Enumeration and unitarity of Arthur's representations for exceptional real groups

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Outline

Slides at `http://www-math.mit.edu/~dav/paper.html`

Point of view

Point of view of special session begins with reductive algebraic $\mathbf{G}$ over global field k, seeks to understand automorphic forms: functions on $\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}(k))$.

Analytic view: **understand** $\leftrightsquigarrow$ find Plancherel decomp of $L^2(\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}))$.

My point of view: begin with reduc alg $\mathbf{G}$ over local F; seek to understand reps of $G = \mathbf{G}(F)$.

Analytic view: **understand** $\leftrightsquigarrow$ find unitary reps of G.

First success: HC Plancherel decomp of $L^2(G)$.

Two viewpoints inform each other, but are distinct.

What am I actually doing today?

Stated goal: explain how to list all the representations in Arthur's conjectures for real exceptional G, and explain proof they are unitary.

Actual goals are

1. explain a way to think about Arthur's conjectures

   (really a way think about Langlands' conjectures)
2. explain how to list reps using that point of view, and
3. convince you that the `atlas` software is the most powerful and wonderful tool imaginable for studying reductive groups.

For (3), you can get your free copy of the software at `http://www.liegroups.org/`

Prehistory of Arthur's conjectures

G reductive group over a local field.

$\widehat{G}$	$\supset$	$\widehat{G}_u$	$\supset$	$\widehat{G}_t$	$\supset$	$\widehat{G}_{ds}$
admissible irr reps		unitary irr reps		tempered irr reps		discrete series

Langlands conjecture 1970: parametrization of $\widehat{G}$.

In light of Harish-Chandra's work, Langlands' conjecture mostly reduces to $\widehat{G}_{ds}$.

Using Harish-Chandra parametrization of $\widehat{G}_{ds}$ for groups over $\mathbb{R}$, Langlands proved his conjecture in those cases.

Langlands' conjecture clearly identifies $\widehat{G}_t$.

But it offers no hint about identifying the rest of $\widehat{G}_u$.

Arthur's conjectures

$\Phi(G)$	$\supset$	$\Phi_t(G)$
Langlands params		tempered params

Parameter $\phi \rightsquigarrow \Pi_L(\phi) \subset \widehat{G}$ finite L-packet of ϕ.

Still conjectural for F p-adic.

DIFFICULTY: doesn't find nontempered unitary reps.

Arthur in 1983 introduced Arthur parameters $\Psi_A(G)$:

$\Phi(G)$	$\supset$	$\Psi_A(G)$	$\supset$	$\Phi_t(G)$
Langlands params		Arthur params		tempered params

Conjectured $\psi \rightsquigarrow \Pi_A(\psi) \supset \Pi_L(\psi)$ finite A-packet of ψ.

Conjectured $\Pi_A(\psi)$ consists of unitary reps.

Looked like a great way to address DIFFICULTY.

Still only hope in Mudville

Arthur: should be many sets $\Pi_A(\psi)$ of unitary reps.

Difficulty: no definition of $\Pi_A(\psi)$.

Barbasch-V 1985: defined $\Pi_A(\psi)$ for groups over $\mathbb{C}$;

calculated set $\Pi_A(\psi)$ fairly explicitly;

calculated characters in $\Pi_A(\psi)$, $\rightsquigarrow$ Arthur *desiderata*.

Paper $\rightsquigarrow$ hints about defining $\Pi_A(\psi)$ for groups over $\mathbb{R}$, realized in Adams-Barbasch-V book 1992.

But we failed to prove $\Pi_A(\psi)$ consists of unitary reps.

It's only my point of view, not my heart's desire.

Forty years of shattered dreams and dashed hopes.

But I'm fine now, and not bitter.

L-group

Chevalley-Grothendieck: reductive alg G over alg closed $\overline{k}$ $\leftrightsquigarrow$ based root datum $\mathcal{R}(G) = (X^*, \Pi, X_*, \Pi^\vee)$.

X^* and X_* are dual lattices, with finite subsets Π and $\Pi^\vee$

This is a description made for computers!

Defining G / any k $\rightsquigarrow$ action of $\Gamma = \mathrm{Gal}(\overline{k}/k)$ on $\mathcal{R}(G)$.

Axioms for root data symmetric in $(X^*, \Pi) \leftrightarrow (X_*, \Pi^\vee)$.

Dual root datum is $\mathcal{R}^\vee = (X_*, \Pi^\vee, X^*, \Pi)$.

Gives reductive algebraic dual group ${}^\vee G$ and L-group ${}^L G = {}^\vee G \rtimes \Gamma$ over $\mathbb{Z}$.

Langlands' insight: representation theory/K of $G(k)$ $\leftrightsquigarrow$ group theory of ${}^L G(K)$.

Typically $K = \mathbb{C}$ and k is local.

Complex reps of $G(k)$ $\leftrightsquigarrow$ complex alg geom of ${}^L G(\mathbb{C})$

Langlands' insight over $\mathbb{R}$

complex reps of $G(\mathbb{R})$ $\leftrightsquigarrow$ group theory of ${}^\vee G(\mathbb{C}) \rtimes \{1, \sigma\}$.

How could this work?

First invariant of rep π is infl char $\lambda(\pi) \in \mathfrak{h}^* = X^* \otimes_{\mathbb{Z}} \mathbb{C}$.

Corresponds on ${}^\vee G$ to $\lambda \in {}^\vee\mathfrak{h}$: semisimple element in ${}^\vee\mathfrak{g}$.

Second invariant of π: put λ in real Cartan, get action of complex conjugation.

Corresponds in ${}^L G$ to $y \in {}^\vee G\sigma$ acting on λ.

A Langlands parameter is $(y, \lambda) \in {}^\vee G\sigma \times {}^\vee\mathfrak{g}$ with

$$\lambda \quad \text{semisimple}, \quad y^2 = \exp(2\pi i\lambda), \quad [\lambda, \mathrm{Ad}(y)(\lambda)] = 0.$$

Theorem (Langlands) Each pair (y, λ) as above $\rightsquigarrow$ finite set $\Pi(y, \lambda)$ of irr reps of $G(\mathbb{R})$, depending only on the ${}^\vee G$ conjugacy class of (y, λ). Sets $\Pi(y, \lambda)$ partition $\widehat{G(\mathbb{R})}$.

Reformulating Langlands over $\mathbb{R}$, part 1

$$\lambda \quad \text{semisimple}, \quad y^2 = \exp(2\pi i\lambda), \quad [\lambda, \mathrm{Ad}(y)(\lambda)] = 0.$$

For λ "generic" ($\mathrm{ad}(\lambda)$ has no pos integer eigvals) then $\Pi(y,\lambda)$ is one irr princ series rep of quasisplit G.

Properties of $\Pi(y,\lambda)$ depend on integer eigenspaces.

For $n \in \mathbb{Z}$, define

$$^\vee\mathfrak{g}(\lambda)_n = \{X \in {}^\vee\mathfrak{g} \mid [\lambda, X] = n\}, \quad {}^\vee\mathfrak{e} = \sum_{n\in\mathbb{Z}} {}^\vee\mathfrak{g}(\lambda)_n.$$

Notice that

$$^\vee E = \text{cent in } {}^\vee G \text{ of } y^2 = \exp(2\pi i\lambda) =_{\text{def}} \epsilon.$$

is a (reductive algebraic) pseudolevi subgroup of $^\vee G$. In it

$$^\vee\mathfrak{q} = \sum_{n\geq 0} {}^\vee\mathfrak{g}(\lambda)_n = {}^\vee\mathfrak{l} + {}^\vee\mathfrak{u}$$

is a parabolic subalgebra of $^\vee\mathfrak{e}$, with Levi subgroup

$$^\vee L = \text{cent in } {}^\vee E \text{ of } \lambda = \text{cent in } {}^\vee G \text{ of } \lambda.$$

Reformulating Langlands over $\mathbb{R}$, part 2

λ semisimple, $y^2 = \exp(2\pi i\lambda) = \epsilon$, $[\lambda, \mathrm{Ad}(y)(\lambda)] = 0.$

${}^\vee E$ = cent in ${}^\vee G$ of ϵ,

$$ {}^\vee\mathfrak{q} = \sum_{n\geq 0} {}^\vee\mathfrak{g}(\lambda)_n = {}^\vee\mathfrak{l} + {}^\vee\mathfrak{u}, \text{ Levi } {}^\vee L = {}^\vee G^\lambda $$

Canonical flat of λ is the affine space

$$ \Lambda = \lambda + \sum_{n>0} {}^\vee\mathfrak{g}(\lambda)_n = \mathrm{Ad}({}^\vee Q)(\lambda) \subset \mathrm{Ad}({}^\vee G(\lambda)) \cdot \lambda, $$

Λ is lagrangian in the symplectic manifold $\mathrm{Ad}({}^\vee E) \cdot \lambda$.

$\mathrm{Stab}_{{}^\vee G}(\Lambda) = {}^\vee Q$ (each $\lambda' \in \Lambda$ has ${}^\vee Q(\lambda') = {}^\vee Q(\lambda)$).

$e(Z) =_{\mathrm{def}} \exp(2\pi i Z)$ is const on Λ: $e(\Lambda) = e(\lambda) = y^2 = \epsilon$.

An ABV Langlands parameter is (y, Λ) with

$$ \Lambda \subset {}^\vee\mathfrak{g} \text{ canonical flat}, \quad y \in {}^\vee G\sigma, \quad y^2 = e(\Lambda) $$

Last ingredient is ${}^\vee K = {}^\vee G^y$, symm subgp of reductive ${}^\vee E$.

Easy: classical Langlands params are in bijection with ${}^\vee K$ orbits on partial flag variety ${}^\vee E/{}^\vee Q$.

What have we gained?

An ABV Langlands parameter is (y, Λ) with

$$\Lambda \subset {}^\vee\mathfrak{g} \text{ canonical flat}, \quad y \in {}^\vee G\sigma, \quad y^2 = e(\Lambda) = \epsilon$$

$\rightsquigarrow$ pseudolevi ${}^\vee E = {}^\vee G^\epsilon \supset$ symmetric ${}^\vee K = {}^\vee G^\epsilon$, parabolic ${}^\vee Q = {}^\vee G^\Lambda$.

$\rightsquigarrow$ ${}^\vee\mathcal{Q} \simeq {}^\vee E/{}^\vee Q$ partial flag variety of ${}^\vee E$-conjugates of ${}^\vee\mathfrak{q}$.

Matsuki (1979), following Wolf (1969): ${}^\vee K$ acts on ${}^\vee\mathcal{Q}$ with finitely many orbits, the orbit of ${}^\vee\mathfrak{q}(\Lambda)$ corresponding precisely to the ${}^\vee G$-orbit of ABV Langlands parameters (y, Λ).

To repeat: Langlands parameters $\overset{\text{bijection}}{\leftrightsquigarrow}$ ${}^\vee K$ orbits on ${}^\vee\mathcal{Q}$.

Classical Langlands params are closed (smooth) ${}^\vee G$ orbits; no interesting geometry.

Theorem (Adams-Barbasch-V) There is a natural bijection

(simple ${}^\vee K$-eqvt perverse sheaves on ${}^\vee\mathcal{Q}$)

$\leftrightsquigarrow$(irr reps of infl char Λ of pure inner forms of $G(\mathbb{R})$).

Map to Langlands parameters is the support of a perverse sheaf, which must be the closure of a single ${}^\vee K$-orbit.

Instructions on how to think

reps of pure inner forms of $G(\mathbb{R}) \leftrightsquigarrow {}^\vee K$-eqvt sheaves on ${}^\vee Q$.

${}^\vee K$ is symmetric in cplx reductive ${}^\vee E$, pseudolevi in ${}^\vee G$;
${}^\vee Q$ is partial flag variety for ${}^\vee E$; and
dual E is endoscopic for G.

How should we think about this?

Look at reps of one infl char λ: fixes ${}^\vee E$ and ${}^\vee Q$.

The few choices for ${}^\vee K \leftrightsquigarrow$ few choices for block of reps; ignore.

What's easy is $\mathcal{V}(G, \lambda)$ = virtual reps of inner forms of G.

This is a finite rank lattice: two bases irr reps or std reps.

Also easy: $\mathcal{V}(Q, {}^\vee K)$ = virtual eqvt constr shves on Q.

Also finite rank: bases eqvt perv shves or eqvt loc sys on orbits.

Lattices $\mathcal{V}(G, \lambda)$ and $\mathcal{V}(Q, {}^\vee K)$ are naturally dual.

irr reps and perverse sheaves are dual bases (up to sgn).

std reps and loc sys on orbits are dual bases (up to sgn).

Virtual rep of G = $\mathbb{Z}$-linear functional on sheaves on Q.

Why is this the right way to think?

Virtual rep of G = $\mathbb{Z}$-linear functional on sheaves on Q.

Langlands/Shelstad/Arthur... approach to harmonic analysis uses stable reps: virtual representations with distn chars constant on each intersection (G-conjugacy class)$\cap G(\mathbb{R})$.

Theorem (ABV) Virtual rep of G is stable $\iff$ functional on sheaves depends only on stalk dims (not local sys).

Two $\mathbb{Z}$-bases for such linear functionals on sheaves:

1. dimension of stalk along one ${}^\vee K$ orbit on Q.
2. mult in char cycle of conormal to one ${}^\vee K$-orbit.

Basis (1) is the Langlands-Shelstad basis for stable characters: alternating sum of std reps in one L-packet.

Basis (2) is how ABV defined Arthur packets.

This is a hot link to pictures for $PGL(2,\mathbb{R})$.

p-adic Weil group

k p-adic local field, $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$ Galois group.

Residue field $\mathbb{F}_q$ for k gives short exact sequence

$$1 \to I_k \to \Gamma_k \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q \to 1.$$

Inertia grp $I_k \leftrightsquigarrow$ ramification of field exts.

$\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) = \widehat{\mathbb{Z}} =$ completion of $\langle \overline{\mathrm{Fr}} \rangle \simeq \mathbb{Z}$.

Here Frobenius elt $\overline{\mathrm{Fr}}$ acts on $x \in \overline{\mathbb{F}}_q$ by $\mathrm{Fr}(x) = x^q$.

Weil group W_k is dense subgroup $\langle I_k, \mathrm{Fr} \rangle$ of Γ_k:

$$1 \to I_k \to W_k \to \langle \overline{\mathrm{Fr}} \rangle \to 1.$$

If $w \in W_k$ maps to $\overline{\mathrm{Fr}}^k$, define $|w| = q^k$.

How Galois element w acts on finite residue fields.

Langlands' insights p-adically

G reduc alg / k p-adic, $\Gamma_k = \mathrm{Gal}(\overline{k}/k) \rightsquigarrow {}^L G = {}^\vee G \rtimes \Gamma$.

${}^\vee G$ is still defined over $\mathbb{C}$, since that's the field for our G-reps.

Langlands param: continuous $\phi\colon W_k \to {}^\vee G$ making

$$\begin{array}{ccc} W_k & \xrightarrow{\phi} & {}^\vee G \\ \searrow & \circlearrowleft & \swarrow \\ & \Gamma & \end{array}$$

Also need $\phi(\mathrm{Fr})$ semisimple.

Langlands understood there should be a ${}^\vee G$-conj class of such ϕ for each rep of inner form of G.

But more data is needed...

Weil-Deligne group $W_k' = W_k \ltimes \mathbb{C}$, $w \cdot z = |w|z$.

Deligne-L parameter $= \phi'\colon W_k' \to {}^L G$

$= (\phi, N_D)$ $\quad \phi\colon W_k \to {}^L G$ Langlands parameter

$N_D \in {}^\vee\mathfrak{g}, \quad \mathrm{Ad}(\phi(w))(N_D) = |w| N_D.$

ϕ is sometimes called the infinitesimal character of ϕ'.

More about p-adic Langlands

Fix Frobenius element $\mathrm{Fr} \in W_k$.

Deligne-Langlands parameter is triple $\phi' = (y, \phi_0, N_D)$:

1. $\phi_0 \colon I_k \to {}^L G$ describes ramification;
2. $y = \phi(\mathrm{Fr}) \in {}^\vee G \cdot \mathrm{Fr}$ normalizes ϕ_0, respects Fr action on I_k;

Condition: y action on $\phi_0(I_k) \leftrightsquigarrow$ Fr action on I_k.

Get reductive algebraic ${}^\vee G^{\phi_0}$, semisimple aut $\mathrm{Ad}(y)$ of ${}^\vee G^{\phi_0}$,

$$ {}^\vee G^{\phi_0, y} = {}^\vee G^{\phi} \subset {}^\vee G^{\phi_0} \qquad \text{twisted pseudolevi in } {}^\vee G^{\phi_0}. $$

$$ {}^\vee \mathfrak{e}_n = \quad q^n \text{ eigenspace of } \mathrm{Ad}(y) \text{ on } {}^\vee \mathfrak{g}^{\phi_0}. $$

$$ {}^\vee \mathfrak{e} = \sum_{n \in \mathbb{Z}} {}^\vee \mathfrak{e}_n, \quad {}^\vee E = \langle {}^\vee G^{\phi}, {}^\vee E_0 \rangle, \quad {}^L E = \langle E, \mathrm{im}(\phi) \rangle. $$

Reductive group ${}^\vee E$ is like ${}^\vee E = \mathrm{Cent}(y^2)$ in real case, or "unramification" in Mishta Ray's talk.

Geometry for DL parameter reduces to ${}^L E$, where it looks like geometry for unramified reps of E.

Last condition on a Deligne-Langlands parameter is

3. $N_D \in {}^\vee \mathfrak{e}_1 = q$-eigenspace of $\mathrm{Ad}(y)$ on ${}^\vee \mathfrak{g}^{\phi_0}$.

Finishing about p-adic Langlands

DL parameter is pair $\phi' = (\phi, N_D)$, $\phi\colon W_k \to {}^L G$.

ϕ given by pair (y, ϕ_0), $\phi_0\colon I_k \to {}^L G$, $y = \phi(\mathrm{Fr})$.

Get $\mathbb{Z}$-graded reductive ${}^\vee E$ with zero level Levi ${}^\vee L = {}^\vee G^\phi$.

${}^\vee L$ acts on ${}^\vee\mathfrak{e}_1$ with finitely many orbits; ${}^\vee G^\phi \cdot N_D \leftrightsquigarrow {}^\vee G$ orbit of Deligne-Langlands parameters (y, ϕ_0, N_D).

Local Langlands conjecture: There is a natural bijection (simple ${}^\vee G^\phi$-eqvt perverse sheaves on ${}^\vee\mathfrak{e}_1$) $\leftrightsquigarrow$ (irr reps of pure inner forms of G of infl char ϕ).

Map to Deligne-Langlands parameters is support of perverse sheaf: closure of one ${}^\vee G^\phi$ orbit.

Now repeat everything said about real local Langlands conjecture, with ${}^\vee L$-orbits on ${}^\vee\mathfrak{e}_1$ replacing ${}^\vee K$-orbits on $\mathcal{Q}$, and local Langlands conjecture replacing Theorem.

This is also a link to pictures for $PGL(2, k)$.

Backhanded apology

In describing Deligne-Langlands parameters, I tried hard to avoid introducing $SL(2)$.

This was deliberate: Deligne defn of W_k' had no $SL(2)$.

Unfortunately the literature on Langlands' conjectures is replete with $SL(2)$s.

I believe the ones used for W_K' are a mistake.

I'm not sure about the "Arthur $SL(2)$."

Perhaps it's the L-group of $PGL(2)$, and
Arthur parameter $=$ functorial lift(trivial of $PGL(2)$).

But I do not know how to make this idea precise.

This is all to say that I am likely to misstate Arthur's conjectures in very serious ways.

Sorry!

Arthur parameters over $\mathbb{R}$

Recall: Langlands parameter is $\phi_0 = (y_0, \lambda_0) \in {}^\vee G\sigma \times {}^\vee\mathfrak{g}$,
$y^2 = {}^\vee e = \exp(2\pi i \lambda)$

Definition (Arthur). Arthur parameter is $\psi = (y_0, \lambda_0, f)$ with

1. $\phi_0 = (y_0, \lambda_0)$ tempered Langlands parameter;
2. $f \colon SL(2) \to {}^\vee G$ algebraic; and
3. image of f commutes with y_0 and λ_0

From ψ can construct another parameter $\phi(\psi) = (y, \lambda)$,

$$y = y_0 \cdot f\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \lambda = \lambda_0 + df\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

Change in $\lambda \rightsquigarrow \phi(\psi)$ nontempered.

Then $\phi(\psi)$ is the Langlands parameter Arthur attaches to ψ, so that one of his desiderata is $\Pi_A(\psi) \supset \Pi_L(\phi(\psi))$.

Arthur packets over $\mathbb{R}$: ABV version

$\psi = (y_0, \lambda_0, f)$ Arthur parameter $\rightsquigarrow$

$$y = y_0 \cdot f\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \lambda = \lambda_0 + df\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad N_A = df\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$\lambda \rightsquigarrow {}^\vee e = \exp(2\pi i \lambda) \in {}^\vee G \rightsquigarrow {}^\vee E = {}^\vee G^{{}^\vee e}$ pseudolevi in ${}^\vee G$.

$\lambda \rightsquigarrow {}^\vee Q$ partial flag variety of ${}^\vee E$-conjugates of ${}^\vee\mathfrak{q}(\lambda)$.

$y \rightsquigarrow {}^\vee K = {}^\vee G^{\,y} \subset {}^\vee E$, symm reductive subgrp of ${}^\vee E$.

$N_A \in {}^\vee\mathfrak{u}(\lambda) \leftrightsquigarrow {}^\vee K$-orbit ${}^\vee\mathcal{O}^\theta$ of nilp elts in ${}^\vee\mathfrak{e}/{}^\vee\mathfrak{k}$.

Recall ABV version of LLC: $\big($simple ${}^\vee K$-eqvt perv sheaves on ${}^\vee Q\big) \leftrightsquigarrow \big($ irr reps of infl char λ of inner forms of $G(\mathbb{R})\big)$. Map to Langlands params is support of perverse sheaf.

Definition (ABV). Arthur packet $\Pi_A(\psi) \leftrightsquigarrow$ perv sheaves whose char cycle contains conormal to ${}^\vee K \cdot \mathfrak{q}(\lambda)$.

Motivation: equivalent to require $(\mathfrak{q}(\lambda), N_A)$ in char cycle.

In terms of $({}^\vee\mathfrak{e}, {}^\vee K)$-modules, these are annihilated by kernel of map to diff ops on ${}^\vee Q$, of largest possible GK dimension.