Our story so far

Here is a quick outline of the mathematical background so far.

We began with a pinned complex reductive group $(G \supset B \supset T, \{X_\alpha\})$ represented as

$$(n, \{\alpha_i \mid 1 \leq i \leq \ell\}, \{\alpha_i^\vee \mid 1 \leq i \leq \ell\})$$

with $\alpha_i \in \mathbb{Z}^n$ (thought of as the lattice $X^*(T)$; these are the simple roots), $\alpha_i^\vee \in \mathbb{Z}^n$ (thought of as the lattice $X_*(T)$; there are the simple coroots); and we fixed an "inner class" of real forms of G: an involutive automorphism of the based root datum; that is, an $n \times n$ integer matrix δ such that

$$\delta^2 = 1, \qquad \delta(\{\alpha_i \mid 1 \leq i \leq \ell\}) = \{\alpha_i \mid 1 \leq i \leq \ell\},$$
$$\delta^t(\{\alpha_i^\vee \mid 1 \leq i \leq \ell\}) = \{\alpha_i^\vee \mid 1 \leq i \leq \ell\}.$$

Then δ defines an *extended group*

$$^\delta G =_{\text{def}} G \rtimes \{1, \delta\}.$$

A *strong involution* means an element $x \in {}^\delta G \backslash G$ such that $x^2 \in Z(G)$.

A strong involution defines an involutive automorphism

$$\theta_x = \theta = \text{Ad}(x) \in \text{Aut}(G), \qquad \theta(g) = xgx^{-1}.$$

The "strong" in the name means that we are keeping track not only of θ but also of the representative x: θ determines only the coset $xZ(G)$. We write

$$K_x = K = \text{centralizer in } G \text{ of } x = \text{fixed points of } \theta_x.$$

This is a complex (possibly disconnected) reductive subgroup of G, which is the complexified maximal compact subgroup of the real form $G(\mathbb{R})$ corresponding to θ_x (by Cartan). Recall that Harish-Chandra's work relates topological (including unitary) representations of $G(\mathbb{R})$ to $(\mathfrak{g}, K)$-modules; it is these latter (algebraic) objects that the software studies.

What the software calls an "equivalence class of real forms" is a G-conjugacy class of strong involutions x.

A strong involution x_1 is called *fundamental* if

$$x_1 = t_1\delta \qquad (t_1 \in T);$$

equivalently, if $\theta_1 = \theta_{x_1}$ preserves $T \subset B$.

Theorem A. (Steinberg; see Timothy's talk 9/23). *Every G-conjugacy class of strong involutions has at least one fundamental representative.*

A strong involution x is called *normal* if it preserves the base torus T:

$$\text{Ad}(x)T = T, \qquad \theta_x(T) = T.$$

Theorem B. *Every B-conjugacy class of strong involutions has a normal representative, which is unique up to conjugation by T.*

I'm not certain to whom to attribute this. I think it is contained in a paper of Matsuki in 1979, and surely it's written in Richardson and Springer's 1990 paper. But I think it's older than any of these.

The software deals exclusively with T-conjugacy classes of normal strong involutions. I'll try now to explain why this is OK. So fix a fundamental strong involution x_1, with corresponding symmetric subgroup $K_1 \subset G$.

You're familiar with the fact that the easiest $\mathfrak{g}$-modules are highest weight modules, which are defined using a single Borel subalgebra (like our preferred one $\mathfrak{b}$). In the same way, it will turn out that all $(\mathfrak{g}, K_1)$-modules are something like "highest weight modules" defined using an arbitrary Borel subalgebra $\mathfrak{b}' \subset \mathfrak{g}$; the construction of these modules depends only on the K_1-orbit of the Borel subalgebra $\mathfrak{b}'$. We know that G acts transtively on Borel subalgebras, and that the stabilizer of the base $\mathfrak{b}$ is B; so

$$G/B \simeq \text{Borel subalgebras of } \mathfrak{b}' \subset \mathfrak{g}, \qquad gB \mapsto \mathfrak{b}' = \mathrm{Ad}(g)(\mathfrak{b}).$$

Consequently

$$\text{double cosets } K_1\backslash G/B \simeq K_1\text{-orbits of Borel subalgebras } \mathfrak{b}' \subset \mathfrak{g}.$$

In order to write down representations, the first thing the software must do is write down the double cosets $K_1\backslash G/B$. This is accomplished by the command `KGB`, which the October 7 video examined a little bit. Here I'll state a bit more of the theory underneath the software.

Theorem C. *With notation as above, any Borel subgroup $B' \subset G$ contains a θ-stable maximal torus T', unique up to conjugation by $K_1 \cap B'$. If we identify T' with T canonically (using an inner automorphism carrying (B', T') to (B, T)) then the action of θ_1 on $T' \simeq T$ is of the form $w\delta$, for some $w \in W(G,T)$. The element w is a "twisted involution" in W, meaning that $w\delta(w) = 1$.*

The discussion above shows that we can identify the double cosets $K_1\backslash G/B$ with the K_1-orbits of Borel subgroups of G. The software takes a somewhat different perspective. Of course G acts transtively on the set of G-conjugates x of the strong involution x_1, and the stabilizer of x_1 is K_1; so

$$K_1\backslash G \simeq \text{strong involutions } x \text{ conjugate to } x_1, \qquad K_1 g \mapsto x = g^{-1}x_1 g.$$

Consequently

$$\text{double cosets } K_1\backslash G/B \simeq B\text{-orbits of strong involutions conjugate to } x_1.$$

Theorem D. *The double cosets $K_1\backslash G/B$ may be identified with T-conjugacy classes of normal strong involutions x conjugate to x_1*

$$x = n\delta, \qquad n \in N_G(T), \quad n'\delta(n) \in Z(G).$$

The image of n in $W = N_G(T)/T$ is a twisted involution, meaning $w\delta(w) = 1$. The action of x on T is by $\theta_x = w\delta$, an involutive automorphism of the root datum.

In the software, `KGBElt` is a built-in type representing a T-conjugacy class of normal strong involutions. The command `KGB(RealForm)` produces the (finite) list of KGBElts in the conjugacy class of strong involutions which *is* the RealForm. This is illustrated in the video from October 7. This list always begins with the fundamental strong involutions. These can be recognized by the fact that the *length* of the KGBElt (the second column of the output of print_KGB(RealForm)) is zero.

Here is an example of getting the software to write the root datum automorphism (always an $n \times n$ matrix) of a KGBElt

```
atlas> set G=GL(3,R)
Variable G: RealForm
atlas> set X=KGB(G)
Variable X: [KGBElt]
atlas> #X
Value:  4
{now I know that there are four normal strong involutions in the
conjugacy class for GL(3,R), numbered 0 to 3.}
atlas> set x0=X[0]
Variable x0:  KGBElt
atlas> set x3=X[3]
Variable x3:  KGBElt
atlas> involution(x0)
Value:
| 0, 0, -1 |
| 0, -1, 0 |
| -1, 0, 0 |
atlas> involution (x3)
Value:
| -1, 0, 0 |
| 0, -1, 0 |
| 0, 0, -1 |
```

Now suppose we have a T-conjugacy class normal strong involutions x, all defining the same involutive automorphism θ_x of T. Define

$$T^{\theta_x} = T \cap K_x = \text{fixed points of } \theta_x \supset T_0^{\theta_x} = \text{identity component}$$

$$T^{-\theta_x} \supset T_0^{-\theta_x} = \text{identity component} = \{s\theta_x(s)^{-1} \mid s \in T\}.$$

Proposition E.

1) The torus T is the product of the groups $T_0^{\theta_x}$ and $T_0^{-\theta_x}$.

2) The intersection $T_0^{\theta_x} \cap T_0^{-\theta_x}$ consists of elements of order (one or) two, and therefore has cardinality at most $2^{\dim T}$.

3 The T-conjugacy class of the normal strong involution $x = n\delta$ is the coset $T_0^{-\theta_x}x$.

Suppose now that $x' = t'n\delta$ is a representative for another T conjugacy class of normal strong involutions conjugate to x_1. By the proposition, we may assume $t' \in T_0^{\theta_x}$. We know that

$$x_1^2 = z \in Z(G),$$

and therefore

$$x^2 = (x')^2 = z.$$

This last equation implies in particular that $(t')^2 = 1$; so t' must be an element of order 2 in $T_0^{\theta_x}$. It follows that

Corollary F. *In the map from $K_1\backslash G/B$ to δ-twisted involutions $w \in W$ defined by Theorem C above, the fiber over w has cardinality at most $2^{\dim(T^{\theta_x})}$.*

This Corollary explains why $K_1\backslash G/B$ is finite.

I'll end by explaining the penultimate column in the output of `print_KGB`: it is a nonnegative integer indexing the W-twisted-conjugacy class of the twisted involution w; equivalently, the W-conjugacy class of $w\delta$ in the "extended Weyl group" $W \rtimes \{1, \delta\}$. The numbering of these conjugacy classes is consistent across the inner class of real forms; so if G is not quasisplit, there will be some "missing numbers" since not all the twisted involutions appear. The class 0 always means δ.