(Small) Resolutions of Closures of K-Orbits in Flag Varieties

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atlas Seminar

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Introduction

Barbasch-Evens

The Morphism μ

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- G, $T \subseteq B$, K

$v \in V$ $\quad \dot{v}$ $\quad K \times B$-orbit in G

- $X = G/B$, $V = K \backslash G/B$
- $\mathcal{Q}_v = K\dot{v}B/B$, $X_v = \overline{\mathcal{Q}_v} \subseteq X$
- $K_v = \mathrm{stab}_K(\dot{v}B/B)$

Fact (Vogan 1983)

The set of irreducible Harish-Chandra modules with infinitesimal character ρ corresponds to

$$\mathcal{D} = \left\{(v, \mathcal{L}) \mid v \in V,\ \mathcal{L} \in (K_v/K_v^0)^\wedge\right\}.$$

- $\mathrm{IC}(\delta)$ extends $\delta \in \mathcal{D}$ to X_v.
- $\mathrm{IC}(\delta)^{2i} = \sum_{\gamma \in \mathcal{D}} (\text{coefficient of } q^i \text{ in } \mathrm{KLV}_{\gamma,\delta})\gamma$
- Say X_v is **Q**-*smooth* if for every $i > 0$, we have $\mathrm{IC}(v, 1)^{2i} = 0$ and $\mathrm{IC}(v, 1)^0 = \sum_{u \leq v} (u, 1)$.

$G_{\mathbf{R}} = U(p, q)$

$B \subseteq P_I \subseteq G$ $\quad I \subseteq \{1, \dots, n-1\}$

- $G = GL(n, \mathbf{C})$, $T \subseteq B$, $K = GL(p, \mathbf{C}) \times GL(q, \mathbf{C})$
- $G/B = \{(0 \subseteq F^1 \subseteq \cdots \subseteq F^{n-1} \subseteq \mathbf{C}^n)\} = \{\text{Borel subgroups}\}$
- $\mathbf{C}^\bullet$ corresponds to B and $\mathbf{C}^{-\bullet}$ corresponds to B^-
- $X = G/P = \mathrm{Gr}_k(\mathbf{C}^n) = \{0 \subseteq E^k \subseteq \mathbf{C}^n\}$ $\quad P = P_{\hat{k}}$
- $V^k_{p,q} = \{(a, b) \in \mathbf{Z}^2_{\geq 0} \mid a \leq p,\ b \leq q,\ a + b \leq k\} = K\backslash G/P$
- $X_{a,b} = \{E^k \mid \dim(\mathbf{C}^p \cap E^k) \geq a,\ \dim(\mathbf{C}^{-q} \cap E^k) \geq b\}$
- $Z_{a,b} = \{(A^a, B^b, E^k) \mid A^a \subseteq \mathbf{C}^p \cap E^k,\ B^b \subseteq \mathbf{C}^{-q} \cap E^k\}$

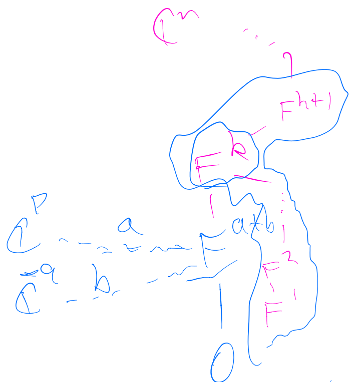
$\cong X_{\hat{a}} \times X_{\hat{b}} \times X_{\hat{k}}$

Proposition (Barbasch-Evens 1994)

The map $\mathrm{pr} : Z_{a,b} \to X_{a,b}$ *by* $(A, B, E) \mapsto E$ *is a resolution of singularities.*

$\mathrm{pr}^{-1}(E) \cong \mathrm{pt}$ $\qquad \mathrm{pr}^{-1}(E_{a',b'}) \cong \mathrm{Gr}_a(\mathbf{C}^{a'}) \times \mathrm{Gr}_b(\mathbf{C}^{b'})$

$a' \geq a,\ b' \geq b$ $\qquad X_{a',b'} \subseteq X_{a,b}$



$$Gr_{a+b}(\mathbb{C}^n) = X_{\widehat{a+b}}$$

$$F^{a+b} = (\mathbb{C}^p \cap F^{a+b}) + (\mathbb{C}^q \cap F^{a+b})$$

$$Z_{a+b} = X_{(\widehat{a+b}) \cap (\hat{b})}$$

$$\begin{tikzcd}
G_{v_0} \times^R P \arrow[r, "\mu'"] \arrow[d] & G_v \arrow[d] & \\
G_{v_0} \times^R P/B \arrow[r, "\mu"] \arrow[d] & X_v \subseteq G/B & \\
Z_{a,b} = X^J_{v_0} \arrow[r] & X_{a,b} \subseteq G/P &
\end{tikzcd}$$

$G_{v_0} \times^R P$ $\qquad$ $K \times P$ - orbits

G_{v_0} $\qquad$ $K \times R$ - orbits

$$K \backslash G_{v_0}/R \longleftrightarrow K \backslash G_{v_0} \times^R P/P = Y$$

$y \in$

$y * w_J = y$

$$KyR \times^R P \longrightarrow KyP$$

$$\boxed{\max_{y \in Y} \ell(y * w_J) + \dim(P/R) - \ell(y * w_I)}$$

Small Resolutions

- A resolution $\xi : Z \to Y$ is *small* means for every $r > 0$, we have
$$\operatorname{codim}_Y \{y \in Y \mid \dim(\xi^{-1}(y)) \geq r\} > 2r.$$
- If ξ is a small resolution, then $\xi_* \underline{\mathbf{Q}}_Z \cong \mathrm{IC}_Y$.
- Note that Y is normal if and only if $\xi_* \mathcal{O}_Z \cong \mathcal{O}_Y$.
- Barbasch-Evens construct a small resolution for every $X_{a,b}$ and show that $X_{a,b}$ is normal.
- Let $\xi : Z \to Y$ be a small resolution, $\iota : Y \subseteq X$ inclusion into a smooth variety, and $f = \iota \circ \xi$. Then
$$SS(\mathrm{IC}_Y) \subseteq f_\pi({}^t f'^{-1}(Z)),$$

$$\begin{tikzcd} TZ \arrow[r, "f'"] \arrow[rr, bend left, "Tf"] \arrow[dr] & Z \times_X TX \arrow[r, "f_\pi"] & TX \arrow[d, "\pi"] \\ & Z \arrow[r, "f"] & X \end{tikzcd} \qquad \begin{tikzcd} T^\vee Z & Z \times_X T^\vee X \arrow[l, "{}^t f'"'] \arrow[r, "f_\pi"] \arrow[d] & T^\vee X \arrow[d, "\pi"] \\ & Z \arrow[ul, "\subseteq"] \arrow[r] & X \end{tikzcd}$$

where $\pi : T^\vee X \to X$ is the cotangent bundle.
- Bressler-Finkelberg-Lunts 1990 show that all Schubert varieties in $\mathrm{Gr}_k(\mathbf{C}^n)$ have trivial characteristic cycles. Barbasch-Evens and more recently Timchenko 2019 show all $X_{a,b}$ have trivial characteristic cycles.

μ

- G, $T \subseteq B$, K connected
- $V = K\backslash G/B$, $W = B\backslash G/B$ $\cong N(T)/T$
- $G_v = \overline{K\dot{v}B} \subseteq G$, $G_w = \overline{B\dot{w}B} \subseteq G$
- $Z = G_{v_0} \times^{R_1} G_{w_1} \times^{R_2} \cdots \times^{R_m} G_{w_m}/R$
- $\mu : Z \to G/P$ by $[g_0, \ldots, g_m R/R] \mapsto g_0 \cdots g_m P/P$
- Define $v \star w$ by the image of $\mu : G_v \times^R G_w/B \to G/B$.

$[g_0 r, g_1] = [g_0, r g_1]$

$Z = G_v \times^R G_w/B = (G_v \times G_w)/(R\times B)$

$(g_0, g_1).(r, b) = (g_0 r, r^{-1} g_1 b)$

$R \subseteq P$

$(W,*)$ is monoid acts on V $\quad V=W$

$u = v * w$

$R_i \leftrightarrow J_i$

$J_1 = \tau_W(v_0) \cap \tau(w_1^{-1})$

$= \{s \in S \mid v_0 * s = v_0\}$

$\mathrm{im}(\mu) = G_u/B$

Proposition

Let $v = v_0 \star w_1 \star \cdots \star w_m \star w_I$, *where* P_I *is the standard parabolic subgroup corresponding to the subset of simple reflections* I. *The map* $\mu : Z \to G_v/P$ *is generically finite if and only if*

$w_I = \max(W_I)$

$v * s * s \overset{?}{=} v * s \qquad G_v(P_s P_s) = G_v(P_s)$

$$\ell(v) = \ell(v_0) + \sum_{i=1}^{m} \ell(w_i) - \sum_{i=1}^{m} \ell(w_{J_i}) + \ell(w_I) - \ell(w_J),$$

where R *corresponds to* J.

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