ A_G \hookrightarrow S_G. \tag{4.25)(c}$$

Clearly π respects the action Ad of G. What is less obvious is

Proposition 4.26. *The principal symbol map π of (4.25) is an algebra homomorphism. In particular, if A_L is completely prime, then the induced Dixmier algebra A_G is as well.*

If $H^1(G/Q, \mathcal{S}_G) = 0$, then π is an isomorphism of graded algebras.

Proof. Suppose $S \in A^1_{\mathfrak{g},p}$. Write $\Sigma = \alpha^{12}(S)$ for the corresponding element of $A^2_{\mathfrak{g},p}$. If u belongs to $U_n(\mathfrak{g})$, then (4.21) shows that

$$S(u) \equiv u\Sigma(1) \pmod{A_{G/Q,n+p-1}}.$$

By the definition of α^{12}, this is the same as

$$S(u) \equiv uS(1) \pmod{A_{G/Q,n+p-1}}. \tag{4.27)(a}$$

Write

$$S(1) = \sum u_i \otimes E_i, \tag{4.27)(b}$$

with u_i in $U_p(\mathfrak{g})$. It is immediate from the definition that the value at the identity of the principal symbol is given by

$$\pi_p(S)(e) = \sum p_i \otimes E_i; \tag{4.27)(c}$$

here p_i is the image of u_i in $S^p(\mathfrak{g}/\mathfrak{q})$.

Suppose S' belongs to $A_{\mathfrak{g},q}$. Write

$$S'(1) = \sum v_j \otimes E'_j, \tag{4.27)(d}$$

with v_j in $U_q(\mathfrak{g})$. If q_j is the image of v_j in $S^q(\mathfrak{g}/\mathfrak{q})$, then

$$\pi_q(S')(e) = \sum q_j \otimes E'_j. \tag{4.27)(e}$$

By (4.23) applied to S',

$$S'(u_i) = \sum_j u_i v_j \otimes E'_j \qquad \pmod{A_{G/Q,p+q-1}}. \tag{4.27)(f}$$

By Lemma 4.11(iv),

$$(S'S)(1) = \sum_{i,j} u_i v_j \otimes E'_j E_i \qquad \pmod{A_{G/Q,p+q-1}}. \tag{4.27)(g}$$

Consequently

$$\begin{aligned} \pi_{p+q}(S'S)(e) &= \sum_{i,j} p_i q_j \otimes E'_j E_i \\ &= \pi_q(S')(e)\pi_p(S)(e). \end{aligned} \tag{4.27)(h}$$

The proposition follows from (4.27)(h) and the Ad(G)-equivariance of π. Q.E.D.

There is a variant of this symbol calculus that is of interest for the Dixmier conjecture. (Because we will never need both at the same time, we will not use a different notation for the filtrations in the variant.) Suppose that A_L is endowed with a good filtration indexed by $1/2\mathbb{N}$. Recall from the discussion before Conjecture 2.3 that this amounts to the following requirements:

$$\begin{gathered} A_{L,p}A_{L,q} \subset A_{L,p+q}; \\ A_{L,p} \text{ is } \mathrm{Ad}(L)\text{-invariant}; \\ \bigcup_p A_{L,p} = A_L; \\ \phi_L(U_n(\mathfrak{l})) \subset (A_L)_n; \\ (R_L, \psi_L) = (\mathrm{gr}\ A_L, \mathrm{gr}\ \phi_L) \text{ is a graded orbit datum.} \end{gathered} \tag{4.28}$$

We want to construct a good filtration of the induced Dixmier algebra (A_G, ϕ_G).

To begin, filter $A_{G/Q}$ by

$$A_{G/Q,p} = \sum_n U_n(\mathfrak{g}) \otimes A_{Q,p-n}. \tag{4.29)(a}$$

(Of course the underlying algebra of A_Q is just A_L, so A_Q inherits the filtration of A_L.) This filtration is Ad(Q)-stable, and

$$U_n(\mathfrak{g}) \cdot A_{G/Q,p} \subset A_{G/Q,n+p}. \tag{4.29)(b}$$

The associated graded object therefore carries an action Ad of Q and a graded $S(\mathfrak{g})$-module structure. A little thought shows that it is the graded tensor product

$$\mathrm{gr}\ A_{G/Q} = S(\mathfrak{g}) \otimes_{S(\mathfrak{q})} \mathrm{gr}\ A_Q. \tag{4.29)(c}$$

This is exactly the algebra $R_{G/Q}$ attached to R_L in the construction of induced orbit data (cf. (3.12)). We call

$$\mathcal{R}_G = G \times_Q R_{G/Q} \tag{4.29)(d}$$

a *good symbol sheaf.* The space of global sections is a graded algebra R_G that we call a *good symbol algebra.* By (3.13) and (3.14),

$$R_G = Ind_{orb}(Q \uparrow G)(R_L). \tag{4.29)(e}$$

Of course R_G is an orbit datum for G by Proposition 3.15.

Now we can filter $A^2_{\mathfrak{g}}$ by

$$A^2_{\mathfrak{g},p} = \{\, \Sigma \in A^2_{\mathfrak{g}} \mid \Sigma(U(\mathfrak{g})) \subset A_{G/Q,p}\,\}. \tag{4.30)(a}$$

Exactly as in (4.23) and (4.24), one shows that this defines a filtered algebra structure on $A^2_{\mathfrak{g}}$. The induced filtration on A_G is

$$A_{G,p} = \mathrm{Ind}_{alg}(Q \uparrow G)(A_{G/Q,p}), \tag{4.30)(b}$$

and

$$\bigcup_p A_{G,p} = A_G. \tag{4.30)(c}$$

Again we get an exact sequence of sheaves on G/Q

$$0 \to \mathcal{A}_{G,p-1} \to \mathcal{A}_{G,p} \to \mathcal{R}^p_G \to 0. \tag{4.31)(a}$$

This gives rise to an exact sequence

$$0 \to A_{G,p-1,gd} \to A_{G,p,gd} \to (R_G)^p. \tag{4.31)(b}$$

The last map in this sequence is called a *good symbol map*, and is denoted γ_p. We therefore have a graded G-equivariant injection

$$\gamma : \mathrm{gr}\ A_G \to R_G. \tag{4.31)(c}$$

Exactly as in Proposition 4.26, one shows that γ is a homomorphism of algebras. We have proved

Proposition 4.32. *Suppose the Dixmier algebra (A_L, ϕ_L) is endowed with a good filtration indexed by $1/2\mathbb{N}$ (cf. (4.28) with associated graded orbit datum (R_L, ψ_L). Write (R_G, ψ_G) for the induced orbit datum for G. Then the induced Dixmier algebra (A_G, ϕ_G) has a natural good filtration, and the associated graded orbit datum injects into (R_G, ψ_G). If the cohomology $H^1(G/Q, \mathcal{R}_G)$ vanishes, then this injection is an isomorphism.*

Evidently this result has some relationship to Conjecture 2.3. We will discuss this in detail in section 5.

5. Sheets of Dixmier algebras.

We can now say something about what a "sheet of Dixmier algebras" ought to look like. Of course such a sheet ought to consist of the Dixmier algebras corresponding via Conjecture 2.3 to a sheet of geometric orbit data. Our goal, however, is to make a little progress toward proving that conjecture. We fix therefore a sheet of geometric orbit data attached to a Levi subgroup L of G and a rigid unipotent orbit datum (R_u, ψ_u) for $L/Z(L)_0$ (Definition 3.20). Thus

$$X_u = \mathrm{Spec}\ R_u \text{ is a unipotent Poisson variety for } L/Z(L)_0 \tag{5.1)(a}$$

(cf. [21]). One consequence is that the action of $\mathbb{C}^\times$ on the support Σ_u must lift to X_u, at least after passing to a finite cover of $\mathbb{C}^\times$. This means that R_u is graded by $k^{-1}\mathbb{N}$ for some positive integer k. As was already mentioned in section 2, Moeglin has observed that k is at most 2. (This is a consequence of the Jacobson-Morozov theorem.) Consequently

$$R_u \text{ is graded by } 1/2\mathbb{N}; \tag{5.1)(b}$$

this gradation is compatible with the standard one on $S(\mathfrak{g})$ and the map ψ_u.

The first serious step is to attach to (R_u, ψ_u) a Dixmier algebra for $L/Z(L)_0$. There is a conjecture for how to do this in section 5 of [21]; we reproduce a strengthened version here.

Definition/Conjecture 5.2. Suppose G is a semisimple group, and (R, ψ) is a rigid geometric (and therefore unipotent) orbit datum (Definition 2.2 and Definition 3.20). Recall the natural grading of R by $1/2\mathbb{N}$, and the natural Poisson structure $\{,\}$ on R. A *(rigid) unipotent Dixmier algebra* associated to (R, ψ) is a Dixmier algebra (A, ϕ) (Definition 2.1) having an $\mathrm{Ad}(G)$-invariant filtration by $1/2\mathbb{N}$, subject to the following conditions.

i) $\phi(U_n(\mathfrak{g})) \subset A_n$.
ii) The pair $(\mathrm{gr}\ A, \mathrm{gr}\ \phi)$ is isomorphic as an algebra with G-action to (R, ψ).

These two properties conjecturally determine exactly one Dixmier algebra. An immediate consequence is

iii) A is completely prime.

As a conjectural consequence of (i) and (ii), one should have four additional properties.

iv) The infinitesimal character of A corresponds to a weight in the rational span of the roots.
v) The kernel of ϕ is a weakly unipotent primitive ideal in $U(\mathfrak{g})$.
vi) Suppose a and b in A_p and A_q have images r and s in R^p and R^q. Then the Poisson bracket $\{r, s\}$ is the image of $ab - ba$.
vii) A admits a transpose antiautomorphism $(a \to {}^t a)$ of order 1, 2, or 4 (Definition 2.8). This preserves the filtration, and the associated graded antiautomorphism acts by $\exp(ip\pi)$ on R^p.

(The notion of *weakly unipotent* primitive ideal may be found in [19], section 8. The condition is that the infinitesimal character cannot be made shorter by tensoring with a finite-dimensional representation. The prototypical example is the augmentation ideal.)

This conjecture is very easy if $R = \mathbb{C}$ (corresponding to the zero nilpotent orbit). If R corresponds to a minimal non-zero nilpotent orbit (which is rigid except in type A) then the conjecture is true because of Joseph's work on the Joseph ideal (as supplemented by Garfinkle). The complete conjecture is known only in a handful of other cases, although there is some evidence for it in many other infinite families of examples. Some of the most powerful results in the direction of (ii) are those in [16].

Now assume that we are in the setting (5.1), and that we have a unipotent Dixmier algebra

$$(A_u, \phi_u) \tag{5.3)(a}$$

for $L/Z(L)_0$ attached to (R_u, ϕ_u) in the sense of Conjecture 5.2. To every character λ of $\mathfrak{z}(\mathfrak{l})$ we can attach an orbit datum

$$(R_L(\lambda), \psi_L(\lambda)) \tag{5.3)(b}$$

for L; here $R_L(\lambda)$ is isomorphic to R_u, and $\psi_L(\lambda)$ is given as in (2.5)(e) and (f). Similarly, we can construct a Dixmier algebra

$$(A_L(\lambda), \phi_L(\lambda)) \tag{5.3)(c}$$

for L. Here $A_L(\lambda)$ is just A_u as an algebra with L action, and

$$\begin{aligned} \phi_L(\lambda)(X) &= \lambda(X) \qquad (X \in \mathfrak{z}(\mathfrak{l})), \\ \phi_L(\lambda)(Y) &= \phi_u(Y) \qquad (Y \in [\mathfrak{l}, \mathfrak{l}]). \end{aligned} \tag{5.3)(d}$$

Definition 5.4. Suppose we are in the setting (5.1), and that the Dixmier algebra (A_u, ϕ_u) for $L/Z(L)_0$ exists. Fix a parabolic subgroup $Q = LU$ with Levi factor L. Form a sheet of orbit data

$$(R_G(\lambda), \psi_G(\lambda)) = \mathrm{Ind}_{orb}(Q \uparrow G)(R_L(\lambda), \psi_L(\lambda)) \tag{5.4)(a}$$

(Definition 3.22). (Recall that, according to Conjecture 3.19, this sheet is independent of the choice of Q.) The *sheet of Dixmier algebras for G attached to $(Q, (A_u, \phi_u))$* is

$$(A_G(\lambda), \phi_G(\lambda)) = \mathrm{Ind}_{Dix}(Q \uparrow G)(A_L(\lambda), \phi_L(\lambda))$$

(Definition 4.7). According to Conjecture 4.18, these algebras do not depend on the choice of Q.

By Proposition 4.32 (and hypothesis (ii) in Definition/Conjecture 5.2) each $A_G(\lambda)$ carries a natural good filtration indexed by $1/2\mathbb{N}$. These filtrations have the property that there is natural injection of graded orbit data

$$(\mathrm{gr}\ A_G(\lambda), \mathrm{gr}\ \phi_G(\lambda) \hookrightarrow (R_G(0), \psi_G(0)). \tag{5.5)(a}$$

An elementary argument also provides natural good filtrations of the orbit data in the sheet, with the property that

$$(\mathrm{gr}\ R_G(\lambda), \mathrm{gr}\ \psi_G(\lambda)) \hookrightarrow (R_G(0), \psi_G(0)). \tag{5.5)(b}$$

Both of these injections ought to be isomorphisms. Because of Proposition 4.32 (and an easier analogue for orbit data) this would be a consequence of

Conjecture 5.6. In the setting of Definition 5.4, form the sheaf $\mathcal{R}_G(0)$ on G/Q as in (3.13). Then the higher cohomology of G/Q with coefficients in $\mathcal{R}_G(0)$ vanishes.

If $R_u = \mathbb{C}$, then the sheaf $\mathcal{R}_G(0)$ is the sheaf of functions on the cotangent bundle of G/Q. In that case Conjecture 5.6 is true, by a well-known result of Elkik.

The correspondence taking $(R_G(\lambda), \psi_G(\lambda))$ to $(A_G(\lambda), \phi_G(\lambda))$ is our candidate for a Dixmier correspondence on geometric orbit data (cf. Conjecture 2.3). To define it, we need to know the existence of unipotent Dixmier algebras attached to rigid orbit data (Conjecture 5.2). For it to be well-defined on a single sheet, we need both $R_G(\lambda)$ and $A_G(\lambda)$ to be independent of Q (Conjectures 3.19 and 4.18). For it to be well-defined on all geometric orbit data, we need the geometric sheets to be disjoint (Conjecture 3.24). For it to satisfy Conjecture 2.3, we need a cohomology vanishing result (Conjecture 5.6).

6. Relation with bimodule induction.

In this section we will resume our study of induced Dixmier algebras, focusing on special conditions on the inducing algebra that permit a further analysis of their structure. The main tool is the idea of induction of Harish-Chandra bimodules, which we now recall.

Definition 6.1. Suppose $Q = LU$ is a parabolic subgroup of G, and B_L is a Harish-Chandra bimodule for L. Write $Q^{op} = LU^{op}$ for the opposite parabolic subgroup. Define a new bimodule B'_L by subtracting from the left and right actions of $\mathfrak{l}$ the character δ defined after (4.2)(a). Make B'_L into a left $\mathfrak{q}$-module by making $\mathfrak{u}$ act by zero, and into a right $\mathfrak{q}^{op}$-module by making $\mathfrak{u}^{op}$ act by zero; we denote this object $B_{\mathfrak{q},\mathfrak{q}^{op}}$. Consider

$$B_{\mathfrak{g},bi} = B_{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{q},\mathfrak{q}^{op}}(U(\mathfrak{g}) \otimes U(\mathfrak{g}), B_{\mathfrak{q},\mathfrak{q}^{op}}).$$

Here $\mathfrak{q}$ acts on the left on the first $U(\mathfrak{g})$ factor and $\mathfrak{q}^{op}$ on the right on the second to define the Hom. $B_{\mathfrak{g}}$ has the structure of a $\mathfrak{g}$-bimodule: the left action of $\mathfrak{g}$ comes from the right action on the first $U(\mathfrak{g})$, and the right action from the left action of $\mathfrak{g}$ on the second $U(\mathfrak{g})$. Put

$$B_{G,bi} = B_G = \mathrm{Ad}(G)\text{-finite part of } B_{\mathfrak{g}};$$

this is the Harish-Chandra bimodule for G *induced* by B_L. We write

$$B_G = \mathrm{Ind}_{bi}(Q \uparrow G)(B_L).$$

Induction of Harish-Chandra bimodules is a very simple and well-understood process. It is an exact functor, and (as algebraic representations under Ad) we have

$$B_G \simeq \mathrm{Ind}_{alg}(L \uparrow G)(B_L) \tag{6.2}$$

(notation (A.8)). Because this is induction from a reductive subgroup, it is much better behaved than the induction appearing in (say) Corollary 4.16. The reader may wonder why we did not use it in section 4 to construct induced Dixmier algebras. The reason is that bimodule induction does not in general take algebras to algebras.

Nevertheless, there is much to be gained (computability, for example) whenever we can relate induction of Dixmier algebras (which are, among other things, Harish-Chandra bimodules) to Ind_{bi}. Examples for SL(2) show that the relationship cannot be quite trivial. To understand it, we will first interpret bimodule induction in terms of endomorphisms between modules (Corollary 6.5). Once that is done, it is convenient to extend the induction construction of Definition 4.7 from Dixmier algebras to arbitrary Harish-Chandra bimodules (Definition 6.7). Once the two kinds of induction are described in parallel, it becomes clear that they are related by something like the Shapovalov form on a Verma module (Definition 6.11). Establishing an isomorphism between them therefore comes down to proving some irreducibility results for (appropriately generalized) Verma modules (Theorem 6.12). The idea of this argument comes from [5]; much more sophisticated incarnations of it appear in Joseph's work relating primitive ideals to highest weight modules (cf. [11] and [12], section 1.3).

Suppose $V_{\mathfrak{l}}$ and $W_{\mathfrak{l}}$ are modules for $\mathfrak{l}$. Assume that

$$B_L \subset \mathrm{Hom}(V_{\mathfrak{l}}, W_{\mathfrak{l}}) \tag{6.3)(a}$$

is a Harish-Chandra bimodule of maps. Here the left action of $\mathfrak{l}$ comes from the action on $W_{\mathfrak{l}}$, etc. (It is easy to see that any Harish-Chandra bimodule arises in this way; we do not require $V_{\mathfrak{l}}$ and $W_{\mathfrak{l}}$ to be particularly nice.) Write $V_{\mathfrak{l}}'$ for $V_{\mathfrak{l}}$ with the action of $\mathfrak{l}$ twisted by $-\delta$, and then $V_{\mathfrak{q}^{op}}$ for the extension to $\mathfrak{q}^{op}$ on which $\mathfrak{u}^{op}$ acts by zero. Similarly define $W_{\mathfrak{q}}$. Then

$$B_{\mathfrak{q},\mathfrak{q}^{op}} \subset \mathrm{Hom}(V_{\mathfrak{q}^{op}}, W_{\mathfrak{q}}). \tag{6.3)(b}$$

Define

$$V_{\mathfrak{g},ind} = U(\mathfrak{g}) \otimes_{\mathfrak{q}^{op}} V_{\mathfrak{q}^{op}} \simeq U(\mathfrak{u}) \otimes_{\mathbf{C}} V_{\mathfrak{q}^{op}} \tag{6.3)(c}$$

$$W_{\mathfrak{g},pro} = \mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), W_{\mathfrak{q}}) \simeq \mathrm{Hom}_{\mathbf{C}}(U(\mathfrak{u}^{op}, W_{\mathfrak{q}}). \tag{6.3)(d}$$

Lemma 6.4. *In the setting (6.3), there is a natural identification of $U(\mathfrak{g})$-bimodules*

$$\begin{aligned}\mathrm{Hom}_{\mathbf{C}}(V_{\mathfrak{g},ind}, W_{\mathfrak{g},pro}) &\simeq \mathrm{Hom}_{\mathfrak{q},\mathfrak{q}^{op}}(U(\mathfrak{g}) \otimes U(\mathfrak{g}), \mathrm{Hom}(V_{\mathfrak{q}^{op}}, W_{\mathfrak{q}}))\\ &\simeq \mathrm{Hom}_{\mathbf{C}}(U(\mathfrak{u}^{op}) \otimes U(\mathfrak{u}), \mathrm{Hom}(V_{\mathfrak{q}^{op}}, W_{\mathfrak{q}})).\end{aligned}$$

We will not give the (easy and well-known) formal proof, but it is helpful to recall the form of the isomorphism. If ϕ on the left corresponds to Φ on the right, then for u and u' in $U(\mathfrak{g})$ and v in $V_{\mathfrak{q}}^{op}$,

$$[\phi(u \otimes v)]\,(u') = (\Phi(u' \otimes u))v.$$

Here $u \otimes v$ is an element of $V_{\mathfrak{g},ind}$. The term in square brackets on the left is therefore an element of $W_{\mathfrak{g},pro}$; that is, it is a map from $U(\mathfrak{g})$ to $W_{\mathfrak{q}}$. We specify it by specifying its value at u'.

Definition 6.5. Suppose we are in the setting of (6.3). A map

$$\phi : V_{\mathfrak{g},ind} \to W_{\mathfrak{g},pro}$$

is said to be of *type* B_L if the corresponding map

$$\Phi : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to \mathrm{Hom}(V_{\mathfrak{q}^{op}}, W_{\mathfrak{q}}))$$

(Lemma 6.4) takes values in $B_{\mathfrak{q},\mathfrak{q}^{op}}$. By Definition 6.1,

$$B_{\mathfrak{g},bi} \simeq \{\, \phi : V_{\mathfrak{g},ind} \to W_{\mathfrak{g},pro} \mid \phi \text{ is of type } B_L \,\}.$$

Definitions 6.1 and 6.5 combine (like all good definitions) to give a result.

Proposition 6.6. *Suppose we are in the setting (6.3). Then the induced Harish-Chandra bimodule* $B_G = \mathrm{Ind}_{bi}(Q \uparrow G)(B_L)$ *consists precisely of the* $\mathrm{Ad}(G)$*-finite maps from* $V_{\mathfrak{g},ind}$ *to* $W_{\mathfrak{q},pro}$ *that are of type* B_L.

We now outline the extension to bimodules of the construction of section 4.

Definition 6.7. In the setting of (6.3), extend $W'_{\mathfrak{l}}$ to a module $W_{\mathfrak{q}^{op}}$ for $\mathfrak{q}^{op}$ by making $\mathfrak{u}^{op}$ act by zero. Define

$$W_{\mathfrak{g},ind} = U(\mathfrak{g}) \otimes_{\mathfrak{q}^{op}} W_{\mathfrak{q}^{op}}. \tag{6.7)(a}$$

In analogy with (4.2), identify B_L with a $U(\mathfrak{q}^{op})$-bimodule

$$B_{Q^{op}} \subset \mathrm{Hom}(V_{\mathfrak{q}^{op}}, W_{\mathfrak{q}^{op}}). \tag{6.7)(b}$$

As in Definition 4.6, we say that a map T from $V_{\mathfrak{g},ind}$ to $W_{\mathfrak{g},ind}$ is of *type* B_L if for every u in $U(\mathfrak{g})$ there are elements u_i in $U(\mathfrak{g})$ and E_i in $B_{\mathfrak{q}^{op}}$ such that for every v in $V_{\mathfrak{q},op}$,

$$T(u \otimes v) = \sum u_i \otimes E_i v. \tag{6.7)(c}$$

The collection of all such maps is a $U(\mathfrak{g})$-bimodule

$$B_{\mathfrak{g},Dix} \subset \mathrm{Hom}(V_{\mathfrak{g},ind}, W_{\mathfrak{g},ind}). \tag{6.7)(d}$$

Define

$$B_{G,Dix} = \mathrm{Ad}(G)\text{-finite maps from } V_{\mathfrak{g},ind} \text{ to } W_{\mathfrak{g},ind} \text{ of type } B_L. \tag{6.7)(e}$$

The proof of Corollary 4.17 shows that $B_{G,Dix}$ is a Harish-Chandra bimodule for G depending only on Q and B_L (and not on the choices of $V_{\mathfrak{l}}$ and $W_{\mathfrak{l}}$). We call it the *Dixmier induced Harish-Chandra bimodule*, and write

$$B_{G,Dix} = \mathrm{Ind}_{Dix}(Q^{op} \uparrow G)(B_L). \tag{6.7)(f}$$

As in Corollary 4.16, one gets an isomorphism of G-modules for Ad

$$B_{G,Dix} \simeq \mathrm{Ind}_{alg}(Q^{op} \uparrow G)(U(\mathfrak{g}) \otimes_{\mathfrak{q}^{op}} B_{Q^{op}}). \tag{6.7)(g}$$

Here $\mathfrak{q}^{op}$ acts on $B_{Q^{op}}$ on the left to define the tensor product, and Q^{op} acts by $\mathrm{Ad} \otimes \mathrm{Ad}$ on the inducing representation.

Recall that what we want is to compare $B_{G,bi}$ and $B_{G,Dix}$. We will do this by comparing the larger bimodules $B_{\mathfrak{g},bi}$ and $B_{\mathfrak{g},Dix}$. To do that, we determine their bimodule structure under the center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$. As in section 4, it is harmless and convenient to assume that $\mathfrak{z}(\mathfrak{l})$ acts by characters λ and ρ on $W'_{\mathfrak{l}}$ and $V'_{\mathfrak{l}}$

respectively. We assume also that the Harish-Chandra bimodule B_L has finite length. Fix characters β and γ of $\mathfrak{z}(\mathfrak{l})$. Then the subspace of $B_{\mathfrak{g},Dix}$ transforming on the left by β and on the right by γ is (cf. (4.9))

$$(B_{\mathfrak{g},Dix})_{\beta\gamma} \simeq \mathrm{Hom}(U(\mathfrak{u})_{\gamma-\rho}, U(\mathfrak{u})_{\beta-\lambda}) \otimes B_{Q^{op}}. \tag{6.8)(a}$$

The whole bimodule is assembled from these pieces by the prescription

$$B_{\mathfrak{g},Dix} \simeq \prod_{\gamma} \left(\sum_{\beta} (B_{\mathfrak{g},Dix})_{\beta\gamma} \right). \tag{6.8)(b}$$

Similarly, Lemma 6.4 gives

$$(B_{\mathfrak{g},bi})_{\beta\gamma} \simeq \mathrm{Hom}(U(\mathfrak{u}^{op})_{-\beta+\lambda} \otimes U(\mathfrak{u})_{\gamma-\rho}, B_{\mathfrak{q},\mathfrak{q}^{op}}. \tag{6.8)(c}$$

Now the indicated weight spaces (for $\mathrm{ad}(\mathfrak{z}(\mathfrak{l})$ in $U(\mathfrak{u})$ and $U(\mathfrak{u}^{op})$ are finite-dimensional. As an $\mathfrak{l}$-bimodule, $B_{\mathfrak{q},\mathfrak{q}^{op}}$ is equal to $B_{Q^{op}}$. We may therefore rewrite this last equation as

$$(B_{\mathfrak{g},bi})_{\beta\gamma} \simeq \mathrm{Hom}(U(\mathfrak{u})_{\gamma-\rho}, (U(\mathfrak{u}^{op})_{-\beta+\lambda})^*) \otimes B_{Q^{op}}. \tag{6.8)(d}$$

By Definition 6.5,

$$B_{\mathfrak{g},bi} \simeq \prod_{\gamma} (\prod_{\beta} (B_{\mathfrak{g},bi})_{\beta\gamma}). \tag{6.8)(e}$$

Lemma 6.9. *Suppose B_L is a Harish-Chandra bimodule of finite length for L. Then as algebraic representations of* $\mathrm{Ad}(L)$*, the bimodule weight spaces $(B_{\mathfrak{g},bi})_{\beta\gamma}$ and $(B_{\mathfrak{g},Dix})_{\beta\gamma}$ for $\mathfrak{z}(\mathfrak{l})$ contain the same representations with the same (finite) multiplicities.*

Proof. This follows from (6.8)(a) and (d), using the isomorphism of L-modules $\mathfrak{u} \simeq (\mathfrak{u}^{op})^*$ provided by the Killing form. Q.E.D.

Our next task is the construction of a bimodule map from $B_{\mathfrak{g},Dix}$ to $B_{\mathfrak{g},bi}$. We need a little more notation in the setting of (6.3). We have

$$U(\mathfrak{g}) = U(\mathfrak{q}) \oplus U(\mathfrak{g})\mathfrak{u}^{op}; \tag{6.10)(a}$$

write

$$\pi : U(\mathfrak{g}) \to U(\mathfrak{q}) \tag{6.10)(b}$$

for the corresponding projection on the first factor. Write

$$I_{\mathfrak{q},\mathfrak{q}^{op}} : W_{\mathfrak{q}^{op}} \to W_{\mathfrak{q}} \tag{6.10)(c}$$

for the "identity map" of the underlying $\mathfrak{l}$-modules.

Definition 6.11. Suppose $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and $\mathfrak{q}^{op} = \mathfrak{l} + \mathfrak{u}^{op}$ are opposite parabolic subalgebras of $\mathfrak{g}$. Suppose $V_{\mathfrak{l}}$ and $W_{\mathfrak{l}}$ are $\mathfrak{l}$-modules; we use other notation as in (6.3) and (6.10). Define a homomorphism of $\mathfrak{q}^{op}$ modules

$$j_{\mathfrak{q}^{op}} : W_{\mathfrak{q}^{op}} \to W_{\mathfrak{g},pro}, \qquad (j_{\mathfrak{q}^{op}}(w))(x) = \pi(x) \cdot (I_{\mathfrak{q},\mathfrak{q}^{op}} w). \tag{6.11)(a}$$

Here x is in $U(\mathfrak{g})$ and w is in $W_{\mathfrak{q}^{op}}$. It is easy to check that $j_{\mathfrak{q}^{op}}(w)$ really belongs to $W_{\mathfrak{g},pro}$, and that the map is a $\mathfrak{q}^{op}$-module injection. By the universality property of induction, $j_{\mathfrak{q}^{op}}$ induces a $\mathfrak{g}$-module map

$$j : W_{\mathfrak{g},ind} \to W_{\mathfrak{g},pro}, \qquad (j(u \otimes w))(x) = \pi(xu) \cdot (I_{\mathfrak{q},\mathfrak{q}^{op}} w). \tag{6.11)(b}$$

We call j the *canonical intertwining operator*; it is closely connected to the Shapovalov form on a Verma module (see [6] or [10]).

Composition with j induces a $\mathfrak{g}$-bimodule map

$$J : \mathrm{Hom}_{\mathbf{C}}(V_{\mathfrak{g},ind}, W_{\mathfrak{g},ind}) \to \mathrm{Hom}_{\mathbf{C}}(V_{\mathfrak{g},ind}, W_{\mathfrak{g},pro}). \tag{6.11)(c}$$

We claim that this composition sends maps of type B_L to maps of type B_L (Definitions 6.5 and 6.7) and therefore restricts to

$$J : B_{\mathfrak{g},Dix} \to B_{\mathfrak{g},bi}. \tag{6.11)(d}$$

(This will be proved in a moment.) As a bimodule map, J respects the action ad, and so restricts to

$$J : \mathrm{Ind}_{Dix}(Q^{op} \uparrow G)(B_L) \to \mathrm{Ind}_{bi}(Q \uparrow G)(B_L); \tag{6.11)(e}$$

this map we call the *canonical bimodule intertwining operator.*

To prove that J preserves type, suppose that $T \in \mathrm{Hom}_{\mathbf{C}}(V_{\mathfrak{g},ind}, W_{\mathfrak{g},ind})$ is of type B_L; we want to prove that $J(T) = j \circ T$ is as well. Write τ for the map from $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ corresponding to $J(T)$ (Lemma 6.4), and fix u and u' in $U(\mathfrak{g})$. Then $\tau(u' \otimes u)$ is a map from $V_{\mathfrak{q}^{op}}$ to $W_{\mathfrak{q}}$; we have to show that it belongs to $B_{\mathfrak{q},\mathfrak{q}^{op}}$. To compute it, fix v in $V_{\mathfrak{q}^{op}}$. Now (6.6)(c) provides elements u_i in $U(\mathfrak{g})$ and E_i in $B_{\mathfrak{q}^{op}}$ (depending only on u) so that

$$T(u \otimes v) = \sum u_i \otimes E_i v.$$

Tracing through the definitions, we find

$$\begin{aligned}
(\tau(u' \otimes u))v &= (J(T)(u \otimes v))(u') && \text{(by Lemma 6.4)}\\
&= (j(\sum u_i \otimes E_i v))(u')\\
&= \sum \pi(u'u_i) \cdot (I_{\mathfrak{q},\mathfrak{q}^{op}} E_i v) && \text{(by (6.11)(b))}.
\end{aligned}$$

Now composition with $I_{\mathfrak{q},\mathfrak{q}^{op}}$ clearly defines an isomorphism from $B_{\mathfrak{q}^{op}}$ to $B_{\mathfrak{q},\mathfrak{q}^{op}}$. Write F_i for the image of E_i under this map. The action of $\pi(u'u_i)$ on $F_i v$ just corresponds to the left action of $U(\mathfrak{q})$ on $B_{\mathfrak{q},\mathfrak{q}^{op}}$; so we get

$$T(u \otimes v) = \left[\sum \pi(u'u_i) F_i\right] v.$$

Now the term in square brackets belongs to $B_{\mathfrak{q},\mathfrak{q}^{op}}$, as we wished to show.

Theorem 6.12. *Suppose $Q = LU$ is a parabolic subgroup of G, and B_L is a Harish-Chandra bimodule of finite length for L; say*

$$B_L \subset \mathrm{Hom}(V_{\mathfrak{l}}, W_{\mathfrak{l}}).$$

Define $W_{\mathfrak{g},ind}$ and $W_{\mathfrak{g},pro}$ as in (6.3) and Definition 6.7, and the canonical intertwining operator j between them as in Definition 6.11. If j is one-to-one, then the canonical bimodule intertwining operator

$$J : \mathrm{Ind}_{Dix}(Q^{op} \uparrow G)(B_L) \to \mathrm{Ind}_{bi}(Q \uparrow G)(B_L)$$

is an isomorphism.

Proof. Because j is one-to-one, the map

$$J : B_{\mathfrak{g},Dix} \to B_{\mathfrak{g},bi}$$

is obviously one-to-one. By Lemma 6.9, the restriction of J to each $\mathfrak{z}(\mathfrak{l})$-bimodule weight space $(B_{\mathfrak{g},Dix})_{\beta\gamma}$ is an isomorphism. The map J respects the decompositions (6.8)(b) and (e) in the obvious way. (This is easy, but it is not a formal consequence of linearity, since infinite direct products are involved. One has to recall how the decompositions on the level of endomorphisms arise from decompositions of modules. These latter decompositions are respected by the module intertwining operator j.) It follows at once that J is injective.

More precisely, J carries the $\mathrm{ad}(\mathfrak{z}(\mathfrak{l}))$-finite part of the domain isomorphically onto the corresponding part of the range. *A fortiori* the restriction of J to the $\mathrm{Ad}(G)$-finite part is an isomorphism. Q.E.D.

It is worth recording the slightly stronger statement that we actually proved: under the same hypotheses as in Theorem 6.12,

$$(B_{\mathfrak{g},Dix})_{\mathrm{ad}(\mathfrak{z}(\mathfrak{l}))\text{-finite}} \simeq (B_{\mathfrak{g},bi})_{\mathrm{ad}(\mathfrak{z}(\mathfrak{l}))\text{-finite}}. \tag{6.13}$$

One can find in section 8 of [19] various sufficient conditions for the map j of Theorem 6.12 to be an isomorphism. Here is one of them.

Proposition 6.14 ([19], Proposition 8.17). *Suppose $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a parabolic subalgebra of $\mathfrak{g}$, and $W_{\mathfrak{l}}$ is an $\mathfrak{l}$-module. Define $W_{\mathfrak{g},ind}$ and $W_{\mathfrak{g},pro}$ as in (6.3) and (6.6), and the canonical intertwining operator between them as in (6.11). Assume that*

i) The annihilator in $U([\mathfrak{l},\mathfrak{l}])$ of $W_{\mathfrak{l}}$ is a weakly unipotent primitive ideal.
ii) The center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ acts by a character λ on $W_{\mathfrak{l}}$.
iii) If α is a weight of $\mathfrak{z}(\mathfrak{l})$ in $\mathfrak{u}$, then

$$\mathrm{Re} < \alpha, \lambda > \geq 0.$$

Then j is injective.

We can combine this with Theorem 6.12 to get

Corollary 6.15. *Suppose $Q = LU$ is a parabolic subgroup of G, and B_L is a Harish-Chandra bimodule of finite length for L. Assume that*

i) The annihilator in $U([\mathfrak{l},\mathfrak{l}])$ of the left action on B_L is a weakly unipotent primitive ideal.
ii) The center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ acts on the left on B_L by a character λ.
iii) If α is a weight of $\mathfrak{z}(\mathfrak{l})$ in $\mathfrak{u}$, then

$$\mathrm{Re} < \alpha, \lambda > \geq 0.$$

Then the canonical bimodule intertwining operator

$$J : \mathrm{Ind}_{Dix}(Q^{op} \uparrow G)(B_L) \longrightarrow \mathrm{Ind}_{bi}(Q \uparrow G)(B_L)$$

is an isomorphism.

Corollary 6.16. *Suppose $Q = LU$ is a parabolic subgroup of G, and (A_L, ϕ_L) is a Dixmier algebra for L. Assume that*

i) The kernel of ϕ_L in $U([\mathfrak{l},\mathfrak{l}])$ is a weakly unipotent primitive ideal.
ii) The center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ acts on A_L by a character λ.
iii) If α is a weight of $\mathfrak{z}(\mathfrak{l})$ in $\mathfrak{u}$, then

$$\mathrm{Re} < \alpha, \lambda > \geq 0.$$

Then the induced Dixmier algebra (A_G, ϕ_G) is isomorphic as a bimodule to $\mathrm{Ind}_{bi}(Q \uparrow G)(A_L)$. In particular, the induced algebra is isomorphic to $\mathrm{Ind}_{alg}(L \uparrow G)(A_L)$ as an algebraic representation under Ad. *The higher cohomology groups of G/Q^{op} with coefficients in the complete symbol sheaf $\mathcal{A}_G$ (Definition 4.15) are zero.*

The point of the cohomology vanishing result is primarily that if it were *not* true, then our definition of induced Dixmier algebras would be flawed: the higher cohomology groups would have to be taken into account somehow. One expects that in interesting cases $\mathcal{A}_G$ has a filtration with associated graded sheaf of the form $\mathcal{R}_G(0)$ (cf. (5.5)). The present vanishing theorem would in such cases be a consequence of Conjecture 5.6; so we may regard it a as a kind of evidence for that conjecture.

Proof. Only the last assertion requires comment. Write Γ for Zuckerman's functor from $(\mathfrak{g}, L)$-modules to G-modules (passage to the G-finite part). Write Γ^i for its derived functors ([18], Chapter 6). If Z is any algebraic representation of Q^{op}, write

$$\mathcal{Z} = G \times_{Q^{op}} Z$$

for the corresponding sheaf on G/Q^{op}. We have already observed (in the proof of Corollary 4.16) that

$$H^0(G/Q^{op}, \mathcal{Z}) \simeq \Gamma(\mathrm{Hom}_{\mathfrak{q}^{op}}(U(\mathfrak{g}), Z)_{L-finite}).$$

It is easy to extend this fact to higher cohomology:

$$H^i(G/Q^{op}, \mathcal{Z}) \simeq \Gamma^i(\mathrm{Hom}_{\mathfrak{q}^{op}}(U(\mathfrak{g}), Z)_{L-finite}).$$

(One reason for this is that the same $\mathfrak{u}^{op}$-cohomology spaces can be used to compute either side.) To compute the higher cohomology with coefficients in $\mathcal{A}_G$, we must therefore apply Γ^i to

$$\mathrm{Hom}_{\mathfrak{q}^{op}}(U(\mathfrak{g}), U(\mathfrak{g}) \otimes_{\mathfrak{q}^{op}} A_{Q^{op}})_{L-finite}.$$

(We are considering only the ad action of $\mathfrak{g}$.) This is just the L-finite part of $A_{\mathfrak{g},Dix}$. By (6.13), it is isomorphic to the L-finite part of $A_{\mathfrak{g},bi}$. By Definition 6.1, this is

$$\mathrm{Hom}_{\mathfrak{q},\mathfrak{q}^{op}}(U(\mathfrak{g}) \otimes U(\mathfrak{g}), A_{\mathfrak{q},\mathfrak{q}^{op}})_{L-finite}.$$

Again we are interested only in the adjoint action of $\mathfrak{g}$. As a module for this action, $A_{\mathfrak{g},bi}$ is isomorphic to $\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{g}), A_L)$; here $\mathfrak{l}$ acts on $U(\mathfrak{g})$ on the left and on A_L by ad. (This elementary fact is the basis of (6.2).) Therefore

$$H^i(G/Q^{op}, \mathcal{A}_G) \simeq \Gamma^i(\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{g}), A_L))_{L-finite}. \tag{6.17}$$

The module to which Γ^i is applied is a standard injective $(\mathfrak{g}, L)$-module, so the right side is zero for positive i. Q.E.D.

Geometrically, the proof shows that the sheaf $\mathcal{A}_G$ is the pushforward of a sheaf on G/L. Since G/L is affine, the cohomology vanishing follows.

7. The translation principle for Dixmier algebras.

As mentioned in the introduction, there ought to be a theory of modules for induced Dixmier algebras entirely analogous to the Beilinson-Bernstein localization theory. I have not developed such a theory, but this section describes one of its basic results (Corollary 7.14). It is included mostly as evidence that the definition of induced Dixmier algebras is reasonable and interesting. The experts will find no surprises here; but such readers may wish to examine carefully the hypotheses in Theorem 7.9, which are substantially weaker than in some formulations of the translation principle.

We begin by recalling from section 3 of [22] the notion of translation functors for Dixmier algebras. Write $\mathfrak{Z}(\mathfrak{g})$ for the center of $U(\mathfrak{g})$. Suppose α is a character of $\mathfrak{Z}(\mathfrak{g})$ and I_α is the associated maximal ideal. If M is a $\mathfrak{Z}(\mathfrak{g})$-finite (left) $\mathfrak{g}$-module, write

$$_\alpha M = \{\, m \in M \mid \text{ for some } n,\ (I_\alpha)^n \cdot m = 0\,\}. \tag{7.1)(a}$$

This is an exact functor on the category of $\mathfrak{Z}(\mathfrak{g})$-finite $\mathfrak{g}$-modules, and

$$M = \sum_\alpha {_\alpha M}, \tag{7.1)(b}$$

the sum running over all characters of $\mathfrak{Z}(\mathfrak{g})$. We use analogous notation for right modules.

Suppose (A, ϕ) is a Dixmier algebra. Then A is a $\mathfrak{Z}(\mathfrak{g})$-finite bimodule, so there is a finite direct sum decomposition of A

$$A = \sum_{\alpha,\beta} {_\alpha A_\beta}. \tag{7.2)(a}$$

This decomposition is preserved by the $U(\mathfrak{g})$-bimodule structure and (as a consequence) by $\mathrm{Ad}(G)$. The multiplication satisfies

$$({_\alpha A_\beta})({_\gamma A_\delta}) \subset \begin{cases} {_\alpha A_\delta} & \text{if } \beta = \gamma. \\ 0 & \text{if } \beta \neq \gamma \end{cases} \tag{7.2)(b}$$

In particular, each ${}_\alpha A_\alpha$ is a subalgebra. More generally, if M is an A-module,

$$({}_\alpha A_\beta)({}_\gamma M) \subset \begin{cases} {}_\alpha M & \text{if } \beta = \gamma \\ 0 & \text{if } \beta \neq \gamma. \end{cases} \tag{7.2)(c}$$

As a formal consequence of these facts, we get

Lemma 7.3. *Suppose A is a Dixmier algebra and M is a simple A-module. With the notation (7.1) and (7.2), each non-zero ${}_\alpha M$ is an irreducible ${}_\alpha A_\alpha$-module.*

Conversely, suppose N is an irreducible ${}_\alpha A_\alpha$-module. Define

$$M' = A \otimes_{{}_\alpha A_\alpha} N$$

Then ${}_\alpha M' \simeq N$. Consequently M' has a unique maximal proper submodule S. The quotient $M = M'/S$ is a simple A-module, and ${}_\alpha M \simeq N$.

These constructions establish a bijection between the set of simple A-modules M with ${}_\alpha M$ non-zero, and the set of simple ${}_\alpha A_\alpha$-modules.

Next, suppose (π, F) is a finite-dimensional representation of $\mathfrak{g}$. If M is a $\mathfrak{Z}(\mathfrak{g})$-finite $\mathfrak{g}$-module, then so is $F \otimes M$ (cf. [13]). To make the corresponding construction for Dixmier algebras, fix a Dixmier algebra (A, ϕ_A) and form the algebra

$$B = \mathrm{End}(F) \otimes A. \tag{7.4)(a}$$

We can define an algebra homomorphism ϕ_B of $U(\mathfrak{g})$ into B by

$$\phi_B(X) = \pi(X) \otimes 1 + 1 \otimes \phi_A(X). \tag{7.4)(b}$$

To make a Dixmier algebra, we need an action Ad of G. This will exist whenever the adjoint action of $\mathfrak{g}$ on $\mathrm{End}(F)$ exponentiates to G (as it always does if F is irreducible, for example). In that case we can define

$$\mathrm{Ad}(g) \cdot (T \otimes a) = (\mathrm{Ad}(g) \cdot T) \otimes (\mathrm{Ad}(g) \cdot a). \tag{7.4)(c}$$

These definitions make B into a Dixmier algebra.

Lemma 7.5. *Suppose A is a Dixmier algebra and F is a finite-dimensional representation of $\mathfrak{g}$. Assume that*

$$\textit{the adjoint action of } \mathfrak{g} \textit{ on } \mathrm{End}(F) \textit{ exponentiates to } G. \tag{i}$$

Form the Dixmier algebra $B = \mathrm{End}(F) \otimes A$ as in (7.4). Then the map $M \to F \otimes M$ is an equivalence of categories from A-modules to B-modules. The inverse functor is $N \to \mathrm{Hom}_{\mathrm{End}(F)}(F, N)$.

This well-known result is an elementary exercise. (The Dixmier algebra structure is purely decorative; B is really just the ring of $n \times n$ matrices with entries in the ring A.)

Fix now a character α of $\mathfrak{Z}(\mathfrak{g})$ and a finite-dimensional representation F of $\mathfrak{g}$. The *elementary translation functor for modules* attached to α and F is the functor

$$TM = {}_\alpha(F \otimes M) \tag{7.6)(a}$$

on $\mathfrak{Z}(\mathfrak{g})$-finite $\mathfrak{g}$-modules. Assume in addition that F satisfies condition (i) of Lemma 7.5. The *elementary translation functor for Dixmier algebras* attached to α and F is

$$\mathcal{T}A = {}_\alpha(\mathrm{End}(F) \otimes A)_\alpha. \tag{7.6)(b}$$

Proposition 7.7 (Translation Principle for Dixmier algebras). *In the setting of (7.6), suppose A is a Dixmier algebra. Then the translation functor T takes modules for A to modules for $\mathcal{T}A$. This functor has the following additional properties.*

a) T is exact.
b) If M is an irreducible A-module, then TM is irreducible or zero as a $\mathcal{T}A$-module.
c) If N is an irreducible $\mathcal{T}A$-module, then there is a unique irreducible A-module M such that $N = TM$.

This proposition is an immediate consequence of Lemmas 7.3 and 7.5.

The expert reader may be wondering what has been hidden, since results of this form about the translation principle usually require more hypotheses and more proof. The difficulty arises if we want to speak about modules for $U(\mathfrak{g})$ instead of for some Dixmier algebra. If $A = U(\mathfrak{g})/I$, then an A-module is just a $U(\mathfrak{g})$-module annihilated by I. There will always be a natural inclusion

$$\phi : U(\mathfrak{g})/J \to \mathcal{T}A. \tag{7.8}$$

If ϕ is surjective, then a $\mathcal{T}A$-module is just a $U(\mathfrak{g})$-module annihilated by J, and Proposition 7.7 becomes a result about $\mathfrak{g}$-modules. It is exactly such surjectivity results that require additional hypotheses and more difficult proofs.

Since we have cast off the shackles of $\mathfrak{g}$-modules, we take a slightly different view of what is required to make Proposition 7.7 interesting. We will begin with a known Dixmier algebra A, and try to understand the translated algebra $\mathcal{T}A$. Here is such a result.

Theorem 7.9. *Suppose $Q = LU$ is a parabolic subgroup of G, and (A_u, ϕ_u) is a Dixmier algebra for $L/Z(L)_0$. For each character ξ of $\mathfrak{z}(\mathfrak{l})$ define a Dixmier algebra $(A_L(\xi), \phi_L(\xi))$ by (5.3)(d). Define*

$$(A_G(\xi), \phi_G(\xi)) = \mathrm{Ind}_{Dix}(Q \uparrow G)(A_L(\xi), \phi_L(\xi)).$$

Fix a one-dimensional character μ of $\mathfrak{l}$, and assume that μ occurs in the restriction to $\mathfrak{l}$ of a finite-dimensional representation of $\mathfrak{g}$. Let F be the unique irreducible finite-dimensional representation of $\mathfrak{g}$ containing μ as an extremal weight. (This means that F has a $\mathfrak{q}'$-invariant line of weight μ, for some parabolic subalgebra $\mathfrak{q}'$ having Levi factor $\mathfrak{l}$.) We sometimes identify μ with its restriction to $\mathfrak{z}(\mathfrak{l})$.

Fix a character λ of $\mathfrak{z}(\mathfrak{l})$. Assume that

i) $\ker(\phi_u)$ is a weakly unipotent primitive ideal in $U([\mathfrak{l}, \mathfrak{l}])$.
ii) If β is any weight of $\mathfrak{z}(\mathfrak{l})$ in $\mathfrak{g}/\mathfrak{l}$ and $\langle \beta, \mu \rangle > 0$, then

$$\mathrm{Re}\, \langle \beta, \lambda \rangle \geq 0.$$

In particular, (i) implies that $A_G(\lambda)$ has an infinitesimal character, which we denote by α. Let $\mathcal{T}$ be the elementary translation functor associated to α and F^*. Then

$$\mathcal{T}(A_G(\lambda + \mu)) = A_G(\lambda).$$

Proof. We argue as in [19], section 8. Fix a faithful module V_u for A_u. For every character ξ of $\mathfrak{z}(\mathfrak{l})$, V_u becomes a faithful module $V_{\mathfrak{l}}(\xi)$ for the Dixmier algebra $A_L(\xi)$. (The point is that the underlying algebra of $A_L(\xi)$ is just A_u; only the map $\phi_L(\xi)$ is changing.) Just as in (4.3) we can construct $V_{\mathfrak{q}}(\xi)$ and

$$V_{\mathfrak{g}}(\xi) = U(\mathfrak{g}) \otimes_{\mathfrak{q}} V_{\mathfrak{q}}(\xi). \tag{7.10}$$

By Lemma 7.5, $F^* \otimes V_{\mathfrak{g}}(\lambda + \mu)$ is a faithful module for $\mathrm{End}(F^*) \otimes A_G(\lambda + \mu)$. Consequently

$$T(V_{\mathfrak{g}}(\lambda + \mu)) = {}_{\alpha}[F^* \otimes V_{\mathfrak{g}}(\lambda + \mu)] \tag{7.11)(a}$$

is a faithful module for $\mathcal{T}(A_G(\lambda + \mu))$. We will show that

$$T(V_{\mathfrak{g}}(\lambda + \mu)) = V_{\mathfrak{g}}(\lambda). \tag{7.11)(b}$$

This will show that $\mathcal{T}(A_G(\lambda+\mu))$ and $A_G(\lambda)$ are both represented as endomorphism algebras of $V_{\mathfrak{g}}(\lambda)$. It is then very easy to check from the definitions that they are exactly the same endomorphisms; we leave this to the reader.

It therefore remains to check (7.11)(b). By (7.11)(a) and the algebraic version of Mackey's tensor product theorem, the right side of (7.11)(b) is

$$_{\alpha}\left[U(\mathfrak{g}) \otimes_{\mathfrak{q}} \left((F^* \mid_{\mathfrak{q}}) \otimes V_{\mathfrak{q}}(\lambda+\mu)\right)\right]. \tag{7.11)(c}$$

Now F^* has a $\mathfrak{q}$-stable filtration whose subquotients are irreducible representations of $\mathfrak{l}$. We get a filtration of (7.11)(c) whose subquotients are

$$_{\alpha}\left[U(\mathfrak{g}) \otimes_{\mathfrak{q}} \left(E^* \otimes V_{\mathfrak{q}}(\lambda+\mu)\right)\right], \tag{7.11)(d}$$

where E can be any irreducible constituent of $F\mid_{\mathfrak{l}}$.

If E is the μ weight space of F, then (7.11)(d) is $V_{\mathfrak{g}}(\lambda)$. We must therefore show that the other terms are all zero. Explicitly, this means the following. Suppose E is a representation of $\mathfrak{l}$ occurring in F, other than the weight μ. Then we must show that the infinitesimal character α does not occur in

$$U(\mathfrak{g}) \otimes_{\mathfrak{q}} \left(E^* \otimes V_{\mathfrak{q}}(\lambda+\mu)\right). \tag{7.12)(a}$$

To prove this, fix a Cartan subalgebra $\mathfrak{t}$ of $[\mathfrak{l},\mathfrak{l}]$. Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{z}(\mathfrak{l})$ is a Cartan subalgebra of $\mathfrak{g}$. Fix a weight $\alpha_0 \in \mathfrak{t}^*$ corresponding to the infinitesimal character of A_u in the Harish-Chandra correspondence. Then α corresponds to

$$(\alpha_0, \lambda) \in \mathfrak{t}^* + \mathfrak{z}(\mathfrak{l})^* = \mathfrak{h}^*. \tag{7.12)(b}$$

Let μ_1 be the weight of $\mathfrak{z}(\mathfrak{l})$ on E, and α_1 an infinitesimal character for $[\mathfrak{l},\mathfrak{l}]$ in $E^* \otimes V_u$. Then a typical infinitesimal character in (7.12)(a) is

$$(\alpha_1, \lambda+\mu-\mu_1). \tag{7.12)(c}$$

What we are trying to show is that this cannot be equal to α; that is, that the weights (7.12)(b) and (7.12)(c) are not conjugate under the Weyl group. To do that, it is obviously sufficient to show that

$$\operatorname{Re}\langle(\alpha_0,\lambda),(\alpha_0,\lambda)\rangle < \operatorname{Re}\langle(\alpha_1,\lambda+\mu-\mu_1),(\alpha_1,\lambda+\mu-\mu_1)\rangle. \tag{7.13)(a}$$

To prove (7.13)(a), we first use the hypothesis (i) of the Theorem. This implies that the infinitesimal characters occurring in $E^* \otimes V_u$ are all longer than α_0:

$$\langle\alpha_0,\alpha_0\rangle \leq \langle\alpha_1,\alpha_1\rangle. \tag{7.13)(b}$$

Next, any weight of $\mathfrak{z}(\mathfrak{l})$ in F must be of the form

$$\mu_1 = \mu - \sum n_\beta \beta.$$

Here the sum is over weights β of $\mathfrak{z}(\mathfrak{l})$ on $\mathfrak{g}/\mathfrak{l}$ having positive inner product with μ, and n_β is a non-negative integer. If μ_1 is different from μ, then the sum is non-empty. By hypothesis (ii),

$$\operatorname{Re}\langle\sum n_\beta\beta, \lambda\rangle \geq 0.$$

Consequently

$$\begin{aligned}\operatorname{Re}\langle\lambda+\mu-\mu_1,\lambda+\mu-\mu_1\rangle &= \operatorname{Re}\langle\lambda+\sum n_\beta\beta,\lambda+\sum n_\beta\beta\rangle \\ &= \operatorname{Re}\langle\lambda,\lambda\rangle + 2\cdot\operatorname{Re}\langle\sum n_\beta\beta,\lambda\rangle + \langle\sum n_\beta\beta,\sum n_\beta\beta\rangle \\ &> \operatorname{Re}\langle\lambda,\lambda\rangle.\end{aligned} \tag{7.13)(c}$$

Adding (7.13)(b) and (7.13)(c) gives (7.13)(a) and completes the proof. Q.E.D.

It is very easy to refine the argument so that the positivity hypothesis (ii) is needed only for those weights β that are restrictions of roots integral on the infinitesimal character.

Corollary 7.14. *Under the hypotheses of Theorem 7.9, every irreducible module N for $A_G(\lambda)$ is of the form TM, with M a unique irreducible module for $A_G(\lambda+\mu)$. Conversely, if M is an irreducible $A_G(\lambda+\mu)$-module, then TM is an irreducible $A_G(\lambda)$-module or zero.*

The point of this corollary is that it allows one to study problems of irreducibility at "very regular" parameters, where they are typically much easier.

Appendix. Induced bundles.

We assemble here some basic definitions used throughout the paper. Suppose throughout this appendix that G is an (affine) algebraic group, and H is a closed (and therefore affine algebraic) subgroup. Recall that the homogeneous space G/H is a quasiprojective algebraic variety. As a point set, G/H is just the coset space. Its topology is the quotient topology from G: a subset U of G/H is open if and only if its preimage V in G is open. A regular function on such an open set U is by definition a (right) H-invariant regular function on V. The main point in the construction of G/H is that for small enough V there are many such functions.

Suppose now that Z_H is any algebraic variety on which H acts. The *induced bundle* $G\times_H Z_H$ is a bundle over G/H whose fiber at the identity coset eH is Z_H:

$$G\times_H Z_H \to G/H \tag{A.1)(a}$$

This bundle is constructed from the product $G\times Z_H$ in the same way that G/H is constructed from G. That is, we define an equivalence relation $\sim$ on (closed points of) $G\times Z_H$ by

$$(x,z)\sim(xh^{-1},h\cdot z) \qquad (x\in G, z\in Z_H, h\in H). \tag{A.1)(b}$$

As a point set, $G\times_H Z_H$ is the set of equivalence classes for this relation. That is, it is the set of orbits of an action of H on $G\times Z_H$ (called the *right action*) defined by

$$h\cdot_R(x,z)=(xh^{-1},h\cdot z) \tag{A.1)(c}$$

The subscript R is included to distinguish this action from the (left) action of G defined by

$$g\cdot(x,z)=(gx,z) \qquad (x\in G, z\in Z_H, g\in G). \tag{A.1)(d}$$

This action of G commutes with the right action of H, so it is inherited by $G\times_H Z_H$.

A subset U of $G\times_H Z_H$ is defined to be open if and only if its preimage V in $G\times Z_H$ is open. The regular functions on U are defined to be the regular functions on V that are invariant under the right action of H:

$$\mathcal{O}(U)=\{\, f\in\mathcal{O}(V)\mid f(h\cdot_R v)=f(v)\quad (v\in V, h\in H)\,\} \tag{A.1)(e}$$

This makes sense because the preimage in $G\times Z_H$ of any subset of $G\times_H Z_H$ is by definition a union of orbits of the right action of H. That it makes $G\times_H Z_H$ into an algebraic variety with a G action is proved just as for the case of G/H itself.

Here are some properties of the induced bundle construction; all follow fairly easily from the definitions.

Proposition A.2. *Suppose G is an algebraic group, H is a closed subgroup, and Z_H is an algebraic variety on which H acts.*

a) The induced bundle $G\times_H Z_H$ is an algebraic fiber bundle over G/H. The fiber over eH is naturally identified with Z_H, and the isotropy action of H on the fiber is the original action of H on Z_H.

b) Suppose Y is an algebraic variety on which G acts, and $f: Z_H\to Y$ is a morphism respecting the actions of H. Then there is a natural morphism $F=G\times_H f$ from $G\times_H Z_H$ to Y, respecting the actions of G. On the level of equivalence classes,

$$F(x,z)=x\cdot f(z).$$

c) In the setting of (b), suppose in addition that f is proper, and that G/H is a projective variety. Then the map F is proper.

d) Suppose $Z_H \simeq H/K$ is a homogeneous space for H. Then

$$G \times_H Z_H \simeq G/K.$$

In the setting of (b) in the proposition, one can easily describe the fibers (over closed points) of the map F. Of course the image of F is $G \cdot f(Z_H)$, so the fibers outside that set are empty. By G-invariance, it suffices to understand the fiber over a point $y \in f(Z_H)$. Write $K \subset G$ for the isotropy group of the G-action at y. Then

$$F^{-1}(y) \simeq K \times_{H \cap K} f^{-1}(y). \tag{A.3}$$

In particular, F is injective if and only if the following two conditions are satisfied: f is injective; and the stabilizer of every point in $f(Z_H)$ is contained in H.

Suppose now that $\mathcal{M}_H$ is a quasicoherent sheaf on Z_H with an action of H (compatible with the action on Z_H). We can define the *induced sheaf* $\mathcal{M}_G = G \times_H \mathcal{M}_H$ on $Z_G = G \times_H Z_H$ as follows. Consider first the quasicoherent sheaf

$$\mathcal{N} = \mathcal{O}_G \otimes \mathcal{M}_H \tag{A.4)(a}$$

on $G \times Z_H$. (This can be thought of informally as "functions on G with values in $\mathcal{M}_H$;" that description is accurate on any open set that is a product of an open set in G with one in Z_H.) The sheaf $\mathcal{N}$ carries a right action of H compatible with the right action of H on $G \times Z_H$, and a commuting (left) action of G. With notation as in (A.1)(e), we define

$$\mathcal{M}_G(U) = \{\, m \in \mathcal{N}(V) \mid h \cdot_R m = m \quad (h \in H). \tag{A.4)(b}$$

The space M_G of global sections of $\mathcal{M}_G$ has a simple description. Write M_H for the space of global sections of $\mathcal{M}_H$. This is a vector space (generally infinite-dimensional) carrying an algebraic action of H. We may speak of the algebraic functions on G with values in M_H; such a function is required to take values in a finite-dimensional subspace, and to be algebraic (in the obvious sense) as a map to that subspace. That is, it should belong to $(\mathcal{O}_G(G)) \otimes M_H$. Now it is clear from the definitions that

$$M_G = \{\, f : G \to M_H \mid f \text{ is algebraic, and } f(xh) = h^{-1}f(x)\,\}. \tag{A.4)(c}$$

Proposition A.5. *Suppose G is an algebraic group, H is a closed subgroup, and Z_H is an algebraic variety with G-action. Write $Z_G = G \times_H Z_H$. Then the category of quasicoherent sheaves on Z_H with H-action is equivalent to the category of quasicoherent sheaves on Z_G with G-action, by the induction construction $\mathcal{M}_H \mapsto G \times_H \mathcal{M}_H$ of (A.4). This equivalence identifies the subcategories of coherent sheaves.*

We omit the proof. It is worthwhile to describe the inverse functor, however. Let $\mathcal{I}_H$ be the sheaf of ideals defining the subvariety Z_H of $Z_G = G \times_H Z_H$. Then $\mathcal{O}_{Z_H}$ may be identified as a sheaf of $\mathcal{O}_{Z_G}$-modules with $\mathcal{O}_{Z_G}/\mathcal{I}_H$. If $\mathcal{M}_G$ is any quasicoherent sheaf on Z_G, then the "geometric fiber"

$$\mathcal{M}_H = \mathcal{O}_{Z_H} \otimes_{\mathcal{O}_{Z_G}} \mathcal{M}_G \simeq \mathcal{M}_G/\mathcal{I}_H\mathcal{M}_G \tag{A.6}$$

is a quasicoherent sheaf on Z_H. If $\mathcal{M}_G$ carries an action of G, then this fiber inherits an action of H (the largest subgroup of G preserving the ideal $\mathcal{I}_H$). As the notation indicates, the geometric fiber provides a natural inverse for the induced sheaf construction of (A.4). Again we omit the proof.

We conclude by considering the relationship between the induced sheaf construction of (A.4) and the notion of induced representation for algebraic groups. Recall first of all that an *algebraic representation* of G is a pair (π, V) with V a vector space and

$$\pi : G \to GL(V) \tag{A.7)(a}$$

a homomorphism. We require in addition that π be algebraic, in the following sense. For every $v \in V$, there should be a finite-dimensional subspace $E \subset V$, containing v, with the property that

$$\pi(G)E \subset E, \tag{A.7)(b}$$

and the resulting homomorphism

$$\pi_E : G \to GL(E) \tag{A.7)(c}$$

is a morphism of algebraic groups.

Suppose now that (π_H, V_H) is an algebraic representation of H. The *algebraically induced representation of G* is the algebraic representation

$$\mathrm{Ind}_{alg}(H \uparrow G)(\pi_H, V_H) = (\pi_G, V_G) \tag{A.8)(a}$$

defined by

$$V_G = \{\, f : G \to V_H \mid f \text{ is algebraic, and } f(xh) = \pi_H(h^{-1})f(x)\,(x \in G, h \in H)\,\} \tag{A.8)(b}$$

$$(\pi_G(g)f)(x) = f(g^{-1}x). \tag{A.8)(c}$$

(We will drop the maps π from the notation when no confusion can result.) The analogy with (A.4) is clear, and in fact the connection is very close.

Proposition A.9. *Suppose G is an algebraic group, H is a closed subgroup, and (π_H, V_H) is an algebraic representation of H.*

a) The induced representation $V_G = \mathrm{Ind}_{alg}(H \uparrow G)(V_H)$ is an algebraic representation of G.

b) Identify V_H with a quasicoherent sheaf $\mathcal{M}_H$ with an H-action on a point. Then V_G may be identified with the space of global sections of $G \times_H \mathcal{M}_H$ (cf. (A.4)).

c) Suppose that V_H is the space of global sections of a quasicoherent sheaf $\mathcal{M}_H$ on some Z_H as in (A.4). Then V_G may be identified with the space of global sections of the induced sheaf $G \times_H \mathcal{M}_H$ on $G \times_H Z_H$.

d) (Frobenius reciprocity.) Suppose W is any algebraic representation of G. Then there is a natural isomorphism

$$\mathrm{Hom}_G(W, V_G) \simeq \mathrm{Hom}_H(W\,|_H, V_H).$$

e) Suppose $\dim V_H < \infty$, so that $G \times_H V_H$ (defined as in (A.1)) is a vector bundle over G/H. Then V_G may be identified with the space of sections of this vector bundle.

This is very well-known, and we omit the straightforward proof.

REFERENCES

[1] A. Beilinson and J. Bernstein, "Localisation de $\mathfrak{g}$-modules," C. R. Acad. Sci. Paris **292** (1981), 15–18.

[2] W. Borho, "Definition einer Dixmier-Abbildung für $\mathfrak{sl}(n, \mathbb{C})$," Inventiones math. **40**(1977), 143–169.

[3] W. Borho, "Über Schichten halbeinfacher Lie-Algebren," Invent. math. **65** (1981), 283–317.

[4] W. Borho and H. Kraft, "Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen," Comment. Math. Helvetici **54** (1979), 61–104.

[5] N. Conze-Berline and M. Duflo, "Sur les représentations induites des groupes semi-simples complexes," Compositio math. **34** (1977), 307–336.

[6] J. Dixmier, *Algèbres Enveloppantes.* Gauthier-Villars, Paris-Brussels-Montreal 1974.

[7] J. Dixmier, "Ideaux primitifs dans les algèbres enveloppantes," J. Algebra **48** (1978) 96–112.

[8] M. Duflo, "Sur les idéaux induits dans les algèbres enveloppantes," Inventiones math. **67** (1982), 385–393.

[9] R. Hartshorne, *Algebraic Geometry.* Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[10] J. C. Jantzen, *Moduln mit einem Höchsten Gewicht*, Lecture Notes in Mathematics **750**. Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[11] A. Joseph, "Dixmier's problem for Verma and principal series modules," J. London Math. Soc.(2) **20** (1979), 193–204.

[12] A. Joseph, "The primitive spectrum of an enveloping algebra," 13–53 in *Orbites Unipotentes et Représentations III. Orbites et Faisceaux pervers*, Astérisque **173–174** (1989).

[13] B. Kostant, "On the tensor product of a finite and an infinite dimensional representation," J. Func. Anal. **20** (1975), 257–285.

[14] G. Lusztig and N. Spaltenstein, "Induced unipotent classes," J. London Math. Soc.(2) **19** (1979), 41–52.

[15] W. McGovern, "Unipotent representations and Dixmier algebras," Compositio math. **69** (1989), 241–276.

[16] C. Moeglin, "Modèles de Whittaker et idéaux primitifs complètement premiers dans les algèbres de lie semi-simples complexes II," Math. Scand. **63** (1988), 5–35.

[17] R. Rentschler, "Comportement de l'application de Dixmier par rapport à l'anti-automorphism principal pour les algèbres de Lie résolubles," C. R. Acad. Sci. Paris **282** (1976), 555–557.

[18] D. Vogan, *Representations of Real Reductive Lie Groups*. Birkhauser, Boston-Basel-Stuttgart, 1981.

[19] D. Vogan, "Unitarizability of certain series of representations," Ann. of Math. **120** (1984), 141–187.

[20] D. Vogan, "The orbit method and primitive ideals for semisimple Lie algebras," in *Lie Algebras and Related Topics*, CMS Conference Proceedings, volume 5, D. Britten, F. Lemire, and R. Moody, eds. American Mathematical Society for CMS, Providence, Rhode Island, 1986.

[21] D. Vogan, "Noncommutative algebras and unitary representations," in *The Mathematical Heritage of Hermann Weyl*, R. O. Wells, Jr., ed. Proceedings of Symposia in Pure Mathematics, volume 48. American Mathematical Society, Providence, Rhode Island, 1988.

[22] D. Vogan, "Irreducibility of discrete series representations for semisimple symmetric spaces," 191–221 in *Representations of Lie groups, Kyoto, Hiroshima, 1986*, K. Okamoto and T. Oshima, editors. Advanced Studies in Pure Mathematics, volume 14. Kinokuniya Company, Ltd., Tokyo, 1988.

[23] D. Vogan, "Associated varieties and unipotent representations," preprint.

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