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Cells in affine Weyl groups, IV

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This paper continues the series [13], [14], [15]. Our main result is the construction of a bijection between the set of two-sided cells in an affine Weyl group W, and the set of unipotent classes in a certain complex reductive group G, associated with W. (See Theorem 4.8.) We also show that the value of the a-function (see [13]) on a two-sided cell of W is equal to the dimension of the variety of Borel subgroups of G containing an element of the corresponding unipotent class. Our proof is based on the representation theory of affine Hecke algebras [6] and of the algebras J in [14], [15]. One of the main tools is a formula (6.4) expressing the restriction of a standard module of an affine Hecke algebra to a finite dimensional Hecke algebra; this formula is in terms of character sheaves. (This formula is not proved in this paper.) This allows us to use properties of character sheaves (6.7) to get information on modules of an affine Hecke algebra. We also prove that (in the case where G is semisimple) any two-sided cell of W meets some finite parabolic subgroup of W. Several of the results of this paper have been conjectured in [9, Problem V]. On the other hand, we state some new conjectures relating the algebras J with certain equivariant vector bundles.

The classification of two-sided cells of W has been known previously in certain special cases: for G of rank 2, see [13]; for G of type A_n, see [20], [12]; for G of rank 3, see [2], [3].

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We use this opportunity to correct some errors in [14], [15].

Errata for [14]. On p. 536, line -9, replace θ_w by t_w. On p. 539, lines 12, 13, replace "$\tau(\ )$" by "constant term of $\tau(\ )$".

Errata for [15]. On p. 231, lines -2, -3, (2.6 (a)) should read: "If a simple H_K-module is V-tempered, then $\hat{C}_M$ is a V-tempered conjugacy class."; 2.5 (a) should be deleted. The last line (conclusion) of Prop. 2.9

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(p. 232, line -2) should read: "Then $\hat{C}_M$ contains some C-point of G." At the end of 2.10 (p. 234) one should add the sentence: "(a) Fix $V: K^* \to R$ as in 2.8 and let Y be the set of isomorphism classes of simple, V-tempered H_K-modules M such that $\hat{C}_M$ contains some C-points of G." The references to 2.5 (a) on p. 234, line -4, p. 237, line -3, p. 238, line -1, p. 241, line -10 should be replaced by references to 2.10 (a).

1. Notations

1.1. Let (W', S) be a Coxeter group (S is the set of simple reflections). Let Ω be an abelian group acting on W' in such a way that $\omega(S)=S$ for all $\omega \in \Omega$. We form the semidirect product $W=\Omega \cdot W'$; it has multiplication $(\omega_1 \cdot w_1')(\omega_2 \cdot w_2')=\omega_1\omega_2 \cdot \omega_2^{-1}(w_1')w_2'$. Let $l: W' \to N$ be the length function; we extend it to a function $l: W \to N$ by $l(\omega w')=l(w')$.

For $y, w \in W'$ we define the polynomial $P_{y,w} \in C[v]$, (v an indeterminate) as follows: write $y=\omega y'$, $w=\omega' w'$ ($\omega, \omega' \in \Omega$, $y', w' \in W'$) and set $P_{y,w}=P_{y',w'}$ ($P_{y',w'}$ as in [5]) if $\omega=\omega'$ and $P_{y,w}=0$ if $\omega \neq \omega'$.

The preorders $\underset{L}{\leq}$, $\underset{LR}{\leq}$ on W' and the associated equivalence relations $\underset{L}{\sim}$, $\underset{LR}{\sim}$ on W' are defined in [5] (see also [13]) in terms of the $P_{y,w}$. We extend them to W by $y \underset{L}{\leq} w$ (resp. $y \underset{L}{\sim} w) \Longleftrightarrow \exists y', w' \in W'$, $\omega_1, \omega_2 \in \Omega$ with $y=\omega_1 y'$, $w=\omega_2 w'$, $y' \underset{L}{\leq} w'$ (resp. $y' \underset{L}{\sim} w'$); $y \underset{LR}{\leq} w$ (resp. $y \underset{LR}{\sim} w) \Longleftrightarrow \exists y', w' \in W'$, $\omega_1, \omega_2, \omega_3, \omega_4 \in \Omega$ with $y=\omega_1 y' \omega_3$, $w=\omega_2 w' \omega_4$, $y' \underset{LR}{\leq} w'$ (resp. $y' \underset{LR}{\sim} w'$). The equivalence classes for $\underset{L}{\sim}$ (resp. $\underset{LR}{\sim}$) on W are called left cells (resp. two-sided cells). The image of a left cell under $w \to w^{-1}$ is called a right cell.

Let $\mathcal{A}=C[v, v^{-1}]$. We shall often identify $\mathcal{A}$ with the C-algebra of regular functions $C^* \to C$ with v being the obvious inclusion $C^* \subset C$.

Let $k=C(v)$ and let $\bar{k}$ be an algebraic closure of k.

The Hecke algebra H is the free $\mathcal{A}$-module with basis $\tilde{T}_w$ ($w \in W$) with the associative $\mathcal{A}$-algebra structure defined by $\tilde{T}_w \tilde{T}_{w'}=\tilde{T}_{ww'}$ if $l(ww')=l(w)+l(w')$, $(\tilde{T}_r+v^{-1})(\tilde{T}_r-v)=0$ if $r \in S$.

It has a unit element $1=\tilde{T}_e$ where e is the neutral element of W. The elements

$$C_w=\sum_y (-v)^{l(w)-l(y)} P_{y,w}(v^{-2}) \tilde{T}_y \in H, \qquad (w \in W),$$

(see [5]) are well defined and form an $\mathcal{A}$-basis of H. We can write

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z, \quad h_{x,y,z} \in \mathcal{A}$$

(this is a finite sum).

We shall write $H_k = H \underset{\mathcal{A}}{\otimes} k$, $H_{\bar{k}} = H \underset{\mathcal{A}}{\otimes} \bar{k}$.

1.2. Let I be a subset of S. Let W'^I be the subgroup of W generated by I, $\Omega^I = \{\omega \in \Omega \mid \omega(I) = I\}$, $W^I = \Omega^I \cdot W'^I \subset \Omega \cdot W' = W$.

Then the previous definitions are applicable for W^I instead of W. In particular, to W^I corresponds a Hecke algebra H^I. It can be identified with the subalgebra of H, spanned as an $\mathcal{A}$-module by $\tilde{T}_w$, $(w \in W^I)$. For $w \in W^I$, the elements C_w defined in terms of W^I or W coincide. We have $H_k^I \subset H_k$, $H_{\bar{k}}^I \subset H_{\bar{k}}$.

1.3. From now on, we assume that (W', S) is a finite of affine Weyl group. A general reference for 1.3-1.4 is [13], [14].

There are well defined functions $a : W \to N$, $\gamma : W \times W \times W \to N$ such that

$$v^{a(z)} h_{x,y,z} - \gamma_{x,y,z^{-1}} \in v\boldsymbol{Z}[v] \qquad \text{for all} \quad x, y, z \in W$$

and such that for any $z \in W$ there exists $x, y \in W$ with $\gamma_{x,y,z} \neq 0$.

Let $\mathcal{D} = \{w \in W' \mid 2 \deg P_{e,w} = l(w) - a(w)\}$. Then $\mathcal{D}$ is a finite set of involutions in W'. Any left cell of W contains a unique element of $\mathcal{D}$.

Let J be the $\boldsymbol{C}$-vector space with basis $(t_w)_{w \in W}$. It has a unique structure of associative $\boldsymbol{C}$-algebra such that $t_x t_y = \sum_{z \in W} \gamma_{x,y,z} t_{z^{-1}}$. It has a unit element $\sum_{d \in \mathcal{D}} t_d$.

For any subset $Z \subset W$, let J_Z be the subspace of J spanned by t_w $(w \in Z)$. Then J is a direct sum $\bigoplus_{\underline{c}} J_{\underline{c}}$ where $\underline{c}$ runs over the set of two-sided cells of W. Each $J_{\underline{c}}$ is a subalgebra of J (with a different unit element: $\sum_{d \in \mathcal{D} \cap \underline{c}} t_d$) and $J_{\underline{c}} \cdot J_{\underline{c}'} = 0$ for $\underline{c} \neq \underline{c}'$, so the last direct sum is a direct sum of algebras. For any left cell Γ of W, $J_{\Gamma \cap \Gamma^{-1}}$ is a subalgebra of J with unit element t_d where d is the unique element of $\Gamma \cap \mathcal{D}$.

If $\underline{c}$ is a two-sided cell of W, the restriction of $a : W \to N$ to $\underline{c}$ is constant; we set $a(\underline{c}) = a(w)$ where w is any element of $\underline{c}$.

1.4. We have a homomorphism of $\mathcal{A}$-algebras with 1, $\Phi : H \to J \otimes \mathcal{A}$ defined by $\Phi(C_w) = \sum_{\substack{d \in \mathcal{D} \\ z \in W \\ d \underset{L}{\sim} z}} h_{w,d,z} t_z$. It induces a homomorphism of k-algebras $\Phi : H_k \to J \otimes k$ and, by specializing v to 1, a homomorphism of $\boldsymbol{C}$-algebras

$\Phi_1: C[W] \to J$. For any J-module E, we can consider $E \otimes_C k$ as a $J \otimes k$-module in the obvious way, hence as an H_k-module with $h \in H_k$ acting as $\Phi(h)$; this H_k-module is denoted ${}^{\Phi}E$. When W' is finite, $\Phi: H_k \to J \otimes k$, $\Phi_1: C[W] \to J$ are isomorphisms.

1.5. Let I be as in 1.2. Let J^I be the C-algebra defined in the same way as J, replacing W by W^I. Let $a^I, \gamma^I, \mathcal{D}^I$ be defined as $a, \gamma, \mathcal{D}$ in 1.3 for W^I instead of W. As in [14, 1.9] we see that a^I is equal to the restriction of a to W^I. If $x, y, z \in W^I$, then $h_{x,y,z}$ computed in terms of W and W^I coincide; hence $\gamma^I_{x,y,z} = \gamma_{x,y,z}$. Thus J^I may be identified with the subalgebra J_{W^I} of J. We have $\mathcal{D}^I = \mathcal{D} \cap W^I$ hence the unit element of J^I is $\sum_{d \in \mathcal{D} \cap W^I} t_d$.

1.6. Let G be a connected reductive algebraic group over C with Lie algebra $\mathfrak{g}$. We fix a Borel subgroup B_0 of G and a maximal torus T_0 of B_0; let $\pi_0: B_0 \to T_0$ be the canonical projection. Let X be the group of characters $T_0 \to C^*$, written additively; let $R \subset X$ be the set of roots, R^- the set of roots such that the corresponding root subgroup is contained in B_0, $R^+ = R - R^-$, $\Pi \subset R^+$ the corresponding set of simple roots. For $\alpha \in R$, let $\check{\alpha}: X \to Z$ be the corresponding coroot. Let X' be the subgroup of X generated by R. Let (W_0, S_0) be the Weyl group of G with respect to T_0, with simple reflections S_0 corresponding to the simple roots. W_0 acts naturally on X and we form the semidirect product $W = W_0 \cdot X$ (with X normal); the multiplication is given $(wx)(w'x') = (ww')(w'^{-1}(x) + x')$, for $w, w' \in W_0$, $x, x' \in X$. Following Iwahori and Matsumoto, we define $l: W \to N$ by $l(wx) = \sum_{\substack{\alpha \in R^+ \\ w(\alpha) \in R^+}} |\check{\alpha}(x)| + \sum_{\substack{\alpha \in R^+ \\ w(\alpha) \in R^-}} |\check{\alpha}(x) + 1|$, $(w \in W_0,\ x \in X)$. Let $W' = W_0 \cdot X' \subset W$, $\Omega = \{w \in W \mid l(w) = 0\}$, $S = \{w \in W' \mid l(w) = 1\}$. Then Ω is an abelian subgroup of W normalizing both W' and S, $W = \Omega \cdot W'$ (semidirect product) and (W', S) is a Coxeter group (an affine Weyl group). Hence the definitions in 1.1–1.4 are applicable to W. In the rest of the paper, W will be as just defined and $H, J, \cdots$ of 1.1–1.4 will refer to this W. We have $S_0 \subset S$.

1.7. Let $\mathcal{B}$ be the variety of Borel subgroups of G.

G acts on $\mathcal{B}$ transitively by conjugation. For any $x \in X$ there is a G-equivariant line bundle L_x on $\mathcal{B}$ (unique up to isomorphism) with the following property: B_0 acts on the fibre of L_x at B_0 via the character $b_0 \to x(\pi_0(b_0))$. We shall use the notation $X_{\text{dom}} = \{x \in X \mid \check{\alpha}(x) \geq 0 \text{ for all } \alpha \in \Pi\}$.

If $u \in G$ is unipotent and $s \in G$ is semisimple, we denote $\mathcal{B}_u = \{B \in \mathcal{B} \mid u \in B\}$, $\mathcal{B}_u^s = \{B \in \mathcal{B} \mid u \in B, s \in B\}$.

1.8. Let $h \to {}^*h$ be the unique automorphism of the $\mathcal{A}$-algebra H such that ${}^*\tilde{T}_r = -\tilde{T}_r^{-1}$ $(r \in S_0)$, ${}^*\tilde{T}_x = \tilde{T}_x^{-1}$ $(x \in X_{\mathrm{dom}})$. This extends to a k-algebra automorphism of H_k denoted in the same way. If M is an H- (or H_k-)module, then composition with $*$ gives a new module ${}^*M$ (with the same underlying space as M).

2. H-modules

2.1. In this and the following section we shall study certain H-modules using the methods of [6]. Note that in [6], G was assumed to have simply connected derived group; analogous results hold without this hypothesis although some of the proofs in [6] need to be modified (we will return to this question elsewhere). We shall refer to [6] for results on general G, even though in [6] these results are proved only under the assumption above.

2.2. Let u be a unipotent element of G and let s be a semisimple element of G such that $su = us$. Let $\langle s \rangle$ be the smallest closed diagonalizable subgroup of G containing s.

We can choose a homomorphism of algebraic groups $\phi : SL_2(\boldsymbol{C}) \to Z_G^0(s)$ such that $\phi\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = u$. For each $\lambda \in \boldsymbol{C}^*$ we set $s_\lambda = \phi\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in Z_G^0(s)$. We define an action of $\langle s \rangle \times \boldsymbol{C}^*$ on $\mathcal{B}_u$ by $(g_1, \lambda) : B \to g_1 s_\lambda B s_\lambda^{-1} g_1^{-1}$, $(g_1 \in \langle s \rangle$, $\lambda \in \boldsymbol{C}^*)$. (Note that $u \in B \Longleftrightarrow u^\lambda \in B$ for $\lambda \in \boldsymbol{C}^*$.)

We consider $K_0^{\langle s \rangle \times \boldsymbol{C}^*}(\mathcal{B}_u) \otimes \boldsymbol{C}$: equivariant K-homology as in [6], or equivariant K-theory of coherent sheaves as in [17]. (As pointed out in [17], the two kinds of K-theory coincide in our case.) It is naturally a module over $\underline{R}_{\langle s \rangle \times \boldsymbol{C}^*}$, the representation ring of $\langle s \rangle \times \boldsymbol{C}^*$, tensored with $\boldsymbol{C}$, i.e., the algebra of regular functions $\langle s \rangle \times \boldsymbol{C}^* \to \boldsymbol{C}$. This algebra can be mapped onto the algebra $\mathcal{A}$ of regular functions $\boldsymbol{C}^* \to \boldsymbol{C}$ as follows: to a regular function $f : \langle s \rangle \times \boldsymbol{C}^* \to \boldsymbol{C}$ we associate the regular function $f' : \boldsymbol{C}^* \to \boldsymbol{C}$, $f'(\lambda) = f(s, \lambda)$. Therefore we can form the tensor product

$$\mathcal{K}_{u,s} = (K_0^{\langle s \rangle \times \boldsymbol{C}^*}(\mathcal{B}_u) \otimes \boldsymbol{C}) \underset{\underline{R}_{\langle s \rangle \times \boldsymbol{C}^*}}{\otimes} \mathcal{A}.$$

This is a finitely generated projective $\mathcal{A}$-module [6, 5.11].

2.3. For any $r \in S_0$ we consider the $\mathcal{A}$-linear operator $\tau^r : \mathcal{K}_{u,s} \to \mathcal{K}_{u,s}$

defined in [6, 3.2(d)]. (It is more pleasant to use the definition of τ^r in terms of coherent sheaves, as in [17, 6].) For any $x \in X_{\mathrm{dom}}$, we consider the corresponding G-equivariant line bundle L_x on $\mathcal{B}$. Its restriction to $\mathcal{B}_u$ is an $\langle s\rangle \times \boldsymbol{C}^*$-equivariant vector bundle via the homomorphism $\langle s\rangle \times \boldsymbol{C}^* \to G$, $(g_1, \lambda) \to g_1 \cdot s_\lambda$. Tensor product with this line bundle defines an $\mathcal{A}$-linear operator $\theta_x : \mathcal{K}_{u,s} \to \mathcal{K}_{u,s}$.

2.4. There is a unique H-module structure on $\mathcal{K}_{u,s}$ extending the natural $\mathcal{A}$-module structure such that

$$\tilde{T}_r \text{ acts as } v^{-1}(\tau^r - 1), \qquad (r \in S_0)$$
$$\tilde{T}_x \text{ acts as } \theta_x^{-1}, \qquad (x \in X_{\mathrm{dom}}).$$

(This follows from [6, 5.11] where, however, the module structure considered is ${}^*\mathcal{K}_{u,s}$.)

2.5. The finite group $A = Z_G(su)/Z_G^0(su)$ acts naturally on $K_0^{\langle s\rangle \times \boldsymbol{C}^*}(\mathcal{B}_u) \otimes \boldsymbol{C}$ by $\underline{R}_{\langle s\rangle \times \boldsymbol{C}^*}$-linear automorphisms as in [6, 1.3(j)]. (Note that A can be identified with the group of components of the simultaneous centralizer $\tilde{A}$ of s and $\phi(SL_2(\boldsymbol{R}))$; $\tilde{A}$ acts by conjugation on $\mathcal{B}_u$, commuting with the action of $\langle s\rangle \times \boldsymbol{C}^*$.) This induces an action of A on $\mathcal{K}_{u,s}$ by $\mathcal{A}$-linear automorphisms which commute with the H-module structure.

Let ρ be any finite dimensional $\boldsymbol{C}[A]$-module and let ρ^* be its dual. We define $\mathcal{K}_{u,s,\rho} = \mathrm{Hom}_A(\rho, \mathcal{K}_{u,s})$. It is clear that $\mathcal{K}_{u,s,\rho}$ is a finitely generated projective $\mathcal{A}$-module and it inherits an H-module structure from $\mathcal{K}_{u,s}$.

For any $\lambda \in \boldsymbol{C}^*$, let $\boldsymbol{C}_\lambda$ be the $\mathcal{A}$-module $\boldsymbol{C}$ with v acting as multiplication by λ. Let $\mathcal{K}_{u,s,\rho}^\lambda = \mathcal{K}_{u,s,\rho} \underset{\mathcal{A}}{\otimes} \boldsymbol{C}_\lambda$. This is a finite dimensional $\boldsymbol{C}$-vector space and an H-module in a natural way.

2.6. LEMMA. *Consider the action of $\boldsymbol{C}^*$ on $\mathcal{B}$ defined by $\lambda : B \to s_\lambda B s_\lambda^{-1}$ and let $\mathcal{B}^{\boldsymbol{C}^*}$ be its fixed point set.*

(a) *For any connected component Y of $\mathcal{B}^{\boldsymbol{C}^*}$ there exists a homomorphism $m_Y : X \to \boldsymbol{Z}$ with the following property: for any $\gamma \in G$ such that $\gamma^{-1} B_0 \gamma \in Y$, any $\lambda \in \boldsymbol{C}^*$ and any $x \in X$, we have $x(\pi_0(\gamma s_\lambda \gamma^{-1})) = \lambda^{-m_Y(x)}$.*

If Y_1, Y_2 are two connected components of $\mathcal{B}^{\boldsymbol{C}^*}$, we have $m_{Y_1} = m_{Y_2}$ if and only if $Y_1 = Y_2$.

(b) *If Y in* (a) *meets $\mathcal{B}_u$ then $m_Y(x) \geq 0$ for all $x \in X_{\mathrm{dom}}$.*

PROOF. Let L be the centralizer in G of the torus $\{s_\lambda \mid \lambda \in \boldsymbol{C}^*\}$. The connected components of $\mathcal{B}^{C^*}$ are exactly the L-orbits on $\mathcal{B}$, hence the B_0-L double coset of γ in (a) is uniquely determined by Y; from this, (a) follows immediately. The proof of (b) is based on an argument in [10, 2.8]. Let $x\in X_{\mathrm{dom}}$, let $\gamma\in G$ be such that $B=\gamma^{-1}B_0\gamma\in Y\cap\mathcal{B}_u$. We can find an irreducible G-module V with a B-stable vector $\xi\neq 0$ such that $b\xi=x(\pi_0(\gamma b\gamma^{-1}))^{-1}\xi$ for all $b\in B$. We can regard V as an $SL_2(C)$-module, via $\phi: SL_2(C)\to G$. By representation theory of $SL_2(C)$, we can decompose $V=\bigoplus_{i\in Z} V_i$, where $V_i=\{\eta\in V\,|\,s_\lambda\eta=\lambda^i\eta,\ \forall\lambda\in C^*\}$ and the 1-eigenspace of $\phi\begin{bmatrix}1&1\\0&1\end{bmatrix}: V\to V$ is contained in $\bigoplus_{i\geq 0} V_i$. Since ξ is contained in this 1-eigenspace, there exists $i\geq 0$ such that $s_\lambda\xi=x(\pi_0(\gamma s_\lambda\gamma^{-1}))^{-1}\xi=\lambda^i\xi$ for all $\lambda\in C^*$. We have $x(\pi_0(\gamma s_\lambda\gamma^{-1}))=\lambda^{-i}$ for all $\lambda\in C^*$ and (b) follows.

2.7. We now define a decomposition $\mathcal{K}_{u,s}\otimes_{\mathcal{A}} k=\bigoplus_Y \mathcal{K}^Y$ as a direct sum of k-subspaces, stable under all θ_x, $(x\in X_{\mathrm{dom}})$. Consider the $\mathcal{A}$-module

$$\mathcal{K}'=(K_0^{\langle s\rangle\times C^*}(\mathcal{B}_u^s\cap\mathcal{B}^{C^*})\underset{\underline{R}\langle s\rangle\times C^*}{\otimes}\mathcal{A}.$$

This is a finitely generated $\mathcal{A}$-module and it has $\mathcal{A}$-linear operators $\theta_x:\mathcal{K}'\to\mathcal{K}'$ $(x\in X_{\mathrm{dom}})$ defined in the same way as for $\mathcal{K}_{u,s}$ (see 2.3). The inclusion $\mathcal{B}_u^s\cap\mathcal{B}^{C^*}\to\mathcal{B}_u$ defines an $\mathcal{A}$-linear map $\mathcal{K}'\to\mathcal{K}_{u,s}$ compatible with the operators θ_x. From the localization theorem in equivariant K-theory it follows that this is an isomorphism over k. Now the decomposition of $\mathcal{B}_u^s\cap\mathcal{B}^{C^*}$ in connected components defines a decomposition of $\mathcal{K}'$ into a direct sum of $\mathcal{A}$-submodules. Using the isomorphism $\mathcal{K}'\otimes_{\mathcal{A}} k\xrightarrow{\approx}\mathcal{K}_{u,s}\otimes_{\mathcal{A}} k$ we obtain a decomposition of $\mathcal{K}_{u,s}\otimes_{\mathcal{A}} k$ as a direct sum of k-subspaces $Z^{Y'}$ indexed by the connected components Y' of $\mathcal{B}_u^s\cap\mathcal{B}^{C^*}$. Each summand $Z^{Y'}$ is stable under the operators θ_x $(x\in X_{\mathrm{dom}})$. From the definitions it follows that θ_x acts on $Z^{Y'}$ as $h_{Y'}(x)$ times a unipotent transformation, where $h_{Y'}: X\to k^*$ is a certain homomorphism which can be described as follows.

Let $B\in Y'$ and let $\gamma\in G$ be such that $\gamma B\gamma^{-1}=B_0$. Then $h_{Y'}(x)$ is the rational function on C^*:

$$\lambda\longrightarrow x(\pi_0(\gamma ss_\lambda\gamma^{-1}))=\zeta_{Y'}(x)\lambda^{-m_Y(x)}$$

where $\zeta_{Y'}(x)=x(\pi_0(\gamma s\gamma^{-1}))\in C^*$ is independent of λ, Y is the connected component of $\mathcal{B}^{C^*}$ containing Y' and $m_Y(x)$ is as in 2.6 (a). Hence we have

(a) $h_{Y'}(x)=\zeta_{Y'}(x)v^{-m_Y(x)}\in k^*$, with $\zeta_{Y'}(x)\in C^*$.

2.8. LEMMA. *Let ρ be a simple A-module. The following conditions are equivalent.*

(a) $\mathcal{K}_{u,s,\rho}\neq 0$.

(b) $\mathcal{K}_{u,s,\rho}\otimes_{\mathcal{A}}\bar{k}$ *is a simple $H_{\bar{k}}$-module.*

(c) ρ *appears in the natural representation of A on $H^*(\mathcal{B}_u^s, \boldsymbol{C})$.*

(d) ρ *appears in the natural representation of A on $H^*(\mathcal{B}_u^s\cap\mathcal{B}^{\boldsymbol{C}^*}, \boldsymbol{C})$.*

PROOF. Let $\mathcal{O}$ be the set of all $\lambda\in\boldsymbol{C}^*$ such that the eigenspaces of $\mathrm{Ad}(ss_\lambda):\mathfrak{g}\to\mathfrak{g}$ coincide with the simultaneous eigenspaces of the family of automorphisms $\{\mathrm{Ad}(s), \mathrm{Ad}(s_\nu)(\nu\in\boldsymbol{C}^*)\}$ of $\mathfrak{g}$. It is clear that $\mathcal{O}$ is the complement of a finite set in $\boldsymbol{C}^*$. Fix some $\lambda\in\mathcal{O}$. Then $\mathcal{B}_u^s\cap\mathcal{B}^{\boldsymbol{C}^*}=\mathcal{B}_u^{ss_\lambda}$, $Z_G^0(ss_\lambda)=Z_L^0(s)$, ($L$ as in 2.6), and the λ^2-eigenspace of $\mathrm{Ad}(ss_\lambda):\mathfrak{g}\to\mathfrak{g}$ coincides with the set Σ of vectors ξ in the Lie algebra of $Z_G^0(s)$ such that $\mathrm{Ad}(s_\nu)\xi=\nu^2\xi$ for all $\nu\in\boldsymbol{C}^*$. It is well known that $Z_L^0(s)$ has an open orbit on Σ and that orbit contains the nilpotent element $\log u$. It follows that $\log u$ is contained in the open orbit of $Z_G^0(ss_\lambda)$ on the λ^2-eigenspace of $\mathrm{Ad}(ss_\lambda):\mathfrak{g}\to\mathfrak{g}$. Using now [6, 5.12, 5.15(a)] we see that the following three conditions are equivalent:

(e) $\mathcal{K}_{u,s,\rho}^\lambda\neq 0$

(f) $\mathcal{K}_{u,s,\rho}^\lambda$ is a simple H-module

(g) ρ appears in the natural representation of $Z_G(ss_\lambda)\cap Z_G(u)/(Z_G(ss_\lambda)\cap Z_G(u))^0=Z_G(su)/Z_G^0(su)=\tilde{A}/\tilde{A}^0$ on $H^*(\mathcal{B}_u^{ss_\lambda}, \boldsymbol{C})$, ($\tilde{A}$ as in 2.5). As we have seen, we have (g)$\Longleftrightarrow$(d). Since $\mathcal{K}_{u,s,\rho}$ is a projective $\mathcal{A}$-module, we have (a)$\Longleftrightarrow$(e). We obviously have (b)$\Longrightarrow$(a). Hence it remains to prove that (f)$\Longrightarrow$(b) and (c)$\Longleftrightarrow$(d).

Now $\mathcal{B}_u^s\cap\mathcal{B}^{\boldsymbol{C}^*}$ is the fixed point set of a $\boldsymbol{C}^*$-action on $\mathcal{B}_u^s$ which commutes with the action of $\tilde{A}$. It is known (see [7, 1.6]) that this implies the equality $\sum(-1)^iH^i(\mathcal{B}_u^s, \boldsymbol{C})=\sum(-1)^iH^i(\mathcal{B}_u^s\cap\mathcal{B}^{\boldsymbol{C}^*}, \boldsymbol{C})$ as virtual representations of $\tilde{A}/\tilde{A}^0$. On the other hand the cohomology groups of $\mathcal{B}_u^s$, $\mathcal{B}_u^s\cap\mathcal{B}^{\boldsymbol{C}^*}=\mathcal{B}_u^{ss_\lambda}$ vanish in odd degrees [6, 4.1]. The equivalence of (c) and (d) follows.

The proof of (f)$\Longrightarrow$(b) is a standard application of Burnside's theorem since $\mathcal{K}_{u,s,\rho}^\lambda$ is a specialization of $\mathcal{K}_{u,s,\rho}\otimes_{\mathcal{A}}\bar{k}$. This completes the proof.

An A-module ρ satisfying the conditions of the lemma is said to be *admissible.*

2.9. Let $V:\bar{k}^*\to\boldsymbol{R}$ be a homomorphism such that $V(v)=1$, $V(av+b)=0$ for $a\in\boldsymbol{C}$, $b\in\boldsymbol{C}^*$. Such a homomorphism exists; we shall fix one. An $H_{\bar{k}}$-module M of finite dimension over $\bar{k}$ is said to be *V-tempered* if all the eigenvalues ζ of $\tilde{T}_x : M \to M$ satisfy $V(\zeta) \leq 0$ for all $x \in X_{\text{dom}}$.

2.10. Let M be an $H_{\bar{k}}$-module of finite dimension over $\bar{k}$. An element $\xi \in M$ is said to be an *eigenvector of* M if $\xi \neq 0$ and if there exists a homomorphism $\chi_\xi : X \to \bar{k}^*$ such that $\tilde{T}_x \xi = \chi_\xi(x)\xi$ for all $x \in X_{\text{dom}}$. For such ξ there is a unique element $\sigma_\xi \in T_0(\bar{k})$ such that $\chi_\xi(x) = x(\sigma_\xi)$ for all $x \in X = \text{Hom}(T_0, C^*) = \text{Hom}(T_0(\bar{k}), \bar{k}^*)$. We say that M is of *constant type* if there exists a semisimple element $s' \in G$ and a homomorphism of algebraic groups $\phi' : SL_2(C) \to Z_G^0(s')$ such that for any eigenvector ξ of M, $\sigma_\xi \in T_0(\bar{k})$ is conjugate in $G(\bar{k})$ to $\phi'\begin{bmatrix} v & 0 \\ 0 & v^{-1} \end{bmatrix} s'$. (Note that ϕ' extends canonically to a homomorphism $\phi' : SL_2(\bar{k}) \to Z_G^0(s')(\bar{k})$ hence $\phi'\begin{bmatrix} v & 0 \\ 0 & v^{-1} \end{bmatrix} \in Z_G^0(s')(\bar{k})$ is well defined; on the other hand, $s' \in G$ is a "constant element".)

2.11. PROPOSITION. *Let Ψ' be the set of G-conjugacy classes of triples (u, s, ρ) where u (resp. s) is a unipotent (resp. semisimple) element of G, $su = us$, and ρ is a simple, admissible module for $A = Z_G(su)/Z_G^0(su)$. Let Ψ'' be the set of isomorphism classes of H_k-modules M' such that ${}^*M' \otimes \bar{k}$ is a simple V-tempered $H_{\bar{k}}$-module of constant type. The correspondence $(u, s, \rho) \to \mathcal{K}_{u,s,\rho} \otimes_{\mathcal{A}} k$ defines a bijection $\Psi' \xrightarrow{\approx} \Psi''$.*

PROOF. Let $(u, s, \rho) \in \Psi'$. By 2.8, the $H_{\bar{k}}$-module ${}^*\mathcal{K}_{u,s,\rho} \otimes_{\mathcal{A}} \bar{k}$ is simple. It is V-tempered, of constant type, since it is a direct summand of ${}^*\mathcal{K}_{u,s} \otimes_{\mathcal{A}} \bar{k}$ which is V-tempered, of constant type by 2.7 (a), 2.6 (b). Now let (u, s, ρ), (u', s', ρ') be distinct elements of Ψ'. We associate to (u, s) a subset $\mathcal{O} \subset C^*$ as in 2.8 and similarly we associate to (u', s') a subset $\mathcal{O}' \subset C^*$. Then $\mathcal{O} \cap \mathcal{O}'$ is open, dense in C^*. If $\lambda \in \mathcal{O} \cap \mathcal{O}'$, then ${}^*\mathcal{K}_{u,s,\rho}^{\lambda}$, ${}^*\mathcal{K}_{u',s',\rho'}^{\lambda}$ are simple standard H-modules in the sense of [6, 5.12] and are not isomorphic by [6, 5.15(b)]. It follows immediately that the $H_{\bar{k}}$-modules ${}^*\mathcal{K}_{u,s,\rho} \otimes_{\mathcal{A}} \bar{k}$, ${}^*\mathcal{K}_{u',s',\rho'} \otimes_{\mathcal{A}} \bar{k}$ are not isomorphic. Hence our map $\Psi' \to \Psi''$ is injective. We now fix a semisimple element $s' \in G$ and a homomorphism of algebraic groups $\phi' : SL_2(C) \to Z_G^0(s')$. Let $\Psi''_{s',\phi'}$ be the subset of Ψ'' consisting of those $M \in \Psi''$ such that, with the notations of 2.10, all elements σ_ξ (ξ an eigenvector of M) are $G(\bar{k})$-conjugate to $\sigma_0 = \phi'\begin{bmatrix} v & 0 \\ 0 & v^{-1} \end{bmatrix} s'$. Let $u' = \phi'\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and let N' be the number of simple $Z_G(s'u')/Z_G^0(s'u')$-modules ρ' (up to isomorphism) which appear in $H^*(\mathcal{B}_{u'}^{s'}, C)$. By the first part of the proof, $\Psi''_{s',\phi'}$ contains at least N' elements. We now use the classification of simple V-tempered $H_{\bar{k}}$-modules given by [6, 7.12, 8.2]. (Those results are applicable since $\bar{k}$ is (non-canonically) isomorphic to $\boldsymbol{C}$.) We deduce that the number of elements of $\Psi''_{s',\phi'}$ is at most the number of $G(\bar{k})$-conjugacy classes of triples (u_1, σ_1, ρ_1) where σ_1 is a semisimple element of $G(\bar{k})$, conjugate to σ_0, u_1 is a unipotent element of $G(\bar{k})$ such that $\log(u')$ is in the dense orbit of $Z^0_{G(\bar{k})}(\sigma_1)$ on the v^2-eigenspace of $\mathrm{Ad}(\sigma_1): \mathfrak{g}\otimes\bar{k}\to\mathfrak{g}\otimes\bar{k}$ and ρ_1 is a simple module of the component group of $Z_{G(\bar{k})}(\sigma_1)\cap Z_{G(\bar{k})}(u_1)$, which is admissible (relative to $\bar{k}$). The number of such triples (up to conjugacy) is N'. Thus $\Psi''_{s',\phi'}$ contains at most N' elements. It follows that $\Psi'\to\Psi''$ is bijective.

2.12. PROPOSITION. *Let $M'\in\Psi''$ and let (u, s, ρ) be the corresponding element of Ψ'. Let $\chi: Z_{\bar{k}}\to\bar{k}$ be the character by which the centre $Z_{\bar{k}}$ of $H_{\bar{k}}$ acts on the $H_{\bar{k}}$-module $M'\otimes\bar{k}$. Then the conjugacy class of u in G depends only on χ, not on M'.*

PROOF. It is known that the $\bar{k}$-algebra homomorphisms $Z_{\bar{k}}\to\bar{k}$ are in canonical bijection with the semisimple conjugacy classes in $G(\bar{k})$. In our case χ corresponds to the conjugacy class of $\tilde{s}=\phi\begin{bmatrix} v & 0 \\ 0 & v^{-1}\end{bmatrix}^{-1}s^{-1}\in G(\bar{k})$, where ϕ denotes a homomorphism of algebraic groups $SL_2(\boldsymbol{C})\to Z^0_G(s)$, $\phi\begin{bmatrix} 1 & 1 \\ 0 & 1\end{bmatrix}=u$, as well as its extension to $\bar{k}$-rational points. Now $\log(u)\in\mathfrak{g}$ is known to belong to the open orbit of $Z^0_{G(\bar{k})}(\tilde{s})$ on the v^2-eigenspace of $\mathrm{Ad}(\tilde{s}^{-1})$. Hence the $G(\bar{k})$-conjugacy class of u is completely determined by χ. The proposition follows.

3. Deformations of H-modules

3.1. Let $\phi: Sl_2(\boldsymbol{C})\to G$ be a homomorphism of algebraic groups. Let F be the centralizer in G of the image of ϕ; then F is a reductive (not necessarily connected) algebraic group. Let F^1 be a connected component of F. We define a closed subgroup D of F as follows: we choose a semisimple element $s_0\in F^1$, let $\mathcal{T}$ be a maximal torus of $Z^0_F(s_0)$ and let D be the subgroup of F generated by $\mathcal{T}$ and s_0. Then:

(a) D is diagonalizable; it is generated by its connected component $D^1=D\cap F^1=s_0\mathcal{T}=\mathcal{T}s_0$.

(b) Any semisimple element in F^1 is conjugate under F^0 to an element of D^1.

(c) The subset $D^1_{\mathrm{reg}}=\{s\in D^1 \mid Z_F(s)=Z_F(D)\}$ is open dense in D^1.

For any $s\in D^1$, we have a commutative diagram of finite groups:

$$\text{(d)}\quad \begin{array}{ccc} N_F(D^1)/D^0 & \longleftarrow & (N_F(D^1)\cap Z_F(s))/D^0 \\ \uparrow & \nearrow & \downarrow p \\ Z_F(D)/D^0 & \longrightarrow & Z_F(s)/Z_F^0(s) \end{array}$$

with all the maps (=obvious ones) being injective except possibly for p which is surjective.

3.2. Let $u=\phi\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $s_\lambda=\phi\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$, $\lambda\in C^*$. Then $D\times C^*$ acts on $\mathcal{B}_u$ by $(d,\lambda):B\to ds_\lambda Bs_\lambda^{-1}d^{-1}$. We consider as in 2.2, $K_0^{D\times C^*}(\mathcal{B}_u)\otimes C$ as a module over $\underline{R}_{D\times C^*}$ (the representation ring of $D\times C^*$ tensored with C) which may be identified canonically with the algebra of regular functions $D\times C^*\to C$. This algebra can be mapped (by restriction) onto the algebra $\underline{R}^1$ of regular functions $D^1\times C^*\to C$, hence we can form the tensor product $\mathcal{K}=(K_0^{D\times C^*}(\mathcal{B}_u)\otimes C)\underset{\underline{R}_{D\times C^*}}{\otimes}\underline{R}^1$. This is a finitely generated projective $\underline{R}^1$-module [6, 5.11] and one can define $\underline{R}^1$-linear operators $\tau^r:\mathcal{K}\to\mathcal{K}$ $(r\in S_0)$, $\theta_x:\mathcal{K}\to\mathcal{K}$ $(x\in X_{\text{dom}})$ as in 2.3, see [6, 3.2(d)]. These define an H-module structure on $\mathcal{K}$ by the same formulas as in 2.4, (see [6, 5.11]).

We can regard $\mathcal{K}$ as the space of sections of an algebraic vector bundle K over $D^1\times C^*$. The operators of H act on K by vector bundle endomorphisms (inducing identity on the base $D^1\times C^*$). Now the finite group $N_F(D^1)/D^0$ acts on K by vector bundle maps as in [6, 1.3(j)] and at the same time it acts on the base $D^1\times C^*$ by conjugation on the first factor. (Thus K is an $N_F(D^1)/D^0$-equivariant vector bundle.) The restriction of this action to the subgroup $Z_F(D)/D^0$ preserves each fibre of K (i.e. $Z_F(D)/D^0$ acts trivially on $D^1\times C^*$). Hence, for any simple $Z_F(D)/D^0$-module $\mathcal{M}$ we can form the vector bundle $K_{\mathcal{M}}=\mathrm{Hom}_{Z_F(D)/D^0}(\mathcal{M},K)$ over $D^1\times C^*$, with space of sections $\mathcal{K}_{\mathcal{M}}=\mathrm{Hom}_{Z_F(D)/D^0}(\mathcal{M},\mathcal{K})$ which is a finitely generated projective $\underline{R}^1$-module with an H-module structure inherited from $\mathcal{K}$.

3.3. Now let $s\in D^1$ and let K_s be the restriction of K to $\{s\}\times C^*$. The restriction of the action of $N_Z(D^1)/D^0$ to the subgroup $(N_F(D^1)\cap Z_F(s))/D^0$ maps each fibre of K_s into itself; moreover, this action of $(N_F(D^1)\cap Z_F(s))/D^0$ on K_s factors through the quotient group $Z_F(s)/Z_F^0(s)$. Using the diagram 2.2 (d), we see that the restriction of the action of $Z_F(D^1)/D^0$ from K to

K_s coincides with the restriction of the action of $Z_F(s)/Z_F^0(s)$ (on K_s) to the subgroup $Z_F(D^1)/D^0$. Hence, if $\mathcal{M}$ is as in 3.2, the restriction of the vector bundle $K_{\mathcal{M}}$ to $\{s\}\times \boldsymbol{C}^*$ is the vector bundle $K_{s,i_s(\mathcal{M})}=\mathrm{Hom}_{Z_F(S)/Z_F^0(s)}(i_s(\mathcal{M}), K_s)$ where $i_s(\mathcal{M})$ is the $Z_F(s)/Z_F^0(s)$-module induced by $\mathcal{M}$. (Recall that $Z_F(D)/D^0 \subset Z_F(s)/Z_F^0(s)$.) On the other hand, by [6, 5.11], the space of sections of $K_{s,i_s(\mathcal{M})}$ is exactly the H-module $\mathcal{K}_{u,s,i_s(\mathcal{M})}$ in 2.5. (Note that $Z_F(s)/Z_F^0(s) \xrightarrow{\approx} Z_G(su)/Z_G^0(su)$.) Using now the fact that $K_{\mathcal{M}}$ is a vector bundle over $D^1\times \boldsymbol{C}^*$, we deduce that:

(a) for any $h\in H$, there exists a regular function $f: D^1\times \boldsymbol{C}^* \to \boldsymbol{C}$ such that $\mathrm{Tr}_{\mathcal{A}}(h, \mathcal{K}_{u,s,i_s(\mathcal{M})})(\lambda)=f(s,\lambda)$ for all $(s,\lambda)\in D^1\times \boldsymbol{C}^*$.

(Recall that $\mathcal{K}_{u,s,i_s(\mathcal{M})}$ is a finitely generated projective $\mathcal{A}$-module, hence the trace of h is defined as an element of $\mathcal{A}$; we consider it as a regular function $\boldsymbol{C}^* \to \boldsymbol{C}$.)

The same argument shows that:

(b) if $h\in H$ acts as zero on $\mathcal{K}_{u,s,i_s(\mathcal{M})}$ for s in an open dense subset of D^1, then it acts as zero on $\mathcal{K}_{u,s,i_s(\mathcal{M})}$ for all $s\in D^1$.

3.4. Assume now that Ω is finite and $I\subset S$ is such that W'^I is finite (hence W^I is finite and H_k^I is a semisimple algebra). Using the fact that $K_{\mathcal{M}}$ (in 3.3) is a vector bundle and using the rigidity of modules over a semisimple algebra we see that

(a) The restriction of the H_k-module $\mathcal{K}_{u,s,i_s(\mathcal{M})}\otimes k$ to H_k^I is independent of s ($s\in D^1$) up to isomorphism.

Now let $s'\in D^1$, $s''\in D^1_{\mathrm{reg}}$; let ρ' be a simple admissible module for $Z_G(s'u)/Z_G^0(s'u)=Z_F(s')/Z_F^0(s')$ and let $\mathcal{M}$ be a simple submodule of the restriction of ρ' to $Z_F(D^1)/D^0$. We denote ρ'' the simple $Z_G(s''u)/Z_G^0(s''u)$-module corresponding to $\mathcal{M}$ under the natural isomorphism $Z_F(D^1)/D^0 \xrightarrow{\sim} Z_F(s'')/Z_F^0(s'')=Z_G(s''u)/Z_G^0(s''u)$.

From (a) we deduce:

(b) The restriction of the H_k-module $\mathcal{K}_{u,s',\rho'}\otimes k$ to H_k^I is isomorphic to an H_k^I-submodule of the restriction of the H_k-module $\mathcal{K}_{u,s'',\rho''}\otimes k$ to H_k^I.

4. J-modules

4.1. Our next result will relate the set of Ψ of isomorphism classes of simple J-modules to the set Ψ' in 2.11.

4.2. THEOREM. *If $E\in\Psi$, then ${}^{\phi}E$ is isomorphic as an H_k-module to $\mathcal{K}_{u,s,\rho}\otimes_{\mathcal{A}} k$ for a well-defined $(u,s,\rho)\in\Psi'$. The correspondence $E\to(u,s,\rho)$*

defines a bijection $\Psi \xrightarrow{\approx} \Psi'$. *Hence for any* $(u, s, \rho) \in \Psi'$, *there is a unique* $E \in \Psi$ *such that* ${}^{\Phi}E \approx \mathcal{K}_{u,s,\rho} \otimes_{\mathcal{A}} k$ *as* H_k*-modules.*

PROOF. Let $E \in \Psi$. According to [15, 2.11(a)] we have ${}^{\Phi}E \in \Psi''$ (notation of 2.11). Note that $M \to {}^{*}M$ defines a bijection $\Psi'' \xrightarrow{\approx} Y'$ (Y' as in [15, 3.3]). Indeed, by 2.11, in any H_k-module M in Ψ'' we can find an H-submodule M_1 such that M_1 is of finite $\mathcal{A}$-rank equal to $\dim_k M$; if $M = \mathcal{K}_{u,s,\rho} \otimes_{\mathcal{A}} k$, take $M_1 = \mathcal{K}_{u,s,\rho}$. According to [15, 3.4], $E \to {}^{*}({}^{\Phi}E)$ defines a bijection $\Psi \xrightarrow{\approx} Y'$. Hence $E \to {}^{\Phi}E$ defines a bijection $\Psi \to \Psi''$. Combining this with the bijection $\Psi' \xrightarrow{\approx} \Psi''$ in 2.11, we obtain the required bijection $\Psi \xrightarrow{\approx} \Psi'$.

4.3. For $(u, s, \rho) \in \Psi'$, we shall denote by $E(u, s, \rho)$ a simple J-module corresponding to (u, s, ρ) as in 4.2.

Let E be a simple J-module. Since J is a direct sum of algebras $\bigoplus J_{\underline{c}}$, there is a unique two-sided cell $\underline{c} = \underline{c}(E)$ of W such that

$$t_w \neq 0 \text{ on } E \text{ for some } w \in \underline{c}$$
$$t_w = 0 \text{ on } E \text{ for all } w \notin \underline{c}.$$

(We then have $\sum_{d \in \mathcal{D} \cap \underline{c}} t_d = \mathrm{Id}$ on E.) We say that $\underline{c}$ *corresponds* to E.

We now define a map

(a) $\Psi' \to$ set of two-sided cells in W

by $(u, s, \rho) \to \underline{c}(u, s, \rho) =$ two-sided cell corresponding (as above) to the simple J-module $E(u, s, \rho)$.

4.4. PROPOSITION. *Let* $M \in \Psi''$.

(a) *There is a unique integer* $\underline{a}(M) \geq 0$ *such that*

$$v^{\underline{a}(M)} \mathrm{Tr}(C_w, M) \in \boldsymbol{C}[v] \quad \textit{for all} \quad w \in W$$
$$v^{\underline{a}(M)-1} \mathrm{Tr}(C_w, M) \notin \boldsymbol{C}[v] \quad \textit{for some} \quad w \in W.$$

(b) *Let* $\gamma_w \in \boldsymbol{C}$ *be the constant term of* $(-v)^{\underline{a}(M)} \mathrm{Tr}(C_w, M) \in \boldsymbol{C}[v]$. *There is a unique two-sided cell* $\underline{c} = \underline{c}(M)$ *of* W *such that* $\gamma_w \neq 0 \Rightarrow w \in \underline{c}$. *We have* $\underline{a}(M) = a(\underline{c})$ *(see* 1.3).

(c) *Let* $E \in \Psi$ *be such that* $M \cong {}^{\Phi}E$ *(see* 4.2). *Then* $\gamma_w = \mathrm{Tr}(t_w, E)$, $(w \in W)$. *Hence* $\underline{c}(M) = \underline{c}(E)$ *(see* 4.3).

(d) *We have* $\sum_{d \in \mathcal{D} \cap \underline{c}} \gamma_d = \dim_k M$.

(e) *If* $a(w) > \underline{a}(M)$, *then* $C_w = 0$ *on* M.

(f) *Let* $(u, s, \rho) \in \Psi'$ *be such that* $M = \mathcal{K}_{u,s,\rho} \otimes k$ *(see* 2.11). *Then*

$\underline{c}(M)=\underline{c}(u, s, \rho)$, (*see* 4.3).

PROOF. Let $\underline{c}=\underline{c}(E)$, (see 4.3). Then (a), (b), (c), (d) follow from [16, 3.3] and (f) follows from the definitions. We now prove (e). Assume that $C_w\neq 0$ on M. Then $\Phi(C_w)\neq 0$ on $E\otimes k$, hence there exist $d\in\mathcal{D}$, $z\in W$ such that $h_{w,d,z}\neq 0$, $t_z\neq 0$ on E. Now $t_z\neq 0$ on E implies $z\in\underline{c}$ and $h_{w,d,z}\neq 0$ implies $z\underset{LR}{\leq} w$, hence $a(z)\geq a(w)$, (see [13]). Thus we have $a(w)\leq a(\underline{c})=\underline{a}(M)$ and (e) follows.

4.5. REMARK. Let I be a subset of S such that W'^I is finite. Let E be a simple J^I-module and let $M={}^{\Phi}E$ be the corresponding H_k^I-module. We can attach to E a two-sided cell $\underline{c}(E)$ of W^I, exactly as in 4.3. The statements 4.4 (a)-(e) remain true for this E, M, and W^I, instead of W, with the same proof.

4.6. PROPOSITION. *Let $\phi, F, F^1\supset D^1, D$ be as in* 3.1, *let $\mathcal{M}$ be a simple $Z_F(D)/D^0$-module and let $u=\phi\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$. Let $s, s'\in D^1$ and let ρ, ρ' be simple admissible modules of $Z_F(s)/Z_F^0(s)=Z_G(su)/Z_G^0(su)$ and $Z_F(s')/Z_F^0(s')=Z_G(s'u)/Z_G^0(s'u)$ respectively, such that both ρ, ρ' restricted to $Z_F(D)/D^0$ contain $\mathcal{M}$. Then $\underline{c}(u, s, \rho)=\underline{c}(u, s', \rho')$.*

PROOF. If $\sigma\in D^1_{\text{reg}}$ (see 3.1 (c)) then $Z_F(D)/D^0=Z_F(\sigma)/Z_F^0(\sigma)$ and $\mathcal{M}$ is clearly an admissible module of $Z_F(\sigma)/Z_F^0(\sigma)=Z_G(\sigma u)/Z_G^0(\sigma u)$. Since $a: W\to N$ is bounded above [13], we can find $\sigma_0\in D^1_{\text{reg}}$ such that $a(\underline{c}(u, \sigma_0, \mathcal{M}))\geq a(\underline{c}(u, \sigma, \mathcal{M}))$ for all $\sigma\in D^1_{\text{reg}}$. Let $\underline{c}_0=\underline{c}(u, \sigma_0, \mathcal{M})$, $a_0=a(\underline{c}_0)$. By 4.4, we then have

(a) $a_0=\underline{a}(\mathcal{K}_{u,\sigma_0,\mathcal{M}}\otimes k)\geq\underline{a}(\mathcal{K}_{u,\sigma,\mathcal{M}}\otimes k)$.

We shall decompose $i_s(\mathcal{M})$ (see 3.3) as a direct sum of simple $Z_F(s)/Z_F^0(s)$-modules $\rho_1, \rho_2, \cdots, \rho_N$ such $\rho_1, \rho_2, \cdots, \rho_{N_1}$ are admissible and $\rho_{N_1+1}, \cdots, \rho_N$ are not admissible. We shall denote $\mathcal{K}^i=\mathcal{K}_{u,\sigma,\rho_i}\underset{\mathcal{A}}{\otimes} k$. Then $\mathcal{K}^i\in\Psi''$ for $i\in[1, N_1]$ and $\mathcal{K}^i=0$ for $i>N_1$. We have $\mathcal{K}_{u,s,i_s(\mathcal{M})}\underset{\mathcal{A}}{\otimes} k=\mathcal{K}^1\oplus\mathcal{K}^2\oplus\cdots\oplus\mathcal{K}^{N_1}$ as H_k-modules. Let $w\in W$ be such that $a(w)>a_0$. Then for any $\sigma\in D^1_{\text{reg}}$ we have $a(w)>\underline{a}(\mathcal{K}_{u,\sigma,\mathcal{M}}\otimes k)$ (see (a)) hence $C_w=0$ on $\mathcal{K}_{u,\sigma,\mathcal{M}}\otimes k$ (see 4.4 (e)). Using 3.3 (b) it follows that $C_w=0$ on $\mathcal{K}_{u,\sigma,i_\sigma(\mathcal{M})}\otimes k$ for all $\sigma\in D^1$ and, in particular, $C_w=0$ on $\mathcal{K}_{u,s,i_s(\mathcal{M})}\otimes k$. Hence $C_w=0$ on $\mathcal{K}^i$ for $i\in[1, N_1]$. Since w was an arbitrary element of W with $a(w)>a_0$ it follows from 4.4 that

(b) $\underline{a}(\mathcal{K}^i)\leq a_0$ for $i\in[1, N_1]$.

Let

(c) $\delta=\dim(\mathcal{K}_{u,\sigma,i_\sigma(\mathcal{M})}\otimes k)$ (=independent of $\sigma\in D^1$).

We set $h=\sum_{d\in\mathcal{D}\cap c_0} C_d\in H$ and write

(d) $(-v)^{a_0}\mathrm{Tr}(h, \mathcal{K}_{u,\sigma,i_\sigma(\mathcal{M})}\otimes k)=\sum_i \mu_i(\sigma)v^i$, $(\sigma\in D^1)$.

Then $\sigma\to\mu_i(\sigma)$ is a regular function $D^1\to \boldsymbol{C}$, for any integer i (see 3.3 (a)).

We have $\mu_0(\sigma_0)=\delta\neq 0$, see (c). Hence $\mu_0(\sigma)\neq 0$ for σ in an open dense subset Z of D^1_{reg}. From 4.4 it follows that $\underline{a}(\mathcal{K}_{u,\sigma,\mathcal{M}}\otimes k)\geq a_0$ $(\sigma\in Z)$. Combining this with (a), we deduce that $\underline{a}(\mathcal{K}_{u,\sigma,\mathcal{M}}\otimes k)=a_0$ for all $\sigma\in Z$. Using 4.4 (d) we then have $\mu_0(\sigma)=\delta$ for all $\sigma\in Z$. Thus, the regular function $\sigma\to\mu_0(\sigma)$ on D^1 is the constant δ on Z hence it is constant on D^1. In particular,

(e) $\mu_0(s)=\delta=\delta_1+\delta_2+\cdots+\delta_{N_1}$ where $\delta_i=\dim \mathcal{K}^i$.

Using (b) we see that we can assume that $\underline{a}(\mathcal{K}^i)=a_0$ for $i\in[1, N_2]$ and $\underline{a}(\mathcal{K}^i)<a_0$ for $i\in[N_2+1, N_1]$ (for some $N_2\leq N_1$). Then, by 4.4 (a), (d):

$$(-v)^{a_0}\mathrm{Tr}(h, \mathcal{K}^i)\in\begin{cases}\delta_i+v\cdot\boldsymbol{C}[v] & \text{if} \quad i\in[1, N_2]\\ \quad v\cdot\boldsymbol{C}[v] & \text{if} \quad i\in[N_2+1, N_1],\end{cases}$$

hence

$$\mu_0(s)=\delta_1+\delta_2+\cdots+\delta_{N_2}, \text{ (see (d)).}$$

Comparing with (e), it follows that $\delta_1+\delta_2+\cdots+\delta_{N_2}=\delta_1+\delta_2+\cdots+\delta_{N_1}$. Since $\delta_i>0$ for $i\in[1, N_1]$, it follows that $N_1=N_2$. Hence

(f) $(-v)^{a_0}\mathrm{Tr}(h, \mathcal{K}^i)$ has non-zero constant term, $(i\in[1, N_1])$, so that $\underline{a}(\mathcal{K}^i)\geq a_0$. Combining with (b), it follows that $\underline{a}(\mathcal{K}^i)=a_0$. Using (f) and the definition of h, we see that $\underline{c}(\mathcal{K}^i)=\underline{c}_0$. Since $\rho=\rho_i$ for some $i\in[1, N_1]$, we see that $\underline{c}(u, \sigma, \rho)=\underline{c}(\mathcal{K}^i)=\underline{c}_0$. The same argument shows that $\underline{c}(u, s', \rho')=c_0$. The proposition is proved.

4.7. PROPOSITION. *Let $f: M\to M'$ be a non-zero homomorphism of J-modules. Assume that M and M' are finitely generated as J-modules. Then there exists a simple quotient E (resp. E') of the J-modules M (resp. M') such that $E\approx E(u, s, \rho)$, $E'\approx E(u', s', \rho')$ where $(u, s, \rho), (u', s', \rho')\in\Psi'$ and u, u' are conjugate in G.*

PROOF. Let $\mathcal{C}$ be the centre of J. The proof of [15, 1.6(i), (ii)] shows that J is a finitely generated $\mathcal{C}$-module and $\mathcal{C}$ is a finitely generated $\boldsymbol{C}$-algebra. In particular, M, M' are finitely generated $\mathcal{C}$-modules. Since f is also a non-zero homomorphism of $\mathcal{C}$-modules, there exists a maximal ideal $\mathcal{I}$ of $\mathcal{C}$ such that f induces a non-zero homomorphism on the localizations $M_{\mathcal{I}}\to M'_{\mathcal{I}}$. In particular, we have $M_{\mathcal{I}}\neq 0$, $M'_{\mathcal{I}}\neq 0$. From $M_{\mathcal{I}}\neq 0$ and Nakayama's lemma it follows that $M_{\mathcal{I}}/\mathcal{I}\cdot M_{\mathcal{I}}\neq 0$. This is a J-module, let E be a simple quotient of it. Then E is a simple quotient of M and $\mathcal{I}$ acts on E as zero. Similarly, we can find a simple J-module E', quotient of M', such that $\mathcal{I}$ acts on E' as zero. Let $h:\mathcal{C}\to\boldsymbol{C}$ be the algebra homomorphism with kernel $\mathcal{I}$; then $z-h(z)\cdot 1$ acts as zero on E, E' for all $z\in\mathcal{C}$. Let $\tilde{h}:\mathcal{C}\otimes_{\boldsymbol{C}}\bar{k}\to\bar{k}$ be the $\bar{k}$-linear extension of h. The homomorphism $\tilde{\Phi}=\Phi\otimes 1_{\bar{k}}:H_{\bar{k}}\to J\otimes_{\boldsymbol{C}}\bar{k}$ maps the centre $Z_{\bar{k}}$ of $H_{\bar{k}}$ into $\mathcal{C}\otimes\bar{k}$ (see the proof of [15, 1.6(i)]). It follows that $\tilde{\Phi}(\tilde{z})-\tilde{h}(\tilde{\Phi}(\tilde{z}))\cdot 1$ acts as zero on $E\otimes\bar{k}$, $E'\otimes\bar{k}$ for all $\tilde{z}\in Z_{\bar{k}}$. Thus the $H_{\bar{k}}$-modules ${}^{\Phi}E\otimes\bar{k}$, ${}^{\Phi}E'\otimes\bar{k}$ have the same central character. Hence the proposition follows from 2.12.

We shall now state the main result of this paper.

4.8. THEOREM. (a) *The two-sided cell* $\underline{c}(u,s,\rho)$ *(for* $(u,s,\rho)\in\Psi'$*) depends only on the G-conjugacy class of u; hence it can be denoted* $\underline{c}(u)$.

(b) $u\to\underline{c}(u)$ *defines a bijection between the set* $\mathcal{U}(G)$ *of unipotent conjugacy classes in G and the set* Cell(W) *of two-sided cells in W.*

(c) $a(\underline{c}(u))=\dim\mathcal{B}_u$ *for any unipotent element* $u\in G$.

(d) *For any two-sided cell* $\underline{c}$ *of W there exists* $I\subset S$ *such that* W'^I *is finite and* $\underline{c}\cap W'^I\neq\emptyset$.

4.9. In this subsection we assume that G is adjoint and that $\sigma: W\to W$ is an automorphism of (W,S) such that the restriction of σ to X coincides with the restriction to X of an inner automorphism of W (which does not necessarily preserve S_0). Then σ defines isomorphisms $J\to J$ $(t_w\to t_{\sigma(w)})$ and $H_{\bar{k}}\to H_{\bar{k}}$ $(\tilde{T}_w\to\tilde{T}_{\sigma(w)})$, which restricts to an isomorphism $Z_{\bar{k}}\to Z_{\bar{k}}$. Also σ maps each two-sided cell of W onto a two-sided cell of W so it defines a map $\sigma_*:\text{Cell}(W)\to\text{Cell}(W)$. We assert that

(a) *If* 4.8 (b) *holds for G, then* $\sigma_*=$Identity.

We first show that the map $Z_{\bar{k}}\to Z_{\bar{k}}$ defined above (by σ) is the identity. Now $Z_{\bar{k}}$ can be regarded as the coordinate ring of the variety $\mathcal{CV}$ of semisimple classes of $G(\bar{k})$. The restriction of σ to X corresponds to an isomorphism $T_0\to T_0$ or $T_0(\bar{k})\to T_0(\bar{k})$, and it is clear that this induces on $\mathcal{CV}$ (=space of W_0-orbits on $T_0(\bar{k})$) an automorphism $\mathcal{CV}\to\mathcal{CV}$ which corresponds to the automorphism $Z_k \to Z_k$ of the coordinate ring considered above. By our assumption on σ, the isomorphism $T_0 \to T_0$ is given by conjugation by an element of W_0, hence $\mathcal{V} \to \mathcal{V}$ is the identity, hence $Z_k \to Z_k$ is the identity, as stated.

Let E be a simple J-module. Composition with σ gives a new simple J-module ${}^{\sigma}E$. It is clear that the central characters of the H_k-modules ${}^{\Phi}E \otimes k$, ${}^{\Phi}({}^{\sigma}E) \otimes \bar{k}$ are related by composition with the automorphim $Z_k \to Z_k$ considered above. Hence these two central characters coincide. We have $E = E(u, s, \rho)$, ${}^{\sigma}E = E(u', s', \rho')$ for some $(u, s, \rho) \in \Psi'$, $(u', s', \rho') \in \Psi'$. Using 2.12, we see that u, u' are conjugate in G. Let $\underline{c}$ be the two-sided cell of W corresponding to E; then $\sigma(\underline{c})$ is the two-sided cell of W corresponding to ${}^{\sigma}E$. By definition and by 4.8 (a) we have $\underline{c} = \underline{c}(u)$, $\sigma(\underline{c}) = \underline{c}(u')$. Since u, u' are conjugate, it follows that $\underline{c} = \sigma(\underline{c})$ and (a) follows.

4.10. We now return to a general G. Let $\pi : G \to \bar{G}$ be the adjoint quotient. The definitions in §1 are applicable to $\bar{G}$, $\bar{T}_0 = \pi T_0$, $\bar{B}_0 = \pi B_0$ instead of G, T_0, B_0; the group W (defined in terms of $\bar{G}$, $\bar{T}_0$, $\bar{B}_0$) may be identified (as a Coxeter group) with the subgroup W' of the group W defined in terms of G, T_0, B_0, and the character group of $\bar{T}_0$ may be identified with the subgroup X' of X. We have a natural map $j : \mathrm{Cell}(W') \to \mathrm{Cell}(W)$ defined by associating to a two-sided cell of W' the unique two-sided cell of W containing it. It is clear that j is surjective and its fibres are precisely the orbits of the natural action of Ω on $\mathrm{Cell}(W')$ (induced by the conjugation action of Ω on W'). We assert that:

(a) If 4.8 (a) holds for $\bar{G}$ then $j : \mathrm{Cell}(W') \to \mathrm{Cell}(W)$ is bijective.

It is enough to show that Ω acts trivially on $\mathrm{Cell}(W')$. By 4.9 (a) it is enough to show that any $\omega \in \Omega$ acts (by conjugation) on X' in the same way as the conjugation by some element of W'. We can write $\omega = w \cdot x$ ($w \in W_0$, $x \in X$). Then, for $x' \in X'$, we have $\omega x' \omega^{-1} = w x' w^{-1}$ and $w \in W'$, since X is commutative. Thus, (a) follows.

4.11. In the setup of 4.10 we assume that 4.8 (a) holds for $\bar{G}$ and we denote by H' (resp. J') the algebra defined in terms of W' in the same way as H (resp. J) is defined in terms of W. Let $H'_k = H' \otimes_{\mathcal{A}} k$. We can identify H', J', H'_k with subalgebras of H, J, H_k.

Let $(u, s, \rho) \in \Psi'$. Let $\bar{u} = \pi(u)$, $\bar{s} = \pi(s)$.

From the definitions it is easy to see that the restrictions of the H_k-module $\mathcal{K}_{u,s} \otimes k$ to H'_k is isomorphic $\mathcal{K}_{\bar{u},\bar{s}} \otimes k$. ($\mathcal{K}_{\bar{u},\bar{s}}$ is defined in terms of $\bar{G}, \bar{u}, \bar{s}$ in the same way as $\mathcal{K}_{u,s}$ is defined in terms of G, u, s.) It follows that the restriction of the H_k-module $\mathcal{K}_{u,s,\rho}\otimes k$ to H_k' is isomorphic to a direct sum of H_k'-modules $\mathcal{K}_{\bar{u},\bar{s},\bar{\rho}}\otimes k$ for various $\bar{\rho}$. From 4.2, it then follows that the restriction of the J-module $E(u, s, \rho)$ to J' is a direct sum of J'-modules of the form $E(\bar{u}, \bar{s}, \bar{\rho})$ for various $\bar{\rho}$. (Here $E(\bar{u}, \bar{s}, \bar{\rho})$ is defined in terms of $\bar{G}, \bar{u}, \bar{s}, \bar{\rho}$ in the same way as $E(u, s, \rho)$ in terms of G, u, s, ρ.) Hence, if $\underline{c}(\bar{u}) \in \mathrm{Cell}(W')$ is defined as in 4.8 (a) (for $\bar{G}$), there exists $w \in \underline{c}(\bar{u})$ such that $t_w \neq 0$ on $E(u, s, \rho)$. Hence $\underline{c}(u, s, \rho) \supset \underline{c}(\bar{u})$. Since in W there is exactly one two-sided cell containing a given two-sided cell of W', it follows that $\underline{c}(u, s, \rho)$ depends only on u, not on s, ρ. Thus, if 4.8 (a) holds for $\bar{G}$ then it also holds for G.

4.12. Assume that 4.8 holds for $\bar{G}$. We assert that in this case 4.8 holds also for G. We have seen already in (4.11) that 4.8 (a) holds for G. Using 4.10 (a) and the fact that $\pi : G \to \bar{G}$ defines a bijection $\mathcal{U}(G) \to \mathcal{U}(\bar{G})$, we see that 4.8 (b) holds for G. It is immediate that 4.8 (c), (d) for G follow from the corresponding statements for $\bar{G}$. Thus, we are reduced to proving 4.8 in the case where G is adjoint. The proof will be given in Section 7.

5. Truncated induction

5.1. In this section we shall assume that Ω is finite (i.e. G is semisimple) and we fix a subset $I \subset S$ such W'^I (hence W^I) is finite and a two-sided cell $\underline{c}_1$ of W^I. Then $\underline{c}_1$ is contained in a unique two-sided cell $\underline{c}$ of W. We identify $J^I_{\underline{c}_1}$ with the subalgebra $J_{\underline{c}_1}$ of $J_{\underline{c}}$. Let M_1 be a left $J^I_{\underline{c}_1}$-module. We define $i(M_1) = J_{\underline{c}} \otimes_{J^I_{\underline{c}_1}} M_1$. (Note that $J^I_{\underline{c}_1}$ acts by right multiplication on $J_{\underline{c}}$ but its unit element does not act as identity; it also acts on the left on M_1. Traditionally, the tensor product is not defined in this case. However, we shall define it as $J_{\underline{c}} \otimes_{C} M_1$ modulo the subspace spanned by all elements $xy\otimes m_1 - x\otimes ym_1$, $(x \in J_{\underline{c}}, y \in J^I_{\underline{c}_1}, m_1 \in M_1)$.)

We shall regard $i(M_1)$ as a left $J_{\underline{c}}$-module with multiplication defined by $x'(x\otimes m_1) = x'x\otimes m_1$, $(x, x' \in J_{\underline{c}}, m_1 \in M_1)$.

Let M be a left $J_{\underline{c}}$-module. We define

$$r(M) = \{m \in M \mid \varepsilon_1 m = m\}$$

where ε_1 is the unit element of $J^I_{\underline{c}_1}$. Then $r(M)$ is in a natural way a left $J^I_{\underline{c}_1}$-module (restricting the $J_{\underline{c}}$-module structure of M).

5.2. Lemma. *If M_1, M are as in* 5.1, *we have a natural isomorphism*

$$\mathrm{Hom}_{J_c}(i(M_1), M) = \mathrm{Hom}_{J^I_{c_1}}(M_1, r(M)).$$

Proof. The left hand side is the set of all C-linear functions $f : J_c \otimes_C M_1 \to M$ such that $f(xy\otimes m_1) = f(x\otimes ym_1)$, $f(x'x\otimes m_1) = x'f(x\otimes m_1)$ for all $x, x' \in J_c$, $y \in J^I_{c_1}$, $m_1 \in M_1$.

The right hand side is the set of all C-linear functions $f' : M_1 \to M$ such that $f'(ym_1) = yf'(m_1)$ for all $y \in J^I_{c_1}$ (such f' has automatically image contained in $r(M)$). To f we associate f' by $f'(m_1) = f(\varepsilon\otimes m_1)$, ($\varepsilon$=unit element of J_c); this establishes the desired bijection.

5.3. Lemma. *Assume that M_1 is non-zero. Then $i(M_1)$ is a non-zero J_c-module.*

Proof. Assume that we can find C-subspace Z of J_c complementary to $J^I_{c_1}$ and stable under right multiplication by elements of $J^I_{c_1}$. Then $i(M_1)$ is a direct sum of two vector spaces: one is $Z \otimes_{J^I_{c_1}} M_1$ (defined as in 5.1), the other is $J^I_{c_1} \otimes_{J^I_{c_1}} M_1 \cong M_1 \neq 0$, so that $i(M_1) \neq 0$.

It remains to prove the existence of Z. We can decompose $J_c = J' \oplus J''$ by $J' = J_c \cdot \varepsilon_1$, $J'' = J_c(\varepsilon - \varepsilon_1)$ where ε (resp. ε_1) is the unit element of J_c (resp. $J^I_{c_1}$). Then J', J'' are clearly stable under right multiplication by $J^I_{c_1}$ and $J^I_{c_1} \subset J'$. Now J' is a right $J^I_{c_1}$-module (ε_1 acts as identity) and $J^I_{c_1}$ is a semisimple algebra since W^I is finite (see 1.4) hence $J^I_{c_1}$ admits a complement in J' which is a right $J^I_{c_1}$-submodule. This complement plus J'' form the required subspace Z.

5.4. Proposition. *Assume that M_1 is a simple $J^I_{c_1}$-module. Then there exists a simple J_c-module M' such that $r(M')$ contains M_1 as a direct summand.*

Proof. By 5.3, $i(M_1)$ is non-zero. By Zorn's lemma, $i(M_1)$ has some maximal proper submodule. Hence there exists a simple J_c-module M' such that $\mathrm{Hom}_{J_c}(i(M_1), M') \neq 0$. It remains to use 5.2 and the semisimplicity of the $J^I_{c_1}$-module $r(M')$.

5.5. Lemma. *Let Γ_1 be a left cell of W^I contained in $\underline{c}_1$, so that $J^I_{\Gamma_1} = J_{\Gamma_1}$ is a left ideal of $J^I_{c_1}$. Let Γ be the unique left cell of W containing Γ_1, so that J_Γ is a left ideal of J_c. We have $i(J^I_{\Gamma_1}) = J_\Gamma$ as (left) J_c-modules.*

PROOF. The $J_{\underline{c}}$-linear map $J_{\underline{c}}\otimes J_{\Gamma_1}\to J_{\Gamma}$, $x\otimes y\to xy$, factors through a $J_{\underline{c}}$-linear map $f: i(J_{\Gamma_1})\to J_{\Gamma}$. Let d be the unique element of $\mathcal{D}\cap\Gamma$. Then $d\in\Gamma_1$ and the $J_{\underline{c}}$-linear map $J_{\Gamma}\to J_{\underline{c}}\underset{C}{\otimes} J_{\Gamma_1}$, $z\to z\otimes t_d$, induces a $J_{\underline{c}}$-linear map $f': J_{\Gamma}\to i(J_{\Gamma_1})$. One checks easily that f, f' are inverse to each other.

5.6. PROPOSITION. *Let E' be a simple J-module and let $\mathrm{res}_I(^{\Phi}E')$ be the restriction of the H_k-module $^{\Phi}E'$ to the subalgebra H_k^I. Let E'' be a simple J^I-module. Let $\underline{c}'$ (resp. $\underline{c}''$) be the two-sided cell of W (resp. W^I) corresponding to E' (resp. E'').*

(a) *If the simple H_k^I-module $^{\Phi}E''$ appears with non-zero multiplicity in the (semisimple) H_k^I-module $\mathrm{res}_I(^{\Phi}E')$ then $a(\underline{c}'')\le a(\underline{c}')$.*

(b) *If $\underline{c}'=\underline{c}$, $\underline{c}''=\underline{c}_1$ are as in 5.1 (so that $\underline{c}_1\subset\underline{c}$), then the multiplicity of $^{\Phi}E''$ in $\mathrm{res}_I(^{\Phi}E')$ is equal to the multiplicity of the $J_{\underline{c}_1}^I$-module E'' in $r(E')$.*

PROOF. (a) We can find $d\in\mathcal{D}\cap\underline{c}''$ such that $\mathrm{Tr}(t_d, E'')=\gamma\neq 0$. By 4.5, we have $(-v)^{a(\underline{c}'')}\mathrm{Tr}(C_d, {}^{\Phi}E'')=\gamma+$element of $vC[v]$. In particular, $C_d\neq 0$ on $^{\Phi}E''$. Since $^{\Phi}E''$ is isomorphic to an H_k^I-submodule of $\mathrm{res}_I(^{\Phi}E')$ it follows that $C_d\neq 0$ on $^{\Phi}E'$. Using 4.4 (e), it follows that $a(d)\le \underline{a}(^{\Phi}E')=a(\underline{c}')$. But $a(d)=a(\underline{c}'')$ and (a) follows.

(b) Let $E_1'', E_2'', \cdots, E_N''$ be a set of representatives for the isomorphism classes of simple J^I-modules with corresponding two-sided cell $\underline{c}_1$, and let μ_i denote the multiplicity of $^{\Phi}E_i''$ in $\mathrm{res}_I(^{\Phi}E')$. Using 4.5 for W_I and (a) we have for all $w\in\underline{c}_1$: $(-v)^{a(\underline{c}_1)}\mathrm{Tr}(C_w, \mathrm{res}_I(^{\Phi}E'))=\sum_i \mu_i\,\mathrm{Tr}(t_w, E_i'')+$ element of $vC[v]$. The left hand side is $(-v)^{a(\underline{c})}\mathrm{Tr}(C_w, {}^{\Phi}E')$ which by 4.4 (for W) is of the form $\mathrm{Tr}(t_w, E')+$element of $vC[v]$. It follows that

$$\mathrm{Tr}(t_w, E')=\sum_{i=1}^{N}\mu_i\,\mathrm{Tr}(t_w, E_i'')\quad\text{for all}\quad w\in\underline{c}_1.$$

Hence $\mathrm{Tr}(t_w, r(E'))=\sum_{i=1}^{N}\mu_i\,\mathrm{Tr}(t_w, E_i'')$ for all $w\in\underline{c}_1$. Since $J_{\underline{c}_1}^I$ is a semisimle algebra, a $J_{\underline{c}_1}^I$-module of finite dimension is determined up to isomorphism by the traces of t_w ($w\in\underline{c}_1$) on it. It follows that $r(E')$ is isomorphic as a $J_{\underline{c}_1}^I$-module to the direct sum of μ_1 copies of E_1'', μ_2 copies of E_2'', etc. The proposition is proved.

6. Character sheaves

6.1. In this section we assume that Ω is finite (i.e. G is semisimple). We shall describe a relationship between H_k-modules on the one hand

and character sheaves [11] on G on the other hand. We shall recall the definition of the character sheaves that we need. Let $\mathcal{S}(T_0)$ be the set of isomorphism classes of $\boldsymbol{C}$-local systems of rank 1 on T_0 with finite monodromy. Now W_0 acts on T_0 (by conjugation) hence it also acts naturally on the set $\mathcal{S}(T_0)$. For $\mathcal{L}\in\mathcal{S}(T_0)$, let $(W_0)_{\mathcal{L}}$ be the stabilizer of $\mathcal{L}$ in W_0.

Let G_{reg} (resp. $(T_0)_{\text{reg}}$) be the set of regular semisimple elements of G (resp. T_0). Let $\tilde{G}_{\text{reg}}=\{(g, xT_0)\in G_{\text{reg}}\times G/T_0| \, x^{-1}gx\in(T_0)_{\text{reg}}\}$. Then $p:\tilde{G}_{\text{reg}}\to G_{\text{reg}}$, $p(g, xT_0)=g$ is a principal covering with group W_0 (which acts on $\tilde{G}_{\text{reg}}$ by $w\cdot(g, xT_0)=(g, x\dot{w}^{-1}T_0)$, $\dot{w}$=representative of w in the normalizer of T_0). Let $\delta:\tilde{G}_{\text{reg}}\to T_0$, $\delta(g, xT_0)=x^{-1}gx$. Consider the local system $\bar{\mathcal{L}}=p_*\delta^*\mathcal{L}$ on G_{reg}. As in [11, 3.5] we can define a canonical isomorphism $\text{End}(\bar{\mathcal{L}})\cong\boldsymbol{C}[(W_0)_{\mathcal{L}}]$. Hence $(W_0)_{\mathcal{L}}$ acts naturally on $\bar{\mathcal{L}}$ (inducing identity on G_{reg}). For any simple $\boldsymbol{C}[(W_0)_{\mathcal{L}}]$-module $\mathcal{E}$ we set $\bar{\mathcal{L}}_{\mathcal{E}}=\text{Hom}_{(W_0)_{\mathcal{L}}}(\mathcal{E}, \bar{\mathcal{L}})$. We then have a canonical direct sum decomposition $\bar{\mathcal{L}}=\bigoplus_{\mathcal{E}}(\mathcal{E}\otimes\bar{\mathcal{L}}_{\mathcal{E}})$, ($\mathcal{E}$ runs over the simple $\boldsymbol{C}[(W_0)_{\mathcal{L}}]$-modules up to isomorphism) and $\bar{\mathcal{L}}_{\mathcal{E}}$ are irreducible local systems on G_{reg}. We consider the intersection cohomology complex $IC(G, \bar{\mathcal{L}}_{\mathcal{E}})$. It is (up to shift) a "character sheaf" of G of a special kind. Its cohomology sheaves $\mathcal{H}^i IC(G, \bar{\mathcal{L}}_{\mathcal{E}})$ are G-equivariant constructible sheaves on G, (zero for i odd). Hence for each $g\in G$, the group $Z_G(g)/Z_G^0(g)$ acts naturally on the stalk $\mathcal{H}_g^i IC(G, \bar{\mathcal{L}}_{\mathcal{E}})$, so that for any simple $\boldsymbol{C}[Z_G(g)/Z_G^0(g)]$-module ρ the multiplicity $(\rho:\mathcal{H}_g^i IC(G, \bar{\mathcal{L}}_{\mathcal{E}}))$ is well defined.

6.2. Given $\mathcal{L}\in\mathcal{S}(T_0)$, there exists $w\in W_0$ and $I\subset S$ such that the canonical projection $W\to W_0$ defines an isomorphism $W^I\xrightarrow{\approx}(W_0)_{w^*\mathcal{L}}$. (In particular, W^I is finite.)

6.3. Conversely, given $I\subset S$ such that W^I is finite, there exists $\mathcal{L}\in\mathcal{S}(T_0)$ such that $W^I\xrightarrow{\approx}(W_0)_{\mathcal{L}}$; we shall select such an $\mathcal{L}$ and denote it $\mathcal{L}(I)$. We shall identify W^I and $(W_0)_{\mathcal{L}(I)}$ using the isomorphism above. Hence given a simple $\boldsymbol{C}[W^I]$-module $\mathcal{E}$, we can regard it as a $\boldsymbol{C}[(W_0)_{\mathcal{L}(I)}]$-module so that $\bar{\mathcal{L}}(I)_{\mathcal{E}}$ is well defined (see 6.1). Now $\mathcal{E}$ gives rise via $\Phi_1:\boldsymbol{C}[W^I]\xrightarrow{\sim}J^I$ (see 1.4) to a simple J^I-module $\tilde{\mathcal{E}}$ and via $\Phi:H_k^I\xrightarrow{\sim}J^I\otimes k$, to a simple H_k^I-module ${}^{\Phi}\tilde{\mathcal{E}}$. We can now state the following result.

6.4. THEOREM. *Let $(u, s, \rho)\in\Psi'$, and let $I, \mathcal{L}(I), \mathcal{E}, {}^{\Phi}\tilde{\mathcal{E}}$ be as in* 6.3. *Then the simple H_k^I-module ${}^{\Phi}\tilde{\mathcal{E}}$ appears in the restriction of the H_k-module $\mathcal{K}_{u,s,\rho}\otimes k$ to H_k^I with a multiplicity equal to*

(a) $\sum_i[\rho:\mathcal{H}_{su}^{2i}(IC(G, \bar{\mathcal{L}}(I)_{\mathcal{E}}))]$.

To prove this theorem one needs a bridge between the K-theoretic construction of H_k-modules and the intersection-cohomology-theoretic construction of Weyl group representations. Such a bridge is provided by the series [18] where the proof of the theorem will be given.

6.5. In the rest of this section we shall assume that $\Omega=\{e\}$, i.e. G is adjoint. Let I be as in 6.3 and let $\underline{c}_1$ be a two-sided cell in W^I. Following [8] we attach to $\underline{c}_1$ a unipotent class $\mathcal{O}=\mathcal{O}(I, \underline{c}_1)$ in G. Let $\bar{\mathcal{E}}_1$ be the unique special representation of W^I which corresponds to $\underline{c}_1$. We identify W^I with its image under the natural projection $W\to W_0$ and we induce $\mathcal{E}_1$ from this image to W_0. Let i_0 be the smallest integer ≥ 0 such that the resulting induced W_0-module is not disjoint from the standard W_0-module $H^{2i_0}(\mathcal{B}, C)$. Then there is a unique simple W_0-module $\bar{\mathcal{E}}_1$ (up to isomorphism) which appears in both these W_0-modules. As pointed out in [8], [1], $\bar{\mathcal{E}}_1$ corresponds under Springer's correspondence to a unipotent class $\mathcal{O}$ in G and the local system $\boldsymbol{C}$ on it; moreover:

(a) $\dim \mathcal{B}_u=a(\underline{c}_1)=i_0$, $(u\in\mathcal{O})$.

6.6. Let $\phi: SL_2(\boldsymbol{C})\to G$ be a homomorphism of algebraic groups, and let F be the centralizer in G of the image of ϕ. Let F_j^1 $(1\leq j\leq m)$ be connected components of F, one in each conjugacy class of F/F^0, (F^0 is the component of e). To each F_j^1 we associate a diagonalizable subgroup $D_j\subset F$ (in the same way as D is defined in terms of F^1 in 3.1) and let $D_j^1=D_j\cap F_j^1$; we also choose $s_j\in D_j^1$ such that $Z_F(s_j)=Z_F(D_j)$ (see 3.1(c)).

6.7. THEOREM. (a) *In the setup of* 6.6 *let* $u=\phi\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ *and let* $\mathcal{O}_u$ *be the conjugacy class of* u *in* G. *There exists a subset* $I\subset S$ (*with* W^I *finite*) *and a two-sided cell* $\underline{c}_1$ *of* W^I *such that the properties* (a1), (a2), (a3) *below hold.*

(a1) $\mathcal{O}_u=\mathcal{O}(I, \underline{c}_1)$ (*see* 6.5).

(a2) *Let* $j\in[1, m]$ *and* ρ *be such that* $(u, s_j, \rho)\in\Psi'$; *then there exists a simple* W^I*-module* $\mathcal{E}$ *corresponding to* $\underline{c}_1$ *such that* ρ *appears in the* $Z_G(s_ju)/Z_G^0(s_ju)$*-module* $\mathcal{H}^{2i}_{s_ju}(IC(G, \bar{\mathcal{L}}(I)_{\mathcal{E}}))$ *for some* i.

(a3) *Let* $j, j'\in[1, m]$ *and* ρ, ρ' *be such that* $(u, s_j, \rho)\in\Psi'$, $(u, s_{j'}, \rho')\in\Psi'$. *Assume that there exists a simple* W^I*-module* $\mathcal{E}$ *corresponding to* $\underline{c}_1$ *such that* ρ *appears in the* $Z_G(s_ju)/Z_G^0(s_ju)$*-module* $\mathcal{H}^{2i}_{s_ju}(IC(G, \bar{\mathcal{L}}(I)_{\mathcal{E}}))$ *for some* i *and* ρ' *appears in the* $Z_G(s_{j'}u)/Z_G^0(s_{j'}u)$*-module* $\mathcal{H}^{2i'}_{s_{j'}u}(IC(G, \bar{\mathcal{L}}(I)_{\mathcal{E}}))$ *for some* i'. *Then* $j=j'$ *and* $\rho=\rho'$.

(b) *Let* $I, \mathcal{E}$ *be as in* 6.3 *and let* $(u, s, \rho)\in\Psi'$ *be such that* ρ *appears* *in the* $Z_G(su)/Z_G^0(su)$*-module* $\mathcal{H}^{2i}_{su}(IC(G, \bar{\mathcal{L}}(I)_{\mathcal{E}}))$ *for some* i. *Let* $\underline{c}_1$ *be the two-sided cell of* W^I *corresponding to* $\mathcal{E}$ *and let* $\mathcal{O}=\mathcal{O}(I, \underline{c}_1)$ *be as in* 6.5. *Then* $u \in \bar{\mathcal{O}}$ (=*closure of* $\mathcal{O}$).

The statements of the theorem express properties which hold for general character sheaves; these properties can be verified in principle by computation since the cohomology sheaves of character sheaves are relatively well understood. More details of the proof will be given in a future article. Note that the special case of (b) with $W^I=W_0$, $s=e$ is contained in [4].

7. Proof of Theorem 4.8

7.1. In this section we give the proof of Theorem 4.8 assuming the results on character sheaves in Section 6. We may assume that G is adjoint (see 4.12).

7.2. LEMMA. *Let* $I \subset S$ *be such that* W^I *is finite and let* $\underline{c}_1$ *be a two-sided cell of* W^I. *Let* $(u', s', \rho') \in \Psi'$. *Let* $\mathcal{E}$ *be a simple* W^I*-module corresponding to* $\underline{c}_1$. *Assume that the restriction of the* H_k*-module* $\mathcal{K}_{u',s',\rho'} \otimes k$ *to* H_k^I *contains the simple* H_k^I*-module* ${}^{\phi}\tilde{\mathcal{E}}$, (see 6.3). *Then* $u' \in \overline{\mathcal{O}(I, \underline{c}_1)}$. *If* $\dim \mathcal{B}_{u'}=a(\underline{c}_1)$, *then* $u' \in \mathcal{O}(I, \underline{c}_1)$.

PROOF. Using 6.4 we see that ρ' appears in the $Z_G(s'u')/Z_G^0(s'u')$-module $\mathcal{H}^{2i}_{s'u'}(IC(G, \bar{\mathcal{L}}(I)_{\mathcal{E}}))$ for some i, and using 6.7 (b) we see that $u' \in \overline{\mathcal{O}(I, \underline{c}_1)}$. If $u' \in \overline{\mathcal{O}(I, \underline{c}_1)} - \mathcal{O}(I, \underline{c}_1)$ then clearly $\dim \mathcal{B}_{u'} > \dim \mathcal{B}_u$ where $u \in \mathcal{O}(I, \underline{c}_1)$. But $\dim \mathcal{B}_u = a(\underline{c}_1)$ by 6.5 (a) hence $\dim \mathcal{B}_{u'} > a(\underline{c}_1)$. The lemma is proved.

7.3. We shall first prove 4.8 (a), (c) by induction. We fix $\phi, F, F_j^1, D_j, D_j^1, s_j, j \in [1, m]$, as in 6.6 and let $u, \mathcal{O}_u, I, \underline{c}_1 \subset W^I$ be as in 6.7 (a). Assume that $\dim \mathcal{B}_u = a_0$, and that 4.8 (a), (c) hold whenever u is replaced by u' with $\dim \mathcal{B}_{u'} > a_0$. Let $\underline{c}$ be the unique two-sided cell of W containing $\underline{c}_1$.

7.4. LEMMA. *In the setup of* 7.3, *we consider a semisimple element* $s \in Z_G(u)$ *and let* ρ *be an admissible simple module of* $Z_G(su)/Z_G^0(su)$. *Then* $\underline{c}(u, s, \rho)$ (*see* 4.3) *coincides with* $\underline{c}$ (*in* 7.3).

PROOF. By replacing s by a conjugate under $Z_G(u)$ we may assume that $s \in D_j^1$ for some $j \in [1, m]$. (Note that $\underline{c}(u, s, \rho)$ depends only on the

G-conjugacy class of (u, s, ρ).) We have natural homomorphisms (see 3.1 (d)):

$$\begin{array}{l} Z_F(D_j)/D_j^0 \hookrightarrow Z_F(s)/Z_F^0(s) = Z_G(su)/Z_G^0(su) \\ \big\downarrow \wr \\ Z_F(s_j)/Z_F^0(s_j). \end{array}$$

Let $\tilde{\rho}$ be an irreducible representation of $Z_F(s_j)/Z_F^0(s_j)$ such that the corresponding representation of $Z_F(D_j)/D_j^0$ is contained in the restriction of ρ. Using 4.6, we have $\underline{c}(u, s, \rho) = \underline{c}(u, s_j, \tilde{\rho})$. Hence we can assume that $s = s_j$.

By 6.7 (a2) (re-expressed by 6.4) we can find a simple W^I-module $\mathcal{E}$ corresponding to $\underline{c}_1$ such that:

(a) the restriction of $\mathcal{K}_{u,s,\rho} \otimes k$ to H_k^I contains the simple H_k^I-module ${}^{\Phi}\tilde{\mathcal{E}}$.

Here $\tilde{\mathcal{E}}$ is a simple J^I-module (see 6.3) with corresponding two-sided cell $\underline{c}_1$. By 5.4, there exists a simple J-module E with corresponding two-sided cell $\underline{c}$ such that $r(E)$ contains $\tilde{\mathcal{E}}$ as a direct summand. Using 5.6, we see that ${}^{\Phi}\tilde{\mathcal{E}}$ is contained in the restriction of the H_k-module ${}^{\Phi}E$ to H_k^I. By 4.2 we have ${}^{\Phi}E \cong \mathcal{K}_{u',s',\rho'} \otimes k$ for some $(u', s', \rho') \in \Psi'$. By 7.2 we have $u' \in \bar{\mathcal{O}}_u$, hence $\dim \mathcal{B}_{u'} \geq \dim \mathcal{B}_u = a_0$. If $\dim \mathcal{B}_{u'} > a_0$, then by the assumption of 7.3, we have $a(\underline{c}(u', s', \rho')) = \dim \mathcal{B}_{u'}$; but we have $\underline{c}(u', s', \rho') = \underline{c}$, $a(\underline{c}) = a(\underline{c}_1) = \dim \mathcal{B}_u = a_0$ (see 6.5 (a)), so $\dim \mathcal{B}_{u'} = a_0$, a contradiction. Hence we have $\dim \mathcal{B}_{u'} = a_0$ and 7.2 shows that $u' \in \mathcal{O}_u$. Hence we can assume that $u' = u$.

Next, replacing s' by a conjugate under $Z_G(u)$, we may assume that $s' \in D_{j'}^1$ for some $j' \in [1, m]$. Let ρ'' be an irreducible representation of $Z_F(s_{j'})/Z_F^0(s_{j'})$ such that the corresponding representation of $Z_F(D_{j'})/D_{j'}^0$ is contained in the restriction of ρ'. Then, $(u, s_{j'}, \rho'') \in \Psi'$, and using 3.4, we see that the restriction of $\mathcal{K}_{u,s_{j'},\rho''} \otimes k$ to H_k^I contains the restriction of $\mathcal{K}_{u,s',\rho'}$ to H_k^I, and in particular:

(b) the restriction of $\mathcal{K}_{u,s_{j'},\rho''} \otimes k$ to H_k^I contains the simple H_k^I-module ${}^{\Phi}\tilde{\mathcal{E}}$.

Moreover, from 4.6 we see that $\underline{c}(u, s', \rho') = \underline{c}(u, s_{j'}, \rho'')$. Since $\underline{c}(u, s', \rho') = \underline{c}$, it follows that

(c) $\underline{c}(u, s_{j'}, \rho'') = \underline{c}$.

Using (a), (b) and 6.7 (a3) (re-expressed by 6.4) we see that $j = j'$

and $\rho=\rho''$. Hence (c) implies $\underline{c}(u, s, \rho)=\underline{c}$. (Recall that $s_j=s$.) The Lemma is proved.

7.5. As we have seen in the proof of 7.4, we have $a(\underline{c})=a_0$ hence from 7.4 we have $a(\underline{c}(u, s, \rho))=\dim \mathscr{B}_u$. Thus 7.4 provides the inductive step in the proof of 4.8 (a), (c); thus 4.8 (a), (c) are proved.

7.6. Now let $\underline{c}'$ be any two-sided cell of W. The algebra $J_{\underline{c}'}$ must have some simple module E'; we can regard E' as a simple J-module (using the canonical projection $J\to J_{\underline{c}'}$). Using 4.2 we see that $E'\approx E(u, s, \rho)$ for some $(u, s, \rho)\in \Psi'$. We then have $\underline{c}'=\underline{c}(u, s, \rho)$ by the definition of $\underline{c}(u, s, \rho)$. This proves that the map in 4.8 (b) is surjective. Moreover, from 7.4 applied to u, we see that $\underline{c}'=\underline{c}\supset\underline{c}_1$ ($\underline{c}$, $\underline{c}_1$ as in 7.3) hence $\underline{c}'\cap W^I\neq\emptyset$ (I as in 7.2). Thus 4.8 (d) holds.

7.7. It remains to prove that the map in 4.8 (b) is injective.

Let u, u' be representatives of unipotent classes of G which are mapped by the map in 4.8 (b) to the same two-sided cell $\underline{c}\subset W$. We attach $(I, \underline{c}_1\subset W^I)$ to u as in 6.7 (a); we attach similarly $(I', \underline{c}'_1\subset W^{I'})$ to u'. By 7.4 we have $\underline{c}_1\subset\underline{c}$, $\underline{c}'_1\subset\underline{c}$. Let Γ_1 (resp. Γ'_1) be a left cell of W^I (resp. $W^{I'}$) contained in $\underline{c}_1$ (resp. $\underline{c}'_1$). Let Γ (resp. Γ') be the left cell of W containing Γ_1 (resp. Γ'_1). We have $\Gamma\subset\underline{c}$, $\Gamma'\subset\underline{c}$. We now show that:

(a) All simple quotients of the left J-module J_Γ are of form $E(u, s, \rho)$ for some s, ρ such that $(u, s, \rho)\in\Psi'$.

Let E be a simple quotient of J_Γ. We have $E\cong E(\bar{u}, \bar{s}, \bar{\rho})$ for some $(\bar{u}, \bar{s}, \bar{\rho})\in\Psi'$, (see 4.2). By 5.2 and 5.5 we have

$$\mathrm{Hom}_{J_{\underline{c}_1}}(J^I_{\Gamma_1}, r(E))=\mathrm{Hom}_{J_{\underline{c}}}(i(J^I_{\Gamma_1}), E)=\mathrm{Hom}_J(J_\Gamma, E)\neq 0$$

hence $r(E)$ contains some irreducible J^I-module $\tilde{\mathcal{E}}$ with corresponding two-sided cell $\underline{c}_1$. ($\mathcal{E}$ is the associated W^I-module.) Then by 5.6 (b), the restriction of ${}^{\Phi}E$ to H^I_k contains the simple H^I_k-module ${}^{\Phi}\tilde{\mathcal{E}}$. Since $\dim\mathscr{B}_{\bar{u}}=a(\underline{c})=a(\underline{c}_1)$ (by 4.8 (c)) we see from 7.2 that $\bar{u}$ is conjugate to u, and the assertion (a) is established.

Similarly, we have:

(b) all simple quotients of the left J-module $J_{\Gamma'}$ are of form $E(u', s', \rho')$ for some s', ρ' such that $(u', s', \rho')\in\Psi'$.

From 3.1 (k) and the line following 3.1 (l) in [16] it follows that any left cell in $\underline{c}$ has non-empty intersection with any right cell in $\underline{c}$. Hence there exists $w \in \Gamma$ such that $w^{-1} \in \Gamma'$. Then w defines a homomorphism of (left) J-modules $f: J_{\Gamma'} \to J_{\Gamma}$ by $f(t_y) = t_y \cdot t_w$, $(y \in \Gamma)$. This is non-zero since if $d' \in \mathcal{D} \cap \Gamma'$ then $f(t_{d'}) = t_w \neq 0$. The J-modules $J_{\Gamma'}, J_{\Gamma}$ are finitely generated; this follows from [15, 1.6 (i)] and the fact that J_{Γ} is a direct summand of J (as a left J-module). Using 4.7, it follows that there exists a simple quotient of J_{Γ} (necessarily of form $E(u, s, \rho)$, see (a)) and a simple quotient of $J_{\Gamma'}$ (necessarily of form $E(u', s', \rho')$, see (b)) with u, u' conjugate. Thus u, u' are conjugate and the injectivity of the map in 4.8 (b) is established. This completes the proof of 4.8.

8. Finite cells

In this section, we assume that Ω is finite, i.e. G is semisimple.

The following result has been conjectured in [9, Problem V].

8.1. THEOREM. *Let $\mathcal{O}$ be a unipotent class of G, and let $\underline{c}$ be the corresponding two-sided cell of W (see 4.8 (b)). The following conditions are equivalent:*

(a) *$\underline{c}$ is a finite set.*

(b) *$Z_G^0(u_0)$ is a unipotent group $(u_0 \in \mathcal{O})$.*

PROOF. Assume first that (b) holds. Then there are only finitely many triples $(u, s, \rho) \in \Psi'$, up to G-conjugacy with $u \in \mathcal{O}$ hence $J_{\underline{c}}$ has only finitely many simple modules (up to isomorphism). Assume that the centre $\mathcal{C}_{\underline{c}}$ of $J_{\underline{c}}$ is infinite dimensional over $\boldsymbol{C}$. Being a finitely generated $\boldsymbol{C}$-algebra (see [15, 1.6 (ii)]), its ideal of nilpotent elements is finite dimensional over $\boldsymbol{C}$, hence the reduced algebra is still infinite dimensional over $\boldsymbol{C}$. Hence $\mathcal{C}_{\underline{c}}$ admits infinitely many algebra homomorphisms into $\boldsymbol{C}$. For each such homomorphism h there exists a simple $J_{\underline{c}}$ module on which $\mathcal{C}_{\underline{c}}$ acts via h (as in the proof of 4.7). This provides infinitely many simple $J_{\underline{c}}$-modules, a contradiction. Thus $\mathcal{C}_{\underline{c}}$ is finite dimensional over $\boldsymbol{C}$. But $J_{\underline{c}}$ is a finitely generated $\mathcal{C}_{\underline{c}}$-module (see [15, 1.6 (i)]) hence $J_{\underline{c}}$ is finite dimensional over $\boldsymbol{C}$ hence (a) holds.

Conversely assume that (b) does not hold. Then there are infinitely many triples $(u, s, \rho) \in \Psi'$ with $u \in \mathcal{O}$, up to G-conjugacy. Hence $J_{\underline{c}}$ has infinitely many simple modules hence it is infinite dimensional over $\boldsymbol{C}$. Thus $\underline{c}$ is infinite. The theorem is proved.

9. Spherical representations

9.1. PROPOSITION. *Let $(u, s, \rho)\in\Psi'$. The following conditions are equivalent*:

(a) $\{\xi\in\mathcal{K}_{u,s,\rho}\otimes k \mid \tilde{T}_w\xi=v^{l(w)}\xi,\ \forall w\in W_0\}\neq 0$ *i.e.* $\mathcal{K}_{u,s,\rho}\otimes k$ *is a spherical representation of* H_k.

(b) $\rho=1$.

(c) *Let* $W_{\min}=\{w\in W \mid w$ *has minimal length in* $wW_0\}$.

There exists $d\in\mathcal{D}\cap\underline{c}(u, s, \rho)\cap W_{\min}$ *such that* $t_dE(u, s, \rho)\neq 0$.

PROOF. By 6.4, condition (a) is equivalent to the condition that $(\rho:\mathcal{H}^{2i}_{su}IC(G, \boldsymbol{C}))\neq 0$ for some i. We can take $\bar{\mathcal{L}}(I)=\boldsymbol{C}$ in this case.) Clearly, $IC(G, \boldsymbol{C})=\boldsymbol{C}$ since G is smooth; we have $\mathcal{H}^{2i}(\boldsymbol{C})=0$ for $i\neq 0$, $\mathcal{H}^{2i}(\boldsymbol{C})=\boldsymbol{C}$ for $i=0$ and the equivalence of (a), (b) follows.

Let Δ be the one dimensional $H_k^{S_0}$-module defined by $\tilde{T}_w\to v^{l(w)}$.

Let $E=E(u, s, \rho)$. For each $d\in\mathcal{D}\cap\underline{c}(u, s, \rho)$, let $E_d=t_dE$. Then the subspaces E_d form a direct sum decomposition of E, and the action of the elements $\tilde{T}_r$ $(r\in S)$ of H_k in the H_k-module ${}^{\Phi}E$ is given by particularly simple formulas (see [15, 3.8]). In particular, if

$$E'=\bigoplus E_d \quad (d \text{ runs over } \mathcal{D}\cap\underline{c}(u, s, \rho)\cap W_{\min})$$
$$E''=\bigoplus E_d \quad (d \text{ runs over } \mathcal{D}\cap\underline{c}(u, s, \rho)\cap(W-W_{\min}))$$

then $E''\otimes k\subset{}^{\Phi}E$ is stable under $H_k^{S_0}$ and does not contain Δ and $E\otimes k/E''\otimes k$ is (as an $H_k^{S_0}$-module) a multiple of Δ.

Since $E\otimes k={}^{\Phi}E$ is completely reducible as an $H_k^{S_0}$-module, it follows that Δ appears in the $H_k^{S_0}$-module ${}^{\Phi}E$ if and only if $E'\otimes k\neq 0$. Thus the equivalence of (a), (c) is proved.

9.2. COROLLARY [19]. *For any two-sided cell* $\underline{c}$ *of* W, *the set* $\Gamma_{\underline{c}}=\underline{c}\cap W_{\min}$ *is non-empty.*

PROOF. By 4.8, we have $\underline{c}=\underline{c}(u, 1, 1)$ for some unipotent element $u\in G$. It remains to use the equivalence of (b) and (c) in 9.1.

9.3. In [19] it is shown that, given a two-sided cell $\underline{c}$ of W, $\Gamma_{\underline{c}}$ is a single left cell; this is deduced from the fact that the algebra $J_{\Gamma_{\underline{c}}\cap\Gamma_{\underline{c}}^{-1}}$ is commutative.

9.4. PROPOSITION. *The simple modules of* $J_{\Gamma_{\underline{c}}\cap\Gamma_{\underline{c}}^{-1}}$ *are one-dimensional and are naturally in* 1–1 *correspondence with the semisimple conjugacy* *classes of* $Z_G(u)$, *(u as in* 9.2).

PROOF. The first assertion follows from the commutativity of $J_{\Gamma_c\cap\Gamma_c^{-1}}$ [19].

Let d be the unique element of $\mathcal{D}\cap\Gamma_c$. We have $J_{\Gamma_c\cap\Gamma_c^{-1}}=t_dJ_ct_d$.

By a general property of algebras in which the unit element is a sum of orthogonal idempotents, we see that $M\to t_dM$ defines a bijection between the set of simple J_c-modules M (up to isomorphism) such that $t_dM\neq 0$ and the set of simple $t_dJ_ct_d$-modules (up to isomorphism). Using 9.1, we see that the simple modules for $J_{\Gamma_c\cap\Gamma_c^{-1}}$ are precisely $t_dE(u,s,1)$ where s is any semisimple element of $Z_G(u)$ (up to conjugacy).

10. A conjecture

10.1. In this section we shall assume that G is almost simple, simply connected. We shall formulate a conjecture which makes the structure of the algebra J very explicit; it is a generalization of the conjecture [16, 3.15]; we also derive some consequences of this conjecture.

10.2. The following discussion is a slight generalization of that in [16, 2.2]: we replace finite groups by reductive groups. Let F be a reductive (possibly disconnected) algebraic group over C. Let Y be a finite F-set (i.e. set with an algebraic action of F; thus, F^0 acts trivially). An F-vector bundle ($=F$-v.b.) on Y is a collection of finite dimensional C-vector spaces V_y ($y\in Y$) with a given algebraic representation of F on $\bigoplus_{y\in Y} V_y$ such that $gV_y=V_{gy}$ for all $g\in F$, $y\in Y$. There is an obvious notion of direct sum and tensor product of F-v.b. on Y. Let $K_F(Y)$ be the Grothendieck group of the category of F-v.b. on Y. It has as basis the irreducible F-v.b. on Y (i.e. those F-v.b. which are $\neq 0$ and are not direct sums of two F-v.b. which are both $\neq 0$). If V is an irreducible F-v.b. on Y then the set $\{y\in Y\mid V_y\neq 0\}$ is a single F-orbit o in Y and, for $y\in o$, the obvious representation of the isotropy group F_y on V_y is irreducible; this gives a bijection between the set of irreducible F-v.b. on Y (up to isomorphism) and the set of pairs (y,ρ) where $y\in Y$, ρ is an irreducible (algebraic) representation of F_y, modulo the obvious action of F.

Let Y' be another finite F-set and let $\pi: Y\to Y'$ be an F-equivariant map. Let V an F-v.b. on Y. Define $\pi_!V$ to be the F-v.b. on Y' with

$(\pi_! V)_{y'} = \bigoplus_{y \in \pi^{-1}(y')} V_y$, $(y' \in Y')$, with the obvious action of F. This defines a homomorphism $\pi_! : K_F(Y) \to K_F(Y')$. Let V' be an F-v.b. on Y'. Define $\pi^! V'$ to be the F-v.b. on Y with $(\pi^! V')_y = V'_{\pi(y)}$ $(y \in Y)$, with the obvious action of F. This defines a homomorphism $\pi^! : K_F(Y') \to K_F(Y)$.

We now consider the finite F-set $Y \times Y$ with diagonal action of F, and two F-v.b. V, V' on $Y \times Y$. We define a new F-v.b. $V_* V'$ on $Y \times Y$ by $V_* V' = (\pi_{13})_! (\pi_{12}^! V \otimes \pi_{23}^! V')$ where $\pi_{ij} : Y \times Y \times Y \to Y \times Y$ are the various projections. Thus, we have $(V * V')_{y,y'} = \bigotimes_{y'' \in Y} (V_{y,y''} \otimes V_{y'',y'})$. This defines an associative ring structure on $K_F(Y \times Y)$; the unit element is the F-v.b. C_Δ which is C on the diagonal in $Y \times Y$ and is zero outside it. We define for any F-v.b. V on $Y \times Y$ a new F-v.b. $\tilde{V}$ on $Y \times Y$ by $\tilde{V} = i^!(V^*)$ where V^* is the F-v.b. dual to V and $i : Y \times Y \to Y \times Y$ is $i(y, y') = (y', y)$; thus $\tilde{V}_{y,y'} = V^*_{y',y}$; clearly $\tilde{V}$ is irreducible if V is irreducible. Note that $K_F(Y \times Y)$ is a based ring in the sense of [16, 1.1]. We denote the C-algebra $K_F(Y \times Y) \otimes C$ by $\underline{K}_F(Y \times Y)$.

10.3. Let s be a semisimple element of F. Let $\mathcal{F}_s$ be the algebra of C-valued functions on the fixed point set $(Y \times Y)^s$ which are constant on the orbits of $Z_F(s)$ (i.e. of $Z_F(s)/Z_F^0(s)$); the algebra structure is given by $(f * f')(y, y') = \sum_{y'' \in Y} f(y, y'') f'(y'', y')$.

We have an algebra homomorphism $h_s : \underline{K}_F(Y \times Y) \to \mathcal{F}_s$ defined by associating to an F-v.b. V on $Y \times Y$ the function $f(y, y') = \mathrm{Tr}(s, V_{y,y'})$. Now let ρ be a simple $Z_F(s)/Z_F^0(s)$-module which appears in the permutation representation of this finite group on the fixed point set Y^s. We attach to ρ the vector space $E_{s,\rho}$ of all functions $Y^s \to \rho$ which commute with the action of $Z_F(s)/Z_F^0(s)$; this is an $\mathcal{F}_s$-module; if $f \in \mathcal{F}_s$ and $\mu \in E_{s,\rho}$ we define $f\mu \in E_{s,\rho}$ by $(f\mu)(x) = \sum_{y \in Y^s} f(x, y)\mu(y)$, $(x \in Y^s)$. This is in fact, a simple $\mathcal{F}_s$-module and, by varying ρ one gets each simple $\mathcal{F}_s$-module exactly once. Furthermore we can regard $E_{s,\rho}$ as a $\underline{K}_F(Y \times Y)$-module, via h_s; this is a simple $\underline{K}_F(Y \times Y)$-module and $(s, \rho) \to E_{s,\rho}$ defines a bijection between the set of pairs (s, ρ) as above (up to F-conjugacy) and the set of isomorphism classes of simple $\underline{K}_F(Y \times Y)$-modules.

We have $\dim E_{s,\rho}$ = multiplicity of ρ in the permutation representation of $Z_F(s)/Z_F^0(s)$ on Y^s.

10.4. We fix a two-sided cell $\underline{c}$ of W. Let $\mathcal{O}$ be the unipotent class of G corresponding to $\underline{c}$ as in 4.8. Let $\phi : SL_2(C) \to G$ be a homomorphism of algebraic groups such that $u = \phi\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathcal{O}$ and let F be the centralizer in G of the image of ϕ; then F is a maximal reductive subgroup of $Z_G(u)$.

We can now state the following:

10.5. CONJECTURE. *In the setup of* 10.4, *there exists a finite F-set Y and a bijection*

$\Pi : \underline{c} \xrightarrow{\sim}$ *set of irreducible F-v.b. on $Y\times Y$ (up to isomorphism) with the following properties*:

(a) *the* **C***-linear map $J_{\underline{c}} \to \underline{K}_F(Y\times Y)$, $t_w \to \Pi(w)$, is an algebra isomorphism (preserving the unit element).*

(b) $\Pi(w^{-1}) = \widetilde{\Pi(w)}$ $(w \in \underline{c})$.

(c) *For any semisimple element $s\in F$, the $Z_F(s)/Z_F^0(s)$-module carried by the permutation representation of this finite group on the fixed point set Y^s is isomorphic to $\bigoplus_i H^{2i}(\mathcal{B}_u^s, C)$ regarded as a module over $Z_F(s)/Z_F^0(s) = Z_G(su)/Z_G^0(su)$ in a natural way.*

(d) *Under Π, the simple $\underline{K}_F(Y\times Y)$-module $E_{s,\rho}$ (see* 10.3) *corresponds to the simple $J_{\underline{c}}$-module $E(u, s, \rho)$, (see* 4.3).

(An analogous conjecture for W_0 instead of W was formulated in [16, 3.15] and the remark following it; in that case instead of F we had a finite group.) It follows that for $w, w', w'' \in \underline{c}$ we have

$$\Pi(w)*\Pi(w') = \bigoplus_{w''\in \underline{c}} \gamma_{w,w',w''}\widetilde{\Pi(w'')} \qquad \text{in} \quad K_F(Y\times Y)$$

so that the structure constants $\gamma_{w,w',w''}$ of $J_{\underline{c}}$ are completely described in terms of operations with vector bundles.

Note that the number of elements of Y is equal to the Euler characteristic of $\mathcal{B}_u$.

10.6. Let F_y be the isotropy group of $y\in Y$ in F. We can also regard Π as a bijection

(a) $\Pi' : \underline{c} \xrightarrow{\approx}$ set of triples (y, y', ρ) where $y, y' \in Y$, ρ is an irreducible (algebraic) representation of $F_y \cap F_{y'}$, modulo the obvious action of F/F^0. Note that $\Pi'(w) = (y, y', \rho) \Longrightarrow \Pi'(w^{-1}) = (y', y, \rho^*)$.

We have the following consequence of the conjecture 10.5.

(b) For each F-orbit o on Y, the set of triples (y, y', ρ) in (a) with $y \in o$ (resp. $y' \in o$) corresponds under Π' to a left cell (resp. right cell) in $\underline{c}$; this gives bijections between the set of left cells in $\underline{c}$, the set of right cells in $\underline{c}$, and the set of F-orbits in Y.

Hence to each left cell Γ in $\underline{c}$ one can associate a subgroup F_Γ of F containing F^0, well defined up to F-conjugacy; it is the isotropy group of some point in the F-orbit $o \subset Y$ corresponding to Γ.

The following is also a consequence of the conjecture 10.5.

(c) If Γ, o are as above then Π' restricts to a bijection

$$\Pi_\Gamma : \Gamma \cap \Gamma^{-1} \xrightarrow{\cong} \text{set of irreducible } F\text{-v.b. on } o \times o$$
$$\cong \text{set of } F/F^0\text{-orbits of triples } (y, y', \rho) \text{ in (a) with } y \in o,\ y' \in o$$

and this extends to an algebra isomorphism

$$J_{\Gamma \cap \Gamma^{-1}} \xrightarrow{\sim} \underline{K}_F(o \times o) = \underline{K}_F(F/F_\Gamma \times F/F_\Gamma).$$

(I have recently proved the analogue of (c) for W_0, which has been formulated in [16, 3.15] as a conjecture.)

10.7. Let $\Gamma_{\underline{c}}$ be as in 9.2, 9.3. One can expect that for this $\Gamma_{\underline{c}}$ one has $F_{\Gamma_{\underline{c}}} = F$, hence the F-orbit in Y corresponding to $\Gamma_{\underline{c}}$ (see 10.6) is a single element of Y, and $\Pi_{\Gamma_{\underline{c}}}$ defines a bijection between $\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}$ and the set of irreducible representations (up to isomorphism) of F, which extends to an algebra isomorphism $J_{\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}} \to \underline{R}_F$ (the representation ring of F, tensored with $\boldsymbol{C}$). Let ρ be an irreducible representation of F and let $w_\rho \in \Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}$ be the element corresponding to it under our (conjectural) bijection. It is likely that w_ρ can be characterized by the property that $t_{w_\rho} \in J_{\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}}$ acts on the simple module corresponding to the semisimple element $s \in F \subset Z_G(u)$ (see 9.4) by the scalar $\mathrm{Tr}(s, \rho)$.

10.8. The union $\bigcup_{\underline{c}} (\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1})$ is equal to the set of all $w \in W$ such that w has minimal length in $W_0 w W_0$; that set is in natural bijection with X_{dom}. (Each double coset $W_0 w W_0$ contains a unique element X_{dom} and a unique element of minimal length.) Using then 10.7 we obtain a (conjectural) bijection between X_{dom} and the set of pairs (u, ρ), (up to G-conjugacy) with $u \in G$ unipotent and ρ an irreducible representation of $Z_G(u)$.

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