LAST TIME Have irr $(\mathfrak{g}, K)$ module X.

Attached to X the Associated Variety $Ass(X)$
Attached to X the characteristic cycle $Ch(X)$

In particular, $Ch(X)$ is a virtual sum of vector bundles on some K-orbits in $(\mathfrak{g}/\mathfrak{k})^*$

$G_{\mathbb{R}} = SL(2, \mathbb{R}) \quad K_{\mathbb{R}} = SO(2)$

$G = SL(2, \mathbb{C}) \quad K \simeq \mathbb{C}^\times$

Discrete series with K types $n, n+2, n+4, \ldots$
$n \geq 2$

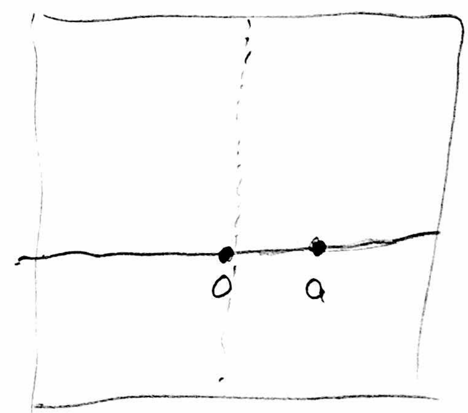
Pick Filtration by $F_0 = n \subset F_1 = n \oplus n+2 \subset n \oplus n+2 \oplus n+4$

$$S(\mathfrak{p}) = \mathbb{C}[X_{-2}, X_{+2}]$$

$\mathfrak{p} = \mathbb{C}X_{-2} \oplus \mathbb{C}X_{+2}$

$$\mathrm{Ann}_{S(\mathfrak{p})}(\mathrm{gr}\, X) = (X_{-2})$$

$$\mathrm{gr}\, X \simeq S(\mathfrak{p})/(X_{-2})$$



$K^a = \pm I$

$\Rightarrow$ Isotropy rep of K^a = trivial rep if n even
= sgn rep if n odd

A Short Summary of Nilpotent Orbits

Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$

PBW Filtration: $gr\, Z(\mathfrak{g}) \cong S(\mathfrak{g})^G$ where G is a connected group with Lie algebra $\mathfrak{g}$.

If X is our irr $(\mathfrak{g}, K)$-module, $Z(\mathfrak{g})$ acts by scalars on X.

So $I_\lambda \subset Z(\mathfrak{g})$ annihilates X.

$gr\, I_\lambda$ annihilates $gr\, X$

$\sqrt{Ann\, X} \supset gr\, I_\lambda$

$gr\, I_\lambda$ has finite codimension in $S(\mathfrak{g})^G$

$\Downarrow$

$\sqrt{gr\, I_\lambda}$ contains $S^+(\mathfrak{g})^G$

$\sqrt{Ann\, X} \supset S^+(\mathfrak{g})^G$

$Ass(X)$ lies in $N^* = \{\lambda \in \mathfrak{g}^* \mid p(\lambda) = 0 \text{ for } p \in S^+(\mathfrak{g})\}$

<u>FACT</u>: N^* is the set of nilpotent elements on $\mathfrak{g}^*$.

$Ass(X) \subset (\mathfrak{g}/\mathfrak{k})^* \Rightarrow Ass(X) \subset (\mathfrak{g}/\mathfrak{k})^* \cap N^*$

<u>FACT 2</u>: $N^* \cap (\mathfrak{g}/\mathfrak{k})^*$ is a finite union of K-orbits

(Kostant-Rallis)

NOW Get info about X from $Ch(X)$

Let Θ be a virtual character of H and π be a finite dimensional rep of H.

Define $[\Theta : \pi]_{quo} = \sum m_i \dim \mathrm{Hom}_H(\pi, p_i)$

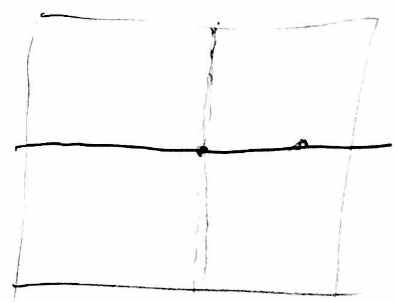
where $\Theta = \sum m_i p_i$ with p_i irreducible

Thm Let X be irr $(\mathfrak{g}, K)$-module, and let $\mathcal{O}$ be an open K-orbit$_\lambda$, with $\mathcal{O} \simeq K/K^\lambda$, $\lambda \in \mathcal{O}$.
in $Ass(X)$

Suppose $\chi(\lambda, X)$ is the virtual representation of $K(\lambda)$ attached to $\mathcal{O}$. If τ is an irr of K, then

$$\dim \mathrm{Hom}_K(\tau, X) \le [\chi(\lambda, X) : \tau]_{K^\lambda, quo}.$$

Example 1: $G_{\mathbb{R}} = SL(2, \mathbb{R})$



$K(\lambda) = \pm I$

$K = \mathbb{C}^\times$

$$\mathrm{Ind}_{K(\lambda)}^{K}(\mathrm{triv}) = V_0 \oplus V_2 \oplus V_{-2} \oplus V_4 \oplus V_{-4} \oplus \cdots$$

$$\mathrm{Ind}_{K(\lambda)}^{K}(\mathrm{sgn}) = V_1 \oplus V_{-1} \oplus V_3 \oplus V_{-3} \oplus \cdots$$

$G_{\mathbb{R}} = Sp(2n, \mathbb{R}) \qquad K_{\mathbb{R}} = U(n)$

$G = Sp(2n, \mathbb{C}) \qquad K = GL(n, \mathbb{C})$

$$K = \left\{ \left[\begin{array}{c|c} h & \\ \hline & {}^t h^{-1} \end{array}\right], \; h \in GL(n, \mathbb{C}) \right\}$$

$$\mathfrak{k} = \left\{ \begin{bmatrix} A & \\ & -A^t \end{bmatrix} \right\} \qquad \mathfrak{p} = \left\{ \left[\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right], \; B, C \text{ symmetric} \right\}$$

Take nilp orbit

$$\underbrace{K \cdot \left[\begin{array}{c|c} 0 & \begin{matrix} I_p & \\ & 0 \end{matrix} \\ \hline \begin{matrix} 0 & \\ & I_q \end{matrix} & 0 \end{array}\right]}_{\lambda_{p,q}} \qquad p + q = n$$

$$K^{\lambda_{p,q}} = \left[\begin{array}{c|c} O(p) & Mat_{p \times q} \\ \hline & O(q) \end{array}\right]$$

$$\mathrm{Ind}_{K^{\lambda_{p,q}}}^{K} (\mathrm{triv}) = \mathrm{Ind}_P^K \, \mathrm{Ind}_{K^{\lambda_{p,q}}}^{P} (\mathrm{triv})$$

$$P = \left[\begin{array}{c|c} GL_p & * \\ \hline & GL_q \end{array}\right]$$

$\mathrm{Ind}_{K \cap P_{\mathfrak{q}}}^{P}(\mathrm{triv}) \underset{\text{as } GL_p \times GL_q \text{ rep}}{\cong} \mathrm{Ind}_{O(p) \times O(q)}^{GL_p \times GL_q}(\mathrm{triv})$

Helgason: This is representations indexed by $2a_1 \geq 2a_2 \geq \cdots \geq 2a_p$
$2b_1 \geq \cdots \geq 2b_q$

$\mathrm{Ind}_{O(p)}^{GL_p}(\mathrm{triv})$ = Direct sum of algebraic reps of GL_p (with multiplicity)

Irr $GL_p(\mathbb{C})$ rep $\Rightarrow$ Irr $GL_p(\mathbb{R})$ representation containing trivial $O(p)$ representation $\Rightarrow$ Spherical $GL_p(\mathbb{R})$ representation

$a_1 \geq a_2 \geq a_3 \geq a_4 \cdots \geq \cdots a_p$

Langlands classification says that finite rep of $GL_p(\mathbb{R})$ is a submodule of

$$\mathrm{Ind}_{\underbrace{MA\bar{N}}_{\text{minimal}}}^{GL_p(\mathbb{R})}(\sigma \otimes e^{\nu} \otimes I)$$

Spherical $\Rightarrow$ $\sigma = \mathrm{triv}$

Finite dimensional: $\sigma \otimes e^{\nu}$ is the hwt of MA in the finite dimensional representation.

- $GL_p(\mathbb{C})$ reps indexed by $a_1 \geq a_2 \geq \cdots \geq a_p$

The hwt of $\underset{(\mathbb{R}^{\times})^n}{\overset{\parallel}{MA}}$ is trivial on M iff a_i is even for all i.

$$\mathrm{Ind}_{K^{\lambda_{p,q}}}^{P}(\mathrm{triv}) = \bigoplus V_{2a_1 \geq 2a_2 \geq \cdots \geq 2a_p, 2b_1 \geq \cdots \geq 2b_q}$$

$$\mathrm{Ind}_{P}^{K}\left(V_{2a_1 \geq \cdots \geq 2a_p, 2b_1 \geq \cdots \geq 2b_q}\right)$$

$$\overset{\text{Borel-Weil}}{=} \begin{cases} V_{2a_1 \geq 2a_2 \geq \cdots 2a_p \geq 2b_1 \geq \cdots \geq 2b_q} & \text{if } a_p \geq b_1 \\ 0 & \text{o.w.} \end{cases}$$

$$\mathrm{Ind}_{K^{\lambda_{p,q}}}^{K} = \bigoplus V_{2a_1 \geq 2a_2 \geq \cdots \geq 2a_n}$$