Unipotent representations of complex reductive groups

David Vogan

Massachusetts Institute of Technology

Automorphic Project
Johns Hopkins/University of Michigan
the interwebs
December 10 2021

Outline

Slides at `http://www-math.mit.edu/~dav/paper.html`

Point of view

Point of view of seminar series begins with reductive algebraic $\mathbf{G}$ over global field k; seeks to understand automorphic forms: functions on $\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}(k))$.

Analytic version: **understand** $\leftrightsquigarrow$ find Plancherel decomp of $L^2(\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}))$.

My point of view begins with reductive algebraic $\mathbf{G}$ over local field F; seeks to understand reps of $G = \mathbf{G}(F)$.

Analytic version: **understand** $\leftrightsquigarrow$ find unitary reps of G.

First success: HC Plancherel decomp of $L^2(G)$.

Two points of view inform each other, but they're distinct.

Going to talk about Langlands' conjectures and Arthur's conjectures, which originate in automorphic forms. I'll still talk about them from my point of view, which will be unfair to the original conjectures.

Tant pis.

Prehistory of Arthur's conjectures

G reductive group over a local field.

$\widehat{G}$	$\supset$	$\widehat{G}_u$	$\supset$	$\widehat{G}_t$	$\supset$	$\widehat{G}_{ds}$
admissible irr reps		unitary irr reps		tempered irr reps		discrete series

Langlands conjecture 1970: parametrization of $\widehat{G}$.

In light of Harish-Chandra's work, Langlands' conjecture mostly reduces to $\widehat{G}_{ds}$.

Using Harish-Chandra's parametrization of $\widehat{G}_{ds}$ for groups over $\mathbb{R}$ and $\mathbb{C}$, Langlands proved his conjecture in those cases.

Langlands' conjecture clearly identifies $\widehat{G}_t$.

But it offers no hint about identifying rest of $\widehat{G}_u$.

Arthur's conjectures

$\Phi(G)$	$\supset$	$\Phi_t(G)$
Langlands params		tempered params

Parameter $\phi \rightsquigarrow \Pi_L(\phi) \subset \widehat{G}$ finite L-packet of ϕ.

Still conjectural for F p-adic.

DIFFICULTY: doesn't find nontempered unitary reps.

Arthur in 1983 introduced Arthur parameters $\Psi_A(G)$:

$\Phi(G)$	$\supset$	$\Psi_A(G)$	$\supset$	$\Phi_t(G)$
Langlands params		Arthur params		tempered params

Conjectured $\psi \rightsquigarrow \Pi_A(\psi) \supset \Pi_L(\psi)$ finite A-packet of ψ.

Conjectured $\Pi_A(\psi)$ consists of unitary reps.

Looked like a great way to address DIFFICULTY.

Still only hope in Mudville

Arthur: should be many sets $\Pi_A(\psi)$ of unitary reps.

Difficulty: no definition of $\Pi_A(\psi)$.

Barbasch-V 1985: defined $\Pi_A(\psi)$ for groups over $\mathbb{C}$;

calculated set $\Pi_A(\psi)$ fairly explicitly;

calculated characters in $\Pi_A(\psi)$, $\rightsquigarrow$ Arthur *desiderata*.

Paper $\rightsquigarrow$ hints about defining $\Pi_A(\psi)$ for groups over $\mathbb{R}$, realized in Adams-Barbasch-V book 1992.

Failed to prove $\Pi_A(\psi)$ consists of unitary reps.

It's only my point of view, not my heart's desire.

Forty years of shattered dreams and dashed hopes.

But I'm fine now, and not bitter.

L-group

Chevalley-Grothendieck: reductive alg G over alg closed $\overline{k}$ $\leftrightsquigarrow$ based root datum $\mathcal{R}(G) = (X^*, \Pi, X_*, \Pi^\vee)$.

reductive alg G over any k $\rightsquigarrow$ action of $\Gamma = \mathrm{Gal}(\overline{k}/k)$ on based root datum.

Axioms for root data are symm in $(X^*, \Pi) \leftrightarrow (X_*, \Pi^\vee)$.

Dual root datum is $\mathcal{R}^\vee = (X_*, \Pi^\vee, X^*, \Pi)$.

Gives reductive algebraic dual group ${}^\vee G$ and L-group ${}^L G = {}^\vee G \rtimes \Gamma$ over $\mathbb{Z}$.

Langlands' insight: representation theory$/K$ of $G(k)$ $\leftrightsquigarrow$ group theory of ${}^L G(K)$.

Langlands' insight over $\mathbb{R}$

complex reps of $G(\mathbb{R}) \rightsquigarrow$ group theory of ${}^\vee G(\mathbb{C}) \rtimes \{1, \sigma\}$.

How could this work?

First invariant of rep π is infl char $\lambda(\pi) \in \mathfrak{h}^* = X^* \otimes_{\mathbb{Z}} \mathbb{C}$.

Corresponds on ${}^\vee G$ to $\lambda \in {}^\vee\mathfrak{h}$: semisimple element in ${}^\vee\mathfrak{g}$.

Second invariant of π: put λ in real Cartan, get action of complex conjugation.

Corresponds in ${}^L G$ to $y \in {}^\vee G\sigma$ acting on λ.

A Langlands parameter is $(y, \lambda) \in {}^\vee G\sigma \times {}^\vee\mathfrak{g}$ with

$$\lambda \quad \text{semisimple,} \qquad y^2 = \exp(2\pi i\lambda).$$

Theorem (Langlands) To each pair (y, λ) as above is attached a finite set $\Pi(y, \lambda)$ of irr reps of $G(\mathbb{R})$, depending only on the ${}^\vee G$ conjugacy class of (y, λ). The sets $\Pi(y, \lambda)$ partition $\widehat{G(\mathbb{R})}$.

More about Langlands over $\mathbb{R}$

parameter is $(y,\lambda) \in {}^\vee G\sigma \times {}^\vee\mathfrak{g}$, $\quad y^2 = \exp(2\pi i\lambda)$

$\lambda \rightsquigarrow {}^\vee e = \exp(2\pi i\lambda) \in {}^\vee G \rightsquigarrow {}^\vee E = {}^\vee G^{{}^\vee e}$ pseudolevi in ${}^\vee G$; Lie algebra ${}^\vee\mathfrak{e}$ = sum of integer eigspaces of $\mathrm{Ad}(\lambda)$.

$\lambda \rightsquigarrow {}^\vee\mathfrak{q}(\lambda) \subset {}^\vee\mathfrak{e}$ parabolic, sum of nonneg integer eigspaces of $\mathrm{Ad}(\lambda)$.

$\lambda \rightsquigarrow {}^\vee Q$ partial flag variety of ${}^\vee E$-conjugates of ${}^\vee\mathfrak{q}$.

$y \rightsquigarrow {}^\vee e = y^2 \rightsquigarrow {}^\vee E = {}^\vee G^{{}^\vee e}$.

$y \rightsquigarrow {}^\vee K = {}^\vee G^{\,y} \subset {}^\vee E$, symm reductive subgrp of ${}^\vee E$.

Matsuki (1979), following Wolf (1969): ${}^\vee K$ acts on ${}^\vee Q$ with finitely many orbits, the orbit of ${}^\vee\mathfrak{q}(\lambda)$ corresponding precisely to the ${}^\vee G$-orbit of Langlands parameters (y,λ).

Theorem (Adams-Barbasch-V) There is a natural bijection (simple ${}^\vee K$-eqvt perverse sheaves on ${}^\vee Q$) $\leftrightsquigarrow$ (irr reps of infl char λ of inner forms of $G(\mathbb{R})$). Map to Langlands parameters is the support of a perverse sheaf, which must be the closure of a single ${}^\vee K$-orbit.

Where are the tempered reps?

Theorem (Langlands) Suppose

$$(y, \lambda) \in {}^\vee G\sigma \times {}^\vee\mathfrak{g}, \qquad y^2 = \exp(2\pi i\lambda)$$

is a parameter. Fix a Cartan ${}^\vee H \subset {}^\vee G$ so that

1. $\lambda \in {}^\vee\mathfrak{h} = X_*({}^\vee H) \otimes_{\mathbb{Z}} \mathbb{C}$, and
2. y normalizes ${}^\vee H$.

Then $\Pi_L(y, \lambda)$ consists of tempered reps $\iff$

$$\lambda + \mathrm{Ad}(y)(\lambda) \in X_*({}^\vee H) \otimes_{\mathbb{Z}} i\mathbb{R}.$$

In terms of the geometry of ${}^\vee K$ acting on ${}^\vee\mathcal{Q}$ (previous slide), tempered implies that $\mathrm{Ad}(y)({}^\vee Q(\lambda)) = {}^\vee Q^{\mathrm{op}}(\lambda)$ is opposite to $Q(\lambda)$, and therefore ${}^\vee K \cdot {}^\vee Q(\lambda)$ is open in ${}^\vee\mathcal{Q}$.

NOTE: often happens that ${}^\vee K \cdot {}^\vee Q(\lambda)$ is open in $\mathcal{Q}$ but the parameter is not tempered.

Tempered means y acts by -1 on real part of λ. Open orbit means y(pos integral roots for λ) = (neg integral roots for λ).

p-adic parameters

G reduc alg / k p-adic, $\Gamma = \mathrm{Gal}(\overline{k}/k) \rightsquigarrow {}^L G = {}^\vee G \rtimes \Gamma$.

Weil-Deligne group $W_k' = W_k \ltimes \mathbb{C}$, $w \cdot z = |w|z$.

Deligne-Langlands parameter $= \phi' \colon W_k' \to {}^L G$

$= (\phi, N_D) \quad \phi \colon W_k \to {}^L G$ (Langlands parameter)

$N_D \in {}^\vee\mathfrak{g}, \quad \mathrm{Ad}(\phi(w))(N_D) = |w| N_D$.

Infinitesimal character of ϕ' is $\lambda = \lambda(\phi') = \phi|_{I_k}$.

Fix Frobenius element $\mathrm{Fr} \in W_k$. Langlands parameter ϕ is

$$\phi = (y = \phi(\mathrm{Fr}), \lambda), \qquad y \in N_{{}^\vee G \cdot \mathrm{Fr}}(\lambda(I_k)).$$

Condition: y action on $\lambda(I_k) \leftrightsquigarrow$ Fr action on I_k.

So $\phi' = (y, \lambda, N_D)$.

More about p-adic parameters

Langlands parameter is triple $\phi' = (y, \lambda, N_D)$:

1. $\lambda\colon I_k \to {}^L G$ describes ramification;
2. $y \in {}^\vee G \cdot \mathrm{Fr}$ normalizes λ, respects Fr action on I_k;

Get reductive algebraic ${}^\vee G^\lambda$, semisimple aut $\mathrm{Ad}(y)$ of ${}^\vee G^\lambda$,

$$ {}^\vee G^{\lambda,y} = {}^\vee G^\phi \subset {}^\vee G^\lambda \qquad \text{twisted pseudolevi in } {}^\vee G^\lambda. $$

Get complex vector space of nilpotent elements

$$ {}^\vee\mathfrak{g}_q^\lambda = \quad q \text{ eigenspace of } \mathrm{Ad}(y). $$

Last condition on parameter is

3. $N_D \in {}^\vee\mathfrak{g}_q^\lambda$.

${}^\vee G^\phi$ acts on ${}^\vee\mathfrak{g}_q^\lambda$ with finitely many orbits; ${}^\vee G^\phi \cdot N_D \leftrightsquigarrow {}^\vee G$ orbit of Deligne-Langlands parameters (y, λ, N_D).

Local Langlands conjecture: There is a natural bijection (simple ${}^\vee G^\phi$-eqvt perverse sheaves on ${}^\vee\mathfrak{g}_q^\lambda$) $\leftrightsquigarrow$ (irr reps of inner forms of G of infl char λ). Map to Deligne-Langlands parameters is support of perverse sheaf: closure of one ${}^\vee G^\phi$ orbit.

What's a tempered p-adic parameter?

Suppose $\phi' = (y, \lambda, N_D)$ Deligne-Langlands parameter, so ${}^\vee G^\lambda$ is reductive, $\mathrm{Ad}(y)$ is semisimple aut of ${}^\vee G^\lambda$, and $N_D \in {}^\vee\mathfrak{g}_q^\lambda$ (q eigenspace of $\mathrm{Ad}(y)$).

Jacobson-Morozov: $\phi_{N_D}\colon SL(2) \to {}^\vee G^\lambda$, $d\phi_{N_D}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = N_D$. Set

$$y_0 = y \cdot \phi_{N_D}\begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}$$

Conjecture (Deligne-Langlands?) $\Pi_L(y, \lambda, N)$ consists of tempered reps $\iff$ $\langle y_0 \rangle \subset {}^L G$ has compact closure.

In terms of geometry of ${}^\vee G^\phi$ on vector space ${}^\vee\mathfrak{g}_q^\lambda$ (previous slide), tempered condition $\implies$ ${}^\vee G^\phi \cdot N_D$ is open in ${}^\vee\mathfrak{g}_q^\lambda$.

NOTE: often happens that ${}^\vee G^\phi \cdot N_D$ is open in ${}^\vee\mathfrak{g}_q^\lambda$ but the parameter is not tempered.

Backhanded apology

In describing Deligne-Langlands parameters, I tried hard to avoid introducing $SL(2)$.

This was deliberate: Deligne defn of W'_k had no $SL(2)$.

Unfortunately the literature on Langlands' conjectures is replete with $SL(2)$s.

I believe the ones used for W'_K are a mistake.

I'm not sure about the "Arthur $SL(2)$."

Perhaps it's the L-group of $PGL(2)$, and
Arthur parameter $=$ functoriality (trivial of $PGL(2)$).

But I do not know how to make this idea precise.

This is all to say that I am likely to misstate Arthur's conjectures in very serious ways.

Sorry!

Arthur parameters over $\mathbb{R}$

Recall: Langlands parameter is $\phi_0 = (y_0, \lambda_0) \in {}^\vee G\sigma \times {}^\vee\mathfrak{g}$, $y^2 = {}^\vee e = \exp(2\pi i\lambda)$

Definition (Arthur). Arthur parameter is $\psi = (y_0, \lambda_0, f)$ with

1. $\phi_0 = (y_0, \lambda_0)$ tempered Langlands parameter;
2. $f\colon SL(2) \to {}^\vee G$ algebraic; and
3. image of f commutes with y_0 and λ_0

From ψ can construct another parameter $\phi(\psi) = (y, \lambda)$,

$$y = y_0 \cdot f\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \lambda = \lambda_0 + df\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

Change in $\lambda \rightsquigarrow \phi(\psi)$ nontempered.

Then $\phi(\psi)$ is the Langlands parameter Arthur attaches to ψ, so that one of his desiderata is $\Pi_A(\psi) \supset \Pi_L(\phi(\psi))$.

Arthur packets over $\mathbb{R}$: ABV version

$\psi = (y_0, \lambda_0, f)$ Arthur parameter $\rightsquigarrow$

$$y = y_0 \cdot f\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \lambda = \lambda_0 + df\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad N_A = df\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$\lambda \rightsquigarrow {}^\vee e = \exp(2\pi i \lambda) \in {}^\vee G \rightsquigarrow {}^\vee E = {}^\vee G^{{}^\vee e}$ pseudolevi in ${}^\vee G$.
$\lambda \rightsquigarrow {}^\vee \mathcal{Q}$ partial flag variety of ${}^\vee E$-conjugates of ${}^\vee \mathfrak{q}(\lambda)$.
$y \rightsquigarrow {}^\vee K = {}^\vee G^y \subset {}^\vee E$, symm reductive subgrp of ${}^\vee E$.

$N_A \in {}^\vee \mathfrak{u}(\lambda) \leftrightsquigarrow {}^\vee K$-orbit ${}^\vee \mathcal{O}^\theta$ of nilp elts in ${}^\vee \mathfrak{e}/{}^\vee \mathfrak{k}$.

Recall ABV version of LLC: (simple ${}^\vee K$-eqvt perv sheaves on ${}^\vee \mathcal{Q}$) $\leftrightsquigarrow$ (irr reps of infl char λ of inner forms of $G(\mathbb{R})$). Map to Langlands params is support of perverse sheaf.

Definition (ABV). Arthur packet $\Pi_A(\psi) \leftrightsquigarrow$ perv sheaves whose char cycle contains conormal to ${}^\vee K \cdot \mathfrak{q}(\lambda)$.

Motivation: equivalent to require $(\mathfrak{q}(\lambda), N_A)$ in char cycle.

In terms of $({}^\vee \mathfrak{e}, {}^\vee K)$-modules, these are annihilated by kernel of map to diff ops on ${}^\vee \mathcal{Q}$, of largest possible GK dimension.

What's in [BV]?

Arthur's conjectures provide infl char map λ from nilpotent adjoint orbits in ${}^\vee\mathfrak{g}$ to infl chars for $\mathfrak{g}$. What [BV] does:

1. Def of order reversing duality d from nilpotent adjoint orbits in ${}^\vee\mathfrak{g}$ to nilpotent coadjoint orbits in $\mathfrak{g}$.
2. Infl char bound: X irr $\mathfrak{g}$-module of infl char λ' and assoc nilp $\mathcal{O}'$. Assume that
   a) $\mathcal{O}' \subset$ closure of $d({}^\vee\mathcal{O})$, and
   b) $\lambda' \in \lambda(\mathcal{O}) + X^*$.

   Then $|\lambda'| \geq |\lambda(\mathcal{O})|$; = only if $\mathcal{O}' = d({}^\vee\mathcal{O})$, and $\lambda' = \lambda(\mathcal{O})$.
3. Characterization of unip A-pkts by W reps in char formulas.
4. Characterization of these W reps using ${}^\vee G$ and ${}^\vee\mathcal{O}$.

Item (2.) $\rightsquigarrow$ Arthur reps π in terms of G: GK dim must be small, infl char small as possible given first cond.

Item (3.) $\rightsquigarrow$ info about characters of Arthur reps.

Item (4.) writes the char info using ${}^LG$, as Arthur asks.

LLC over $\mathbb{R}$, part one

λ infl char $\rightsquigarrow$ in $\mathfrak{h}^* = X^*(\mathbf{H}) \otimes_{\mathbb{Z}} \mathbb{C}$ dominant on G side.

$\lambda \rightsquigarrow$ integral coroots $R^\vee(\lambda) = \{\alpha^\vee \in R^\vee(\mathbf{G}, \mathbf{H}) \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\}$.

$\Pi^\vee(\lambda) =$ simple for $R^\vee(\lambda) \supset \Pi^{\vee,\lambda}$ zero on λ.

$W(\lambda) = W(R^\vee(\lambda))$ integral Weyl group $\supset W^\lambda$ Levi.

λ infl character $\rightsquigarrow$ in ${}^\vee\mathfrak{h} = X_*({}^\vee\mathbf{H}) \otimes_{\mathbb{Z}} \mathbb{C}$ on ${}^\vee G$ side.

$\lambda \rightsquigarrow {}^\vee e = \exp(2\pi i \lambda) \in {}^\vee G \rightsquigarrow {}^\vee E = {}^\vee G^{{}^\vee e}$ pseudolevi.

$R({}^\vee E, {}^\vee H) = R^\vee(\lambda)$, $W({}^\vee E, {}^\vee H) = W(R^\vee(\lambda))$ int Weyl gp.

Modify λ on G by $X^*(\mathbf{H}) \rightsquigarrow {}^\vee E$ unchanged.

Theorem (ABV). (Irr reps M of forms of $\mathbf{G}$ of infl char λ) $\leftrightsquigarrow$ (irr $({}^\vee\mathfrak{e}, {}^\vee K)$-mods ${}^\vee M$ of triv infl char, $\tau({}^\vee M) \supset \Pi^{\vee,\lambda}$).

LLC over $\mathbb{R}$, part two

$\lambda \rightsquigarrow$ integral coroots $R^\vee(\lambda) \rightsquigarrow W(\lambda)$ integral Weyl group.

$\widehat{\mathbf{G}}_\lambda =$ irr reps M of infl char λ; $W(\lambda)$ acts on $\mathbb{Z}\cdot\widehat{\mathbf{G}}_\lambda$.

Action $\rightsquigarrow$ chars, comp series of standard reps,. . . .

Better: W^λ-bi-coinvts in $\mathbb{Z}W(\lambda)$ acts on $\mathbb{Z}\cdot\widehat{\mathbf{G}}_\lambda$.

All tied to irr reps of $W(\lambda)$ containing trivial of W^λ.

$\widehat{{}^\vee E}^\lambda =$ irrs ${}^\vee M$, triv infl char, $\tau({}^\vee M) \supset \Pi^{\vee,\lambda}$; $W(\lambda)$ acts on $\mathbb{Z}\cdot\widehat{{}^\vee E}^\lambda$.

Action $\rightsquigarrow$ characters, comp series of standard reps,. . . .

Better: W^λ-bi-skew in $\mathbb{Z}W(\lambda)$ acts on $\mathbb{Z}\cdot\widehat{{}^\vee E}^\lambda$.

All tied to irr reps of $W(\lambda)$ containing sgn of W^λ.

G and ${}^\vee G$ linked by $W(\lambda)$-invt perfect pairing

$$\mathbb{Z}\cdot\widehat{\mathbf{G}}_\lambda \times \mathbb{Z}\cdot\widehat{{}^\vee E}^\lambda \to \mathrm{sgn}(W(\lambda)).$$

LLC makes $\{M \in \widehat{\mathbf{G}}_\lambda\}$ and $\{{}^\vee M \in \widehat{{}^\vee E}^\lambda\}$ dual bases in pairing.

About the infinitesimal character bound

Infl Arthur param is (λ, N_A) in ${}^\vee\mathfrak{g}$ extending to (N'_A, λ, N_A),

$$[\lambda, N_A] = 2N_A, \qquad [\lambda, N'_A] = -2N'_A, \qquad [N_A, N'_A] = \lambda.$$

$\lambda \rightsquigarrow {}^\vee e = \exp(2\pi i\lambda) \in {}^\vee G \rightsquigarrow {}^\vee E = {}^\vee G^{{}^\vee e}$ pseudolevi in ${}^\vee G$.

LLC $\implies$ (irr G reps of infl char $\lambda' \in \lambda + X^*$) $\rightsquigarrow$ $({}^\vee\mathfrak{e}, {}^\vee K)$-modules of trivial infl char, τ invt $\supset \Pi^{\vee,\lambda}$.

Plan was to sketch proof of bound; but refer to [BV].

Understanding the W reps

Suppose (λ, N_A) is an infinitesimal Arthur param.

This one Yiannis asked about during the talk, and I sketched part of an answer on Microsoft OneNote page
`https://1drv.ms/u/s!AuIZlbpNWacjghE8sRpTGwnC682U`

p-adic Arthur parameters

Recall: Deligne-Langlands parameter is $\phi_0 = (y_0, \lambda, N_D)$ with $\phi = (y_0, \lambda)\colon W_k \to {}^L G$ a Langlands parameter:

$$\lambda\colon I_k \to {}^L G, \qquad y \in N_{{}^\vee G \cdot \mathrm{Fr}}(\lambda), \qquad \mathrm{Ad}(y)(N_D) = qN_D.$$

Then ${}^\vee G^\lambda$ is a reductive subgroup.

Definition (Arthur). Arthur parameter is $\psi = (\phi_0', f)$ with

1. $\phi_0' = (y_0, \lambda, N_D)$ tempered Langlands parameter;
2. $f\colon SL(2) \to {}^\vee G^{\phi_0'}$ algebraic.

Arthur $\psi \rightsquigarrow$ new Langlands parameter

$$\phi(\psi) = (y, \lambda, N_D), \qquad y = y_0 \cdot f\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}.$$

Change in $y \rightsquigarrow \phi(\psi)$ nontempered.

Then $\phi(\psi)$ is the Langlands parameter corresponding to ψ, so that one of the requirements on the Arthur packet is

$$\Pi_A(\psi) \supset \Pi_L(\phi(\psi)).$$

p-adic Arthur packets: ABV version

$\psi = (\phi_0, f) = (y_0, \lambda, N_D, f)$ Arthur parameter.

$$\rightsquigarrow y = y_0 \cdot f\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \phi(\psi) = (y, \lambda),\ N_A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Recall reductive subgp ${}^\vee G^\lambda \supset {}^\vee G^{\phi(\psi)}$ twisted pseudolevi.

$N_D, N_A \in {}^\vee\mathfrak{g}^\lambda_q = q$ eigenspace of $\mathrm{Ad}(y)$.

ABV version of LLC: There is a natural bijection (simple ${}^\vee G^\phi(\psi)$-eqvt perverse sheaves on ${}^\vee\mathfrak{g}^\lambda_q$) $\leftrightsquigarrow$ (irr reps of inner forms of G of infl char λ). Map to Deligne-Langlands parameters is support of perverse sheaf: closure of one ${}^\vee G^{\phi(\psi)}$ orbit.

Definition (ABV). Arthur packet $\Pi_A(\psi_{\mathrm{mod}}) \leftrightsquigarrow$ perv sheaves whose char cycle contains conormal to ${}^\vee G^{\phi(\psi)} \cdot N_D$.

Motivation: equivalent to require (N_D, N_A) in char cycle.

For any perverse sheaf, char cycle contains conormal to support; so $\Pi_A(\psi) \supset \Pi_L(\phi(\psi_A))$.