Structure of Harish-Chandra cells

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September 13, 2021

Outline

Introduction

Kazhdan-Lusztig cells

Harish-Chandra cells

What questions can you ask?

More questions to ask.

Slides at http://www-math.mit.edu/~dav/paper.html

What's a Harish-Chandra cell (A)?

$G(\mathbb{R})$ real reductive $\supset K(\mathbb{R}) = G(\mathbb{R})^\theta$ maxl compact

$G \supset K = G^\theta$ complexifications, $\mathfrak{g} = \mathrm{Lie}(G)$

Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$, $W = W(\mathfrak{g}, \mathfrak{h})$

$\lambda \in \mathfrak{h}^*$ dominant regular integral,

$\mathcal{M}(\mathfrak{g}, K)_\lambda = (\mathfrak{g}, K)$-modules of infinitesimal char λ

$\mathrm{Irr}(\mathfrak{g}, K)_\lambda =$ irr reps, $K\mathcal{M}(\mathfrak{g}, K)_\lambda = \mathbb{Z} \cdot \mathrm{Irr}(\mathfrak{g}, K)_\lambda$ Groth grp.

Weyl group W acts on $K\mathcal{M}(\mathfrak{g}, K)_\lambda$.

Get natural W representation, basis $\mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

W representation $\rightsquigarrow$ Harish-Chandra cells.

Analogous to left reg rep of $W \rightsquigarrow$ left cells (KL).

What's a Harish-Chandra cell (B)?

Preorder $\underset{LR}{\leq}$ on $\mathrm{Irr}(\mathfrak{g}, K)_\lambda$: Kazhdan-Lusztig style definition is

$$Y \underset{LR}{\leq} X \iff \exists w \in W, \ [Y] \text{ appears in } w \cdot X.$$

Representation def is (with F fin-diml rep of G^{ad})

$$Y \underset{LR}{\leq} X \iff \exists F, Y \text{ composition factor of } F \otimes X.$$

Equiv rel $Y \underset{LR}{\sim} X$ means $Y \underset{LR}{\leq} X \underset{LR}{\leq} Y$; complement is $Y \underset{LR}{<} X$.

A Harish-Chandra cell is an $\underset{LR}{\sim}$ equiv class in $\mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

Two irreducibles are in the same cell iff you can get from each to the other by tensoring with finite-dimensional representations.

$C(X) =_{\underset{LR}{\sim}}$ equiv class of X = HC cell $\subset \mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

$\overline{C}(X) =_{\underset{LR}{<}}$ interval below X = HC cone $\subset \mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

$\partial C(X) = \overline{C}(X) - C(X)$.

What's true about Harish-Chandra cells?

An interesting and difficult-to-compute invariant of a representation X is its associated variety $\mathcal{AV}(X)$.

Theorem. (Consequence of rep theory defn of cells.)

1. $Y \underset{LR}{\leq} X \implies \mathcal{AV}(Y) \subset \mathcal{AV}(X)$.
2. $Y \underset{LR}{<} X \implies \mathcal{AV}(Y) \subsetneq \mathcal{AV}(X)$.
3. $Y \underset{LR}{\sim} X \implies \mathcal{AV}(Y) = \mathcal{AV}(X)$.

Theorem. (Consequence of KL defn of cells.)

1. W acts on $\overline{C}_{\mathbb{Z}}(X) = \Big[\sum_{Y \underset{LR}{\leq} X} \mathbb{Z}Y\Big] \supset \partial C_{\mathbb{Z}}(X)$.
2. W acts on $C_{\mathbb{Z}}(X) \simeq \overline{C}_{\mathbb{Z}}(X)/\partial C_{\mathbb{Z}}(X)$.
3. $C_{\mathbb{Z}}(X)$ contains unique special rep $\sigma(X) \in \widehat{W}$.
4. $\mathcal{AV}(X) =$ union of closures of K-forms of $\mathcal{O}(\sigma(X))$.

Nilpotent orbit $\mathcal{O}(\sigma(X))$ is defined by Springer correspondence.

KL cells

Theorem (Kazhdan-Lusztig)

1. KL relations $\underset{L}{\sim}$ and $\underset{LR}{\sim}$ partition W into left cells and two-sided cells $C_L(w) \subset C_{LR}(w)$ $(w \in W)$.
2. $\mathbb{Z}$-module $C_{\mathbb{Z},L}(w)$ carries a representation of W.
3. $C_{\mathbb{Z},LR}(w)$ carries a representation of $W \times W$.
4. $\sum_{C_{LR}} C_{\mathbb{Z},LR} \simeq \mathbb{Z}W$, regular representation of $W \times W$.
5. Two-sided cells C_{LR} partition $\widehat{W}$ into subsets $\Sigma(C_{LR})$ called families: $C_{\mathbb{Z},LR} \simeq \sum_{\sigma \in \Sigma(C_{LR})} \sigma \otimes \sigma^*$.
6. As a representation of the first W, $C_{\mathbb{Z},LR} \simeq \sum_{C_L \subset C_{LR}} C_{\mathbb{Z},L}$.

Lusztig's description of families

For any finite group F, Lusztig in 1979 defined

$$\mathcal{M}(F) = \{\, (x,\xi) \mid x \in F,\ \xi \in \widehat{F^x}\,\}/(\text{conjugation by } F)$$

The group F acts itself by conjugation;

$$\mathcal{M}(F) \simeq \text{simple } F\text{-equivariant coherent sheaves on } F.$$

Theorem (Lusztig) Suppose that Σ is a family in $\widehat{W}$.

1. Σ has exactly one special representation $\sigma_s(\Sigma) \in \widehat{W}$.
2. $\sigma_s \underset{\text{Springer}}{\longleftrightarrow}$ special nilpotent orbit $\mathcal{O}_s(\Sigma) = \mathcal{O}_s(\sigma_s) \subset \mathcal{N}^*/G$.
3. Put $A(\mathcal{O}_s) = \pi_1^G(\mathcal{O}_s)$ (equivariant fundamental group). Put

   $$\{\sigma_s = \sigma_1, \sigma_2 \ldots, \sigma_r\} = \Sigma \cap (\text{Springer}(\mathcal{O}_s));$$

   this is all W-reps in $\Sigma \underset{\text{Springer}}{\longleftrightarrow} \xi_j \in \widehat{A(\mathcal{O}_s)}$. Define

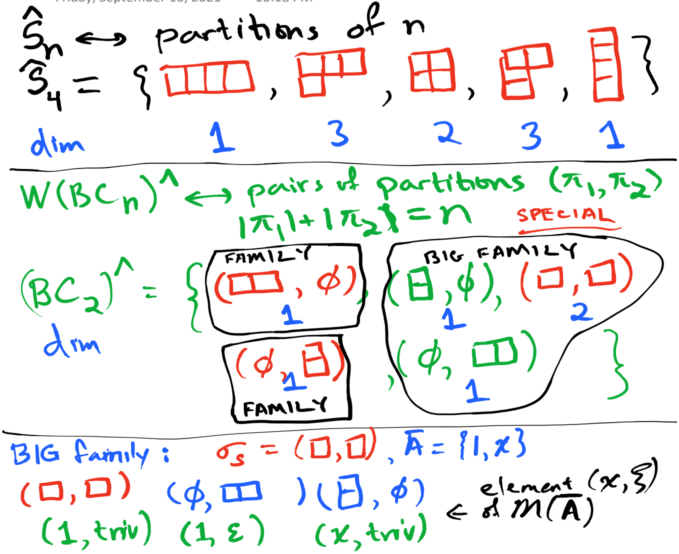
   $$\overline{A} = \overline{A}(\mathcal{O}_s) = A(\mathcal{O}_s)/[\cap_j \ker \xi_j]$$

4. Have inclusion $\Sigma \hookrightarrow \mathcal{M}(\overline{A})$, $\quad \sigma \mapsto (x(\sigma), \xi(\sigma))$ so

   $$x(\sigma_s) = x(\sigma_j) = 1 \in \overline{A}, \quad \xi(\sigma_j) = \xi_j \in \widehat{\overline{A}}.$$

Classical W reps: special in red

Friday, September 10, 2021 10:18 PM



Lusztig's description of left cells

Recall that finite group F gives

$$\mathcal{M}(F) = \{(x,\xi) \mid x \in F,\ \xi \in \widehat{F^x}\}/(\text{conj by } F)$$
$$\simeq \text{irr conj-eqvt coherent sheaves } \mathcal{E}(x,\xi) \text{ on } F.$$

Given subgroup $S \subset F$, const sheaf $\mathcal{S}$ on S is S-eqvt for conj.
Push forward to F-eqvt sheaf supp on F-conjugates of S:

$$i_*(\mathcal{S}) = \sum_{s,\xi} m_S(s,\xi)\mathcal{E}(s,\xi), \qquad m_S(s,\xi) = \dim \xi^{S^s}.$$

Sum runs over S conjugacy classes $s \in S$. Can write this as

$$i_*(\mathcal{S}) = \sum_s \mathcal{E}(s, \mathrm{Ind}_{S^s}^{F^s}(\text{triv})).$$

Theorem (Lusztig) $C_L \subset C_{LR} \leftrightarrow \Sigma \subset \widehat{W}$, $\overline{A}$ finite group,

$$\Sigma \hookrightarrow \mathcal{M}(\overline{A}), \quad \sigma \mapsto (x(\sigma),\xi(\sigma)).$$

1. $\exists$ subgp $\Gamma = \Gamma(C_L) \subset \overline{A}$ so $C_{\mathbb{Z},L} \simeq \sum_{x,\xi} m_\Gamma(x,\xi)\sigma(x,\xi)$
2. $m_\Gamma(1,\text{triv}) = 1$, so special rep σ_s appears once in $C_{\mathbb{Z},L}$.
3. $\exists$ Lusztig left cells with $\Gamma = \overline{A}$, so $C_{\mathbb{Z},L} \simeq \sum_x \sigma(x,\text{triv})$.
4. G classical $\implies$ $\exists$ Springer left cells with $\Gamma = \{e\}$, so $C_{\mathbb{Z},L} \simeq \sum_{\xi\in\widehat{\overline{A}}} \dim(\xi)\sigma(1,\xi)$, Springer reps for $\mathcal{O}_s$ in Σ.

Classical left cells: special in red

Saturday, September 11, 2021 11:11 AM

$\bar A = (\mathbb{Z}/2\mathbb{Z})^m$, $\mathcal{M}(\bar A) = \bar A \times \hat{\bar A}$

C2 family: $\sigma_s = (\square, \square)$, $\bar A = \{1, x\}$

$(\square, \square)$ $(\phi, \square\square)$ $(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \phi)$ element $(x, \xi) \in$ of $\mathcal{M}(\bar A)$

$(1, \mathrm{triv})$ $(1, \varepsilon)$ (x, triv)

Left cells $\leftrightarrow$ $S \subset \bar A$

$$C_S = \{\sigma(x, \xi) \mid x \in S, \ \xi|_S = 1\}$$

$$C_1 = \{\sigma(1, \xi) \mid \xi \in \hat{\bar A}\}$$ SPRINGER

$$C_{\bar A} = \{\sigma(x, \mathrm{triv}) \mid x \in \bar A\}$$ LUSZTIG

C2 cells:

$$C_1 = \{(\square, \square), (\phi, \square\square)\}$$ Springer

$$C_{\bar A} = \{(\square, \square), (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \phi)\}$$ Lusztig

Consequences of Lusztig for HC cells

HC world $(\mathfrak{g}, K)$: $\operatorname{Irr}(\mathfrak{g}, K)_\lambda \supset C =$ HC cell $\rightsquigarrow W$ rep $C_{\mathbb{Z}}$.

$C_{\mathbb{Z}} \supset \sigma_s(C)$ special in $\widehat{W} \rightsquigarrow \mathcal{O}(C)$, $\Sigma(\mathcal{O}) \subset \widehat{W}$, $\overline{A}(\mathcal{O})$ finite.

Theorem. Suppose C is a HC cell in $\operatorname{Irr}(\mathfrak{g}, K)_\lambda$.

$$C_{\mathbb{Z}} = \sum_{\sigma \in \Sigma} m_C(\sigma)\sigma, \quad m_C(\sigma) \in \mathbb{N}, \quad m_C(\sigma_s) = 1.$$

That is, a HC cell is a sum of certain W reps in a single family, including the special W rep with multiplicity one.

Problem: understand the set of W reps on cells.

How many cells are there for each special rep σ of W?

Which other reps from the family of σ appear in each cell?

Conjecture. For each HC cell C as above, $\exists$ $S(C) \subset \overline{A}$ so

$$m_C(\sigma(x, \xi)) = m_{S(C)}(x, \xi).$$

Evidence for conjecture

Theorem (McGovern, Binegar) C a HC cell in $\mathrm{Irr}(\mathfrak{g}, K)_\lambda$.

1. $C_{\mathbb{Z}} = \sum_{\sigma \in \Sigma} m_C(\sigma)\sigma, \quad m_C(\sigma) \in \mathbb{N}, \quad m_C(\sigma_s) = 1.$
2. IF $G(\mathbb{R})$ real form of type A, $SO(n)$, $Sp(2n)$, or an exceptional group, THEN $\exists S(C) \subset \overline{A}$ so

$$m_C(\sigma(x,\xi)) = m_{S(C)}(x,\xi).$$

3. The subgroup $S(C) \subset \overline{A}$ in (2) is always one of Lusztig's Γ attached to a left cell; so
4. in (2), each HC cell is isomorphic to a KL left cell.
5. IF $G(\mathbb{R})$ cplx, so $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_1$, $\overline{A}(\mathcal{O}) = \overline{A_1} \times \overline{A_1}$, THEN $S(C) = (\overline{A_1})_\Delta$; not one of Lusztig's Γ unless $\overline{A}_1 = 1$.

Finding the conjectural $S(C) \subset \overline{A}$

Suppose we have a HC cell C of irr HC modules:

$$C_{\mathbb{Z}} = \sum_{\sigma \in \Sigma} m_C(\sigma)\sigma.$$

"We" (that is, the `atlas` software!) can calculate nonnegative integers $m_C(\sigma(x,\xi))$.

How do we recover $S(C)$ from these integers?

Recall pairs $(x,\xi) \in \mathcal{M}(\overline{A})$ defined up to $\overline{A}$ conjugacy;

$x \in \overline{A}$, and $\xi \in \widehat{\overline{A}^x}$ irrep of its centralizer.

The W-reps $\sigma(x, \mathrm{triv})$ actually exist.

$m_S(x, \mathrm{triv}) = \#S$-conjugacy classes in (class of x) $\cap$ S: class function on $\overline{A}$ giving nice statistic about S.

Gives question: given a finite group A, class function m on A, is there at most one subgroup S having this statistic?

Answer to such a question for general A is always NO.

But Lusztig's $\overline{A}$ is a product of copies of S_m for $m \le 5$...

Homework problem

Suppose A is a product of copies of S_m for $m \leq 5$.
If $S \subset A$ is a subgroup, define a class function on A:

$$m_S(x) = \#\{\, S\text{-conj classes in } (A - \text{conj class of } x) \cap\ S \,\}.$$

Find an algorithm to recover the group S (up to conjugation in A) from the function m_S.

Homework of Roger Zierau[1]

Let $\mathcal{O}$ =(special) nilp in $\mathfrak{sp}(4)^3$ subreg in each factor.

Lusztig's quotient is $\overline{A}(\mathcal{O}) = (\mathbb{Z}/2\mathbb{Z})^3$.

Lusztig's family $\Sigma \subset \widehat{W}$ consists of 27 irreps: those which on each $W(C_2)$ factor are neither trivial nor sign.

There are eight KL left cells, each a different subset of 8 reps from Σ. Correspond (Lusztig thm) to the eight Lusztig subgroups $\Gamma \subset \overline{A}$ that are products of three factors.

There are a total of 16 subgroups S of $\overline{A}$, each defining a different subset $C_S \subset \Sigma$.

Each C_S consists of 5, 6, or 8 reps in Σ.

Theorem (Zierau). For G loc isom to $Sp(2, R)^3$, each HC cell rep is of the form C_S for some $S \subset \overline{A}$; and every subgroup S appears.

[1] Teacher's pet

What questions come next?

Given real reductive G, special orbit $\mathcal{O}$, and corr special rep $\sigma \in \widehat{W}$, would be great to understand

1. how many HC cells for G include σ?
2. Is each cell defined as W rep by some $S \subset \overline{A(\mathcal{O})}$?
3. Which subgroups S appear, and how often?

(1) is a complication of classical question how many $\mathbb{R}$-forms does $\mathcal{O}$ have?

(2) ought to be addressable using ideas from Lusztig's classification of left cells.

(3) is a complication of the classical question how much of $\pi_1(\mathcal{O})$ is defined over $\mathbb{R}$?

Since this is the last slide, you can guess that I have no idea how to answer these questions.

Thanks for the invitation; math with people is the best kind!