Associated Varieties

For irred $(\mathfrak{g}, K)$-module X, want to associate invariants.
Associated variety is good way to measure size.

Commutative Algebra $\Rightarrow$ R commutative ring, M a f.g. R-module.

One measure of size is $\mathrm{Ann}(M) = \{ r \in R \mid rm = 0 \text{ for all } m \in M \}$.

Example: If M is generated by a single element, then $M \cong R/\mathrm{Ann}(M)$.

The associated variety of M is defined to be $V(\mathrm{Ann}(M)) \subset \mathrm{Spec}\, R = \{ p \in \mathrm{Spec}\, R \mid p \supset \mathrm{Ann}(M) \}$.
This gives a geometric invariant.

Want to do the same for $U(\mathfrak{g})$.
PBW filtration on $U(\mathfrak{g})$ with $U(\mathfrak{g})_n = \{ X_1 \cdots X_k \mid X_i \in \mathfrak{g},\ k \le n \}$ gives
$S(\mathfrak{g}) \cong \mathrm{gr}\, U(\mathfrak{g})$ which is commutative.

$\Rightarrow$ Natural to consider compatible filtration on irr $(\mathfrak{g}, K)$-module X.
$\cdots \subset F_{-1}X \subset F_0 X \subset F_1 X \subset \cdots$ with $U(\mathfrak{g})_n F_k X \subset F_{n+k} X$

When the filtration is clear, we write $X_n = F_n X$.
We denote the graded module with respect to $F_\bullet$ by $\mathrm{gr}(X, F)$, but will write $\mathrm{gr}\, X$ when F is clear.

However, our ~~invariants~~ associated variety should not depend too much on the choice of filtration.

Good filtration:
- $X_{-n} = 0$ for $n \gg 0$
- $\bigcup_{n \in \mathbb{Z}} X_n = X$
- $\dim X_n < \infty$ for all n
- $U(\mathfrak{g})_p X_q = X_{p+q}$ for $q \gg 0$.

These conditions are equivalent to $\operatorname{gr} X$ being a finitely generated $S(\mathfrak{g})$ module.

Suppose F, G are two good filtrations.

- Have $s, t \in \mathbb{N}$, s.t. $G_{p-s} X \subset F_p X \subset G_{p+t} X$ for all $p \in \mathbb{Z}$.
- There are finite filtrations

$$0 = \operatorname{gr}(X,F)_{-1} \subset \operatorname{gr}(X,F)_0 \subset \cdots \subset \operatorname{gr}(X,F)_{s+t} = \operatorname{gr}(X,F)$$
$$0 = \operatorname{gr}(X,G)_{-1} \subset \operatorname{gr}(X,G)_0 \subset \cdots \subset \operatorname{gr}(X,G)_{s+t} = \operatorname{gr}(X,G)$$

by graded $S(\mathfrak{g})$ submodules such that $\operatorname{gr}(X,F)_j / \operatorname{gr}(X,F)_{j-1} \cong \operatorname{gr}(X,G)_{s+t-j} / \operatorname{gr}(X,G)_{s+t-j-1}$ (with some shift in grading).

This fact about good filtrations is important since it suggests that one should look for invariants additive on short exact sequences for $S(\mathfrak{g})$ modules.

The above fact says that such invariants will give a well-defined invariant for X.

The associated variety gives such an invariant.

For an exact sequence $0 \to A \to B \to C \to 0$ of finitely generated $S(\mathfrak{g})$ modules,

we have $(\mathrm{Ann}\ A)(\mathrm{Ann}\ C) \subset \mathrm{Ann}\ B \subset \mathrm{Ann}\ A \cap \mathrm{Ann}\ C$.

$$V(\mathrm{Ann}\ B) = V(\mathrm{Ann}\ A) \cup V(\mathrm{Ann}\ C)$$

If we consider the set of closed subvarieties of $\mathrm{Spec}\ S(\mathfrak{g})$ with addition corresponding to union, then this map from finitely generated $S(\mathfrak{g})$ modules to subvarieties is additive.

Thus, this gives a well-defined invariant $\mathrm{Ass}(X)$ on our irreducible $(\mathfrak{g}, K)$ module X.

(Rmk. Also get Gel'fand-Kirillov dimension from this. It is the dimension of the associated variety.)

Up until now, I've been ignoring the "K" part of $(\mathfrak{g}, K)$. How do we make use of it?

All of the facts about filtrations and graded modules can make use of K. Take $\cdots \subset F_{-1}X \subset F_0 X \subset F_1 X \subset F_2 X \subset \cdots$ be filtration such that $F_i X$ is closed under the K action. Then $\mathrm{gr}\, X$ has an induced K-action, and it naturally becomes a $(S(\mathfrak{g}), K)$-module.

All the theorems about good filtrations and associated variety are exactly the same, so K acts on the associated variety, $\mathrm{Ann}(\mathrm{gr}\, X)$, ...
Moreover, one easily sees that $V(\mathrm{gr}\, X) \subset (\mathfrak{g}/\mathfrak{k})^* \subset \mathfrak{g}^*$, where $\mathfrak{g}^* \cong \mathrm{Spec}\, S(\mathfrak{g})$.

So this is the most basic form of the associated variety, so let's compute some (easy) examples.

- X finite dimensional. Then for any good filtration, $X_n = 0$ for $n \gg 0$. Then $\mathrm{Ann}(\mathrm{gr}\, X) \supset S(\mathfrak{g})_n$ for $n \gg 0$. Thus, $V(\mathrm{gr}\, X) =$ zero orbit
- Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be simple such that $\mathfrak{p}$ is an irr $\mathfrak{k}$-rep (i.e. $\mathfrak{g}$ not Hermitian symmetric). Let X be a ladder representation, so that if β is the h.w. of $\mathfrak{p}$ as a K-rep, then there is a K-type μ such that $X|_K \cong V_\mu \oplus V_{\mu+\beta} \oplus V_{\mu+2\beta} \oplus \cdots$
We let $X_0 = V_\mu$, $X_i = U(\mathfrak{g})_i X_0 / U(\mathfrak{g})_{i-1} X_0$.
The ring of functions $\mathbb{C}[\mathrm{Ass}(X)]$ of the associated variety is a graded ring isomorphic to a quotient of $S(\mathfrak{g}/\mathfrak{k}) \cong S(\mathfrak{p})$. All weights in $S(\mathfrak{p})_n$ are at most $n\beta$, so $X_i = V_{\mu+i\beta}$. Then $\mathbb{C}[\mathrm{Ass}\, X] \cong \mathbb{C} \oplus V_\beta \oplus V_{\beta^2} \oplus \cdots$ as a K-rep.
Since any nonzero associated variety must have $\mathbb{C}[\mathrm{Ass}\, X]_1 \cong \mathfrak{p}$, it follows that this is minimal among nonzero associated varieties. (when we discuss nilp orbits, this is the minimal orbit.)

The associated variety also gives information about the K-types of X.

$X_0 =$ lowest K-type. $X_i = U(\mathfrak{g})_i X_0 / U(\mathfrak{g})_{i-1} X_0$ as before.

If λ is a K-type in $\mathbb{C}[\mathrm{Ass}(X)]$ in degree d, then, $\mu + \lambda$ is a K-type in $gr_d X$.

Now we describe a finer invariant: the Characteristic Cycle.

As with the associated variety, we will first discuss the $S(\mathfrak{g})$-version, and then add in the K-part.

If M is a finitely generated $S(\mathfrak{g})$-module, let $C = \mathrm{Ass}(M)$ be the associated variety.

Have $C \longleftrightarrow$ minimal primes $P_1, \dots, P_k$ for irr components.

Comm alg: Have filtration of $S\mathfrak{g}/Q_i$'s. # of $S\mathfrak{g}/P_i$ is fixed. M_{P_i} has finite length as $S(\mathfrak{g})_{P_i}$ mod. Denote by $m(M, P_i)$

Then we can associate to M the formal sum $\sum m(M, P_i) P_i$.

One performs addition on these formal sums by considering only the P_i for the irred components of the union of the associated variety and adding those $m(-,-)$.

(This is the same as taking minimal P_i's).

This is then an additive invariant on f.g $S(\mathfrak{g})$-mod so we get an invariant for X.

To add in K-action, we will construct vector bundles on our irred components and produce a K-equivariant coherent sheaf.

First, note that if $\mathrm{Ann}(M) = P$, then $M_P \simeq \left(\frac{S(\mathfrak{g})}{P}\right)_P^{m(P,M)}$. Since P is the generic point on $V(P)$, this statement says that M is generically free. (free on some open set)

$\Rightarrow$ There is some $f \in S(\mathfrak{g})$ such that $M_f \simeq \left(\frac{S(\mathfrak{g})}{P}\right)_f^{m(P,M)}$.

Then if M is an $(S(\mathfrak{g}), K)$-module, there is a maximal ideal $m(\lambda)$ such that

$\frac{M}{m(\lambda)M} \simeq \left(\frac{S(\mathfrak{g})}{m(\lambda)}\right)^{m(P,M)}$. If $K(\lambda)$ is the stabilizer of $m(\lambda)$, then we get a representation of $K(\lambda)$ on $\left(\frac{S(\mathfrak{g})}{m(\lambda)}\right)^{m(P,M)} \simeq \mathbb{C}^{m(P,M)}$. This corresponds to a K-equivariant vector bundle $K \times_{K(\lambda)} \mathbb{C}^{m(P,M)}$.

To adapt for general M, we use an appropriate filtration to reduce to this case. In particular, we pick a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ such that for each j, there is some f such that $(M_j/M_{j-1})_f$ is free over each $\left(\frac{S(\mathfrak{g})}{P_i}\right)_f$ where P_i is a minimal prime of $\mathrm{Ass}(M)$. For sufficiently generic $m(\lambda)$, we can then produce a virtual character from $\sum_j M_j / m(\lambda) M_j + M_{j-1}$.

We have not talked about nilp orbits, but the important part is:

For $\underset{\text{irr}}{\wedge}$ X we have $\mathrm{Ass}(X)$ is a finite union of K-orbits in the nilp cone.

If $\mathcal{O}_1, \ldots \mathcal{O}_k$ are the maximal orbits (ordering is by containment in closure), with $\lambda_i \in \mathcal{O}_i$, then get virtual reps of $K(\lambda_i)$, defining a virtual vector bundle on each orbit. This is our characteristic cycle.