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Cohomology and group representations

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Contents

This article is based on three lectures ostensibly devoted to "cohomological induction," a method for constructing unitary representations of reductive Lie groups. In fact the lectures concerned mostly more elementary cohomological notions, beginning with de Rham cohomology of compact manifolds. When the manifolds are related to Lie groups, de Rham cohomology is related to Lie algebra cohomology. In this way questions about de Rham cohomology can sometimes be translated into questions about cohomological properties of group representations. Cohomological induction appears at the very end, as a way to construct representations having these cohomological properties.

I am grateful to the organizers for the opportunity to participate in this conference. Tony Knapp's notes are responsible for whatever connection exists between this article and the original lectures.

1. Cohomology of locally symmetric spaces

Suppose G is a connected real reductive algebraic group, and $K \subset G$ is a maximal compact subgroup. The homogeneous space G/K is a Riemannian symmetric space; it is diffeomorphic to $\mathbb{R}^n$. Suppose now that $\Gamma \subset G$ is a torsion-free discrete subgroup. Then Γ acts freely on G/K on the left, so the double coset space

$$X = \Gamma\backslash G/K \qquad (1.1)(a)$$

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is a smooth manifold (in fact a Riemannian locally symmetric space). Since G/K is simply connected, it is the universal cover of X; so

$$\pi_1(X) \simeq \Gamma. \tag{1.1)(b}$$

But even more is true. Because G/K is contractible, X is a "$K(\Gamma, 1)$," an Eilenberg-MacLane space. It may be thought of as a kind of geometric incarnation of the discrete group Γ. According to the original definition of the cohomology of the group Γ, we have

$$H^i(\Gamma, \mathbb{C}) \simeq H^i(X, \mathbb{C}). \tag{1.1)(c}$$

If G/K is a Hermitian symmetric space, then it is a complex Stein manifold. The complex structure is inherited by X. If Γ is cocompact in G, then X has in a natural way the structure of a projective algebraic variety; it is a Shimura variety. (Actually the most interesting Shimura varieties arise from non-cocompact arithmetic subgroups Γ, by compactification of X.) A great deal is known about the cohomology of Shimura varieties; some background may be found in [**9**]. From the point of view of the Langlands program, however, the most basic example of a Riemannian locally symmetric space has $G = GL(n, \mathbb{R})$ and $K = O(n)$. In that case X is not a complex manifold (unless $n = 2$); and there seem to be few ideas about what kind of special extra structure X might carry.

At any rate, we want to study the cohomology of X using the de Rham theorem. The de Rham complex has differential

$$d\colon (\text{complex-valued } p\text{-forms on } X) \to (\text{complex-valued } p+1\text{-forms on } X)\,.$$

Its cohomology groups are $H^p(X, \mathbb{C})$. We want to study this complex in group-theoretic terms. We begin by replacing X by a homogeneous space G/H. The first case to look at is G itself. A p-form on G is a section of $\wedge^p(T^*G)$. Because G is a Lie group, T^*G can be trivialized by left-invariant forms. This leads to a trivialization of p-forms, as follows. Think of the Lie algebra $\mathfrak{g}$ as consisting of the left-invariant vector fields on G. If ω is a p-form on G and $X_1, \dots, X_p \in \mathfrak{g}$ are left-invariant vector fields, then

$$\omega(X_1, \dots, X_p) \in C^\infty(G). \tag{1.2)(a}$$

This construction provides an identification

$$(p\text{-forms on } G) \simeq \mathrm{Hom}_{\mathbb{R}}(\wedge^p \mathfrak{g}, C^\infty(G)). \tag{1.2)(b}$$

(We have been a little vague about the coefficients: for complex valued p-forms one must use complex-valued smooth functions, and for real-valued forms real-valued smooth functions.)

The next problem is to compute the differential. If ω is a p-form on a smooth manifold M and $X_0, \dots, X_p$ are vector fields, then

$$\begin{aligned} d\omega(X_0, \dots, X_p) = &\sum_{i=0}^{p} (-1)^i X_i \cdot \omega(X_0, \dots, \widehat{X_i}, \dots, X_p) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p). \end{aligned} \tag{1.3}$$

(See for example [**18**], Proposition 2.25(f).) Here in the first sum the vector field X_i acts on the smooth function $\omega(X_0, \dots, \widehat{X_i}, \dots, X_p)$. This formula is well suited to the identification (1.2)(b) of forms on G, because the left-invariant forms are closed under Lie bracket. The resulting formula for d on $\mathrm{Hom}_{\mathbb{R}}(\wedge^p \mathfrak{g}, C^\infty(G))$ involves just two things: the action of $\mathfrak{g}$ on $C^\infty(G)$ by differentiation on the right, and the Lie bracket on $\mathfrak{g}$.

Now suppose $H \subset G$ is a closed subgroup. We want to identify p-forms on G/H as "special" p-forms on G. There is a submersion $\pi: G \to G/H$. The corresponding pullback is an inclusion

$$\pi^*: (p\text{-forms on } G/H) \hookrightarrow (p\text{-forms on } G). \tag{1.4}$$

Pullback of forms by smooth maps always commutes with d, so this is an inclusion of complexes. It is not difficult to identify the image.

PROPOSITION 1.5. *In the setting of* (1.4), *a p-form* $\omega \in \mathrm{Hom}_{\mathbb{R}}(\wedge^p \mathfrak{g}, C^\infty(G))$ *comes from G/H if and only if*

1. $\omega(X, \dots) = 0$ *whenever* $X \in \mathfrak{h}$, *and*
2. $\omega \in \mathrm{Hom}_H(\wedge^p \mathfrak{g}, C^\infty(G))$. *Here H acts on $\wedge^p \mathfrak{g}$ by the adjoint action, and on $C^\infty(G)$ by right translation.*

Consequently there is an identification

$$(p\text{-}\mathit{forms\ on\ } G/H) \simeq \mathrm{Hom}_H(\wedge^p \mathfrak{g}/\mathfrak{h}, C^\infty(G)).$$

If Γ is a discrete subgroup of G acting freely and properly discontinuously on G/H (on the left), then there is an identification

$$(p\text{-}\mathit{forms\ on\ } \Gamma\backslash G/H) \simeq \mathrm{Hom}_H(\wedge^p \mathfrak{g}/\mathfrak{h}, C^\infty(\Gamma\backslash G)).$$

In all cases the formula for d is (1.3): it involves the action of $\mathfrak{g}$ on $C^\infty(G)$ or $C^\infty(\Gamma\backslash G)$ by differentiation on the right. The formula for the complex involves also the right translation action of H on $C^\infty(G)$ or $C^\infty(\Gamma\backslash G)$. In order to apply representation theory to this picture, we will try to decompose $C^\infty(\Gamma\backslash G)$ into pieces invariant under these two right actions, and then study the contribution of each piece separately to $H^p(\Gamma\backslash G/H, \mathbb{C})$. Here is a natural formal setting for this study.

DEFINITION 1.6. A *pair* is a pair $(\mathfrak{g}, H)$ where $\mathfrak{g}$ is a finite-dimensional real Lie algebra and H is a Lie group with $\mathfrak{h} \subset \mathfrak{g}$. We also assume given an action Ad of H on $\mathfrak{g}$ by Lie algebra automorphisms, compatible with the adjoint action of H on $\mathfrak{h}$.

If G is a Lie group with Lie algebra $\mathfrak{g}$ and H is a Lie subgroup of G, then $(\mathfrak{g}, H)$ is in a natural way a pair. For us the most important example will be the pairs $(\mathfrak{g}, K)$ with G a reductive Lie group and K a maximal compact subgroup.

DEFINITION 1.7. Suppose $(\mathfrak{g}, H)$ is a pair. A $(\mathfrak{g}, H)$-*module* is a complex vector space V endowed with representations of $\mathfrak{g}$ and H, subject to the following conditions.

1. The action of H on V is *locally finite.* That is, each $v \in V$ belongs to a finite-dimensional H-invariant subspace V_1, and the representation of H on V_1 is smooth.
2. The differential of the action of H (which makes sense by 1) is equal to the restriction to $\mathfrak{h}$ of the action of $\mathfrak{g}$.
3. If $X \in \mathfrak{g}$, $h \in H$, and $v \in V$, then $h \cdot (X \cdot v) = (\mathrm{Ad}(h)X) \cdot (h \cdot v)$.

EXAMPLE 1.8. Suppose H is a closed subgroup of a Lie group G. There are representations of $\mathfrak{g}$ and H on $C^\infty(G)$, by differentiation and translation on the right. These satisfy condition (3) in Definition 1.7, and even a version of (2). (One needs to impose an appropriate topology on $C^\infty(G)$ to make sense of the limit appearing in the definition of derivative.) But condition (1) fails unless H is finite. We can circumvent the problem in the following way. Write ρ for the action of G on $C^\infty(G)$ by right translation:

$$(\rho(g)f)(x) = f(xg) \qquad (g, x \in G).$$

Now define

$$C^\infty(G)_H = \{f \in C^\infty(G) \mid \dim(\langle \rho(h)f \mid h \in H\rangle) < \infty\}.$$

Here $\langle \rho(h)f\rangle$ is the space spanned by all right translates of f by elements of H. The subspace $C^\infty(G)_H$ is preserved by the action of $\mathfrak{g}$, and obviously it satisfies (1) of Definition 1.7. Consequently $C^\infty(G)_H$ is a $(\mathfrak{g}, H)$-module. If Γ is any subgroup of G, then the space $C^\infty(\Gamma\backslash G)_H$ of functions invariant by Γ on the left is a $(\mathfrak{g}, H)$-submodule.

EXAMPLE 1.9. Suppose G is a reductive group, K is a maximal compact subgroup, and $(\pi, \mathcal{H}_\pi)$ is a continuous representation on a Hilbert space. Write $\mathcal{H}_\pi^\infty$ for the space of smooth vectors of π. This is a dense subspace of $\mathcal{H}_\pi$ invariant under the action of G, and it carries a natural representation of the Lie algebra $\mathfrak{g}$. By analogy with the preceding example, we can define

$$\mathcal{H}_{\pi,K}^\infty = \{v \in \mathcal{H}_\pi^\infty \mid \dim(\langle \pi(k)v \mid k \in K\rangle) < \infty\}$$

the space of K-finite smooth vectors of π. This space is invariant under the action of $\mathfrak{g}$ (although not under the action of G), and is therefore a $(\mathfrak{g}, K)$-module, called the *Harish-Chandra module of* π. Because K is compact, it is easy to check that $\mathcal{H}_{\pi,K}^\infty$ is dense in $\mathcal{H}_\pi$.

The construction in the preceding example makes sense for any compact subgroup of any Lie group. What makes it particularly interesting when G is reductive and K is maximal compact are theorems of Harish-Chandra, which say that when π is irreducible and unitary, then $\mathcal{H}_{\pi,K}^\infty$ is algebraically irreducible (as a $(\mathfrak{g}, K)$-module) and determines π.

DEFINITION 1.10. Suppose $(\mathfrak{g}, H)$ is a pair, and V is a $(\mathfrak{g}, H)$-module (Definitions 1.6 and 1.7). The adjoint action Ad of H on $\mathfrak{g}$ preserves $\mathfrak{h}$, and therefore descends to $\mathfrak{g}/\mathfrak{h}$. We can therefore define

$$\Omega^p(\mathfrak{g}, H; V) = \mathrm{Hom}_H(\wedge^p(\mathfrak{g}/\mathfrak{h}), V),$$

the V-valued p-forms for $(\mathfrak{g}, H)$. We want to define a differential making this a complex. For $\omega \in \Omega^p(\mathfrak{g}, H; V)$, we define $d\omega$ by

$$\begin{aligned} d\omega(X_0, \dots, X_p) = &\sum_{i=0}^{p} (-1)^i X_i \cdot \omega(X_0, \dots, \widehat{X_i}, \dots, X_p) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p) \end{aligned}$$

whenever $X_i \in \mathfrak{g}$. The action in the first sum is given by the representation of $\mathfrak{g}$ on the range of ω. It is not difficult to check that $d\omega \in \Omega^{p+1}(\mathfrak{g}, H; V)$; and that $d^2 = 0$ follows from the Jacobi identity for $\mathfrak{g}$. We may therefore define

$$H^p(\mathfrak{g}, H; V) = (\ker d \text{ on } \Omega^p(\mathfrak{g}, H; V))/(\operatorname{im} d \text{ on } \Omega^{p-1}(\mathfrak{g}, H; V)),$$

the *relative Lie algebra cohomology of* $(\mathfrak{g}, H)$ *with coefficients in* V.

PROPOSITION 1.11. *Suppose H is a closed subgroup of the Lie group G, and that $\Gamma \subset G$ is a discrete group acting freely and properly discontinuously on G/H (so that $\Gamma\backslash G/H$ is a manifold). Define $C^\infty(\Gamma\backslash G)_H$ as in Example* 1.8. *Then there is a natural isomorphism*

$$H^p(\Gamma\backslash G/H, \mathbb{C}) \simeq H^p(\mathfrak{g}, H; C^\infty(\Gamma\backslash G)_H).$$

This is a formal consequence of the de Rham theorem, Proposition 1.5, and the definitions.

To make further progress along the lines suggested at the beginning of these notes, we need to decompose $C^\infty(\Gamma\backslash G)_H$ as a $(\mathfrak{g}, H)$-module. The simplest results are available when G is reductive, K is maximal compact, and Γ is cocompact and torsion free. In that case the unitary representation of G on $L^2(\Gamma\backslash G)$ is a Hilbert space direct sum of irreducible representations having finite multiplicity:

$$L^2(\Gamma\backslash G) \simeq \widehat{\bigoplus_{\pi\in\widehat{G}}} m_\pi \mathcal{H}_\pi \tag{1.12)(a}$$

Here m_π is a non-negative integer, the multiplicity of π in $L^2(\Gamma\backslash G)$. (Often it can be identified as the dimension of some classical space of automorphic functions.) For example, if $(\tau, \mathbb{C})$ is the trivial representation of G, then m_τ is the dimension of the space of G-invariant functions in $L^2(\Gamma\backslash G)$. Obviously the only G-invariant functions are constant; and since these do belong to L^2 (since $\Gamma\backslash G$ is assumed to be compact) we get

$$m_\tau = 1 \qquad (\tau = \text{trivial representation of } G) \tag{1.12)(b}$$

In order to apply Proposition 1.11 we need to understand not just the decomposition of L^2 but the more subtle decomposition of $C^\infty(\Gamma\backslash G)$. It turns out that the smooth vectors in each $\mathcal{H}_\pi$ map (by (1.12)(a)) to smooth functions on $\Gamma\backslash G$; so there are inclusions

$$\bigoplus_{\pi\in\widehat{G}} m_\pi \mathcal{H}_\pi^\infty \hookrightarrow C^\infty(\Gamma\backslash G). \tag{1.12)(c}$$

$$\bigoplus_{\pi\in\widehat{G}} m_\pi \mathcal{H}_{\pi,K}^\infty \hookrightarrow C^\infty(\Gamma\backslash G)_K. \tag{1.12)(d}$$

At least in the case of (1.12)(d), one can describe exactly how the sum on the left must be completed to give an isomorphism. This leads to the following fundamental result of Matsushima.

THEOREM 1.13 ([**11**]; see [**2**], Theorem VII.3.2). *Suppose G is a real reductive algebraic group, K is a maximal compact subgroup, and Γ is a torsion-free* *cocompact subgroup. Use the notation of* (1.12). *The inclusion of* (1.12)(d) *and the isomorphism of Proposition* 1.11 *induce an isomorphism*

$$\bigoplus_{\pi \in \widehat{G}} m_\pi H^p(\mathfrak{g}, K; \mathcal{H}^\infty_{\pi,K}) \simeq H^p(\Gamma\backslash G/K, \mathbb{C}).$$

Matsushima's theorem accomplishes in this setting the goal of disassembling the cohomology of the space $\Gamma\backslash G/K$ into contributions of irreducible representations. In the next section we will begin to examine those individual contributions.

2. Cohomology of irreducible representations: the trivial representation

If we recall Harish-Chandra's theorem that the space of smooth K-finite vectors in an irreducible unitary representation of a reductive group G is an algebraically irreducible $(\mathfrak{g}, K)$-module, then Theorem 1.13 suggests

PROBLEM 2.1. Determine the set of irreducible $(\mathfrak{g}, K)$-modules V for which $H^*(\mathfrak{g}, K; V) \neq 0$; and compute the cohomology in those cases.

This problem can be completely solved when $\operatorname{rank} G = \operatorname{rank} K$, and quite a bit is known about it in general. There are only finitely many inequivalent V for which the cohomology is non-zero, and it is not terribly difficult to list the candidates. (In this connection an old result of David Wigner (see [**2**], Theorem I.4.1) says that the cohomology can be non-zero only if the center of the enveloping algebra acts in V as in the trivial representation. This already reduces matters to a finite set of candidates.) Actually computing the cohomology is more difficult, and involves the full strength of the ideas around the Kazhdan-Lusztig conjectures: $\mathcal{D}$-modules, the Beilinson-Bernstein localization theory, and perverse sheaves.

Fortunately for us, Problem 2.1 is not quite the right question. The answer simplifies enormously if we change it to

PROBLEM 2.2. Determine the set of irreducible unitary $(\mathfrak{g}, K)$-modules V for which $H^*(\mathfrak{g}, K; V) \neq 0$; and compute the cohomology in those cases.

To see what kind of answer we can expect, we begin with an example. Suppose

$$G = U(p, q), \qquad K = U(p) \times U(q). \tag{2.3)(a}$$

This means that G is the group of complex-linear transformations of $\mathbb{C}^{p+q}$ preserving the Hermitian form

$$|z_1|^2 + \cdots + |z_p|^2 - |z_{p+1}|^2 - \cdots - |z_{p+q}|^2 \tag{2.3)(b}$$

THEOREM 2.4 ([**17**]). *In the setting of* (2.3), *the set of irreducible unitary* $(\mathfrak{g}, K)$*-modules* V *with* $H^*(\mathfrak{g}, K; V) \neq 0$ *is in one-to-one correspondence with all expressions*

$$\begin{array}{ll} p = p_1 + \cdots + p_r & p_i + q_i \neq 0 \\ & p_i = 0 \Rightarrow q_i = 1 \\ q = q_1 + \cdots + q_r & q_i = 0 \Rightarrow p_i = 1 \end{array}$$

In that case, there is a dimension shift R *(depending on the* p_i *and* q_i*) so that* $H^*(\mathfrak{g}, K; V)$ *may be computed in terms of the cohomology of a compact symmetric* *space:*

$$H^m(\mathfrak{g}, K; V) \simeq H^{m-R}\left(\prod_{i=1}^{r} U(p_i+q_i)/(U(p_i)\times U(q_i)), \mathbb{C}\right).$$

Here $U(p_i+q_i)/(U(p_i)\times U(q_i))$ is the Grassmanian of p_i-planes in $\mathbb{C}^{p_i+q_i}$.

The trivial representation $V = \mathbb{C}$ corresponds in this parametrization to the case $r = 1$; that is, to the expressions $p = p_1$ and $q = q_1$. The dimension shift R is zero, as we will see in Theorem 2.10 below.

We will eventually give a similarly precise and explicit result for any G. For the rest of this section, we will concentrate on the problem of computing the cohomology groups of the Grassmann varieties appearing in the theorem. We begin with a closer look at the complex of Definition 1.10.

DEFINITION 2.5. The pair $(\mathfrak{g}, H)$ (Definition 1.6) is said to be *symmetric* if we are given an involutive automorphism σ of $\mathfrak{g}$ such that σ commutes with $\mathrm{Ad}(H)$, and $\mathfrak{g}^\sigma = \mathfrak{h}$. In this case we write $\mathfrak{q}$ for the -1 eigenspace of σ, so that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \qquad \mathrm{Ad}(H)(\mathfrak{q}) \subset \mathfrak{q}.$$

The fact that σ is a Lie algebra automorphism means that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}, \qquad [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}.$$

Two examples will be important for us: the pairs $(\mathfrak{g}, K)$ with G reductive and K maximal compact; and the pairs $(\mathfrak{a}, 1)$ with $\mathfrak{a}$ an abelian Lie algebra.

PROPOSITION 2.6. *If $(\mathfrak{g}, H)$ is a symmetric pair, then*

$$H^p(\mathfrak{g}, H; \mathbb{C}) = \Omega^p(\mathfrak{g}, H; \mathbb{C}) = \mathrm{Hom}_H(\wedge^p(\mathfrak{g}/\mathfrak{h}), \mathbb{C}) = \mathrm{Hom}_H(\wedge^p \mathfrak{q}, \mathbb{C})$$

(Definition 1.10). That is, the differential in this complex is zero.

PROOF. Suppose that $\omega \in \mathrm{Hom}_H(\wedge^p \mathfrak{q}, \mathbb{C})$. We want to show that $d\omega = 0$. So suppose $X_0, \ldots, X_p \in \mathfrak{q}$. Then

$$\begin{aligned} d\omega(X_0, \ldots, X_p) &= \sum_{i=0}^{p} (-1)^i X_i \cdot \omega(X_0, \ldots, \widehat{X_i}, \ldots, X_p) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p) \end{aligned}$$

The terms in the first sum are all zero since $\mathfrak{g}$ acts trivially on $\mathbb{C}$. The last display in Definition 2.5 show that all the brackets $[X_i, X_j]$ belong to $\mathfrak{h}$; so the terms in the second sum are zero as well. Q.E.D.

COROLLARY 2.7. *If $\mathfrak{g}$ is an abelian Lie algebra, then $H^p(\mathfrak{g}; \mathbb{C}) = \mathrm{Hom}(\wedge^p \mathfrak{g}, \mathbb{C})$.*

DEFINITION 2.8. Suppose G is a reductive Lie group and K is a maximal compact subgroup. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the corresponding Cartan decomposition; thus $\mathfrak{p}$ is the -1 eigenspace of a Cartan involution. It follows from the bracket relations in Definition 2.5 that

$$\mathfrak{u} = \mathfrak{k} + i\mathfrak{p} \subset \mathfrak{g}_{\mathbb{C}}$$

is a real form of $\mathfrak{g}_{\mathbb{C}}$. A *compact dual* for G is a connected compact group U endowed with a subgroup isomorphic to (and denoted) K, with the property that the Lie algebra of U is isomorphic to $\mathfrak{u}$ in a K-equivariant way.

EXAMPLE 2.9. Suppose $G = SO(2,1)$, the group of linear transformations of $\mathbb{R}^3$ preserving the quadratic form $x^2 + y^2 - z^2$, and having determinant one. As maximal compact subgroup we can take $S(O(2) \times O(1)) \simeq O(2)$; the isomorphism sends a matrix $A \in O(2)$ to

$$\begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix} \in S(O(2) \times O(1)).$$

(Here one of the zeros is a 2×1 matrix, and the other is 1×2.) The complexification of G is the group $G_{\mathbb{C}}$ of linear transformations of $\mathbb{C}^3$ preserving the same quadratic form and having determinant 1. Inside $\mathbb{C}^3$ there is another real form $V = \mathbb{R}^2 + i\mathbb{R}$; the quadratic form on $\mathbb{C}^3$ restricts to a positive definite real form on V. The subgroup U of $G_{\mathbb{C}}$ preserving V is isomorphic to $SO(3)$, and it contains K. It is easy to see that U is a compact dual for $SO(2,1)$. Notice that the homogeneous space U/K is $\mathbb{RP}^2$. Consequently

$$H^p(U/K;\mathbb{C}) = \begin{cases} \mathbb{C}, & \text{if } p = 0; \\ 0, & \text{otherwise.} \end{cases}$$

The method of the preceding example is rather general.

THEOREM 2.10. *Suppose G is the group of real points of a reductive algebraic group with $G_{\mathbb{C}}$ connected, and K is a maximal compact subgroup of G. Let U be a maximal compact subgroup of $G_{\mathbb{C}}$ containing K. Then U is a compact dual of G. There are natural isomorphisms*

$$\begin{aligned} H^p(\mathfrak{g},K;\mathbb{C}) &\simeq \mathrm{Hom}_K(\wedge^p\mathfrak{p},\mathbb{C}) \simeq i^p\mathrm{Hom}_K(\wedge^p(i\mathfrak{p}),\mathbb{C}) \\ &\simeq H^p(\mathfrak{u},K;\mathbb{C}) \simeq H^p(U/K;\mathbb{C}). \end{aligned}$$

Notice that this result shows how to compute the relative Lie algebra cohomology with coefficients in the trivial representation as the cohomology of a natural compact compact manifold (in fact a compact symmetric space).

PROOF. Write θ for the Cartan involution of G fixing K. We can always realize G as a subgroup of $GL(n,\mathbb{R})$ in such a way that the θ acts by inverse transpose: $\theta g = {}^t g^{-1}$. Once this is done, $G_{\mathbb{C}}$ becomes a subgroup of $GL(n,\mathbb{C})$, and the complex conjugation action defining the real form is just conjugation of matrices. The complexification of θ is still inverse transpose, which is a holomorphic automorphism of order two commuting with complex conjugation. We may therefore define a new real form σ of $G_{\mathbb{C}}$ by $\sigma g = {}^t\overline{g}^{-1}$. The group U of real points is just $G_{\mathbb{C}} \cap U(n)$, which is compact; so U must be a compact real form of G. By construction U contains K, and it is easy to check that the Lie algebra is $\mathfrak{k} + i\mathfrak{p}$. So U is a compact dual of G. All the isomorphisms in the theorem follow from Proposition 2.6 except for the very last one. For that, Proposition 1.5 shows that $\mathrm{Hom}_K(\wedge^p(i\mathfrak{p}),\mathbb{C})$ may be identified with the space of p-forms on U invariant under left translation. Because U is connected, the action of U by left translation on $H^p(U/K,\mathbb{C})$ is trivial. It follows that every cohomology class is represented by a U-invariant p-form, and the isomorphism we want follows. Q.E.D.

A complete description of the cohomology groups of the space U/K in Theorem 2.10 may be found (at least for connected K) in [**3**], as Theorem V on page 465. The method of the next example applies to the Hermitian symmetric cases; but other ideas are required in general.

EXAMPLE 2.11. Suppose $G = U(p,q)$, $K = U(p) \times U(q)$, and $U = U(p+q)$. Write $n = p+q$. Then U/K is the Grassmann variety of p-planes in $\mathbb{C}^n$. The group $G_{\mathbb{C}}$ may be identified with $GL(n,\mathbb{C})$, so $\mathfrak{g}_{\mathbb{C}}$ consists of all $n \times n$ matrices. We have

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{gl}(p,\mathbb{C}) \times \mathfrak{gl}(q,\mathbb{C}), \tag{2.12)(a}$$

$$\mathfrak{p}_{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \mid A \in M_{p\times q}(\mathbb{C}), B \in M_{q\times p}(\mathbb{C}) \right\}. \tag{2.12)(b}$$

Consequently

$$\mathfrak{p}_{\mathbb{C}} \simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^p, \mathbb{C}^q) \oplus \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^q, \mathbb{C}^p) = \mathfrak{p}_{\mathbb{C}}^- \oplus \mathfrak{p}_{\mathbb{C}}^+; \tag{2.12)(c}$$

the last equality is a definition. The spaces $\mathfrak{p}_{\mathbb{C}}^{\pm}$ are the holomorphic and antiholomorphic tangent spaces for the complex structures on G/K and U/K associated with the Hermitian symmetric structures. We will also use the fact that the standard invariant bilinear form $\langle X, Y\rangle = \mathrm{tr}\, XY$ on $\mathfrak{gl}(n,\mathbb{C})$ restricts to an identification $\mathfrak{p}_{\mathbb{C}}^- \simeq (\mathfrak{p}_{\mathbb{C}}^+)^*$. Consequently

$$\wedge^m \mathfrak{p}_{\mathbb{C}} \simeq \bigoplus_{a+b=m} \left(\wedge^a \mathfrak{p}_{\mathbb{C}}^+\right) \otimes \left(\wedge^b \mathfrak{p}_{\mathbb{C}}^-\right) \simeq \bigoplus_{a+b=m} \mathrm{Hom}\left(\wedge^a \mathfrak{p}_{\mathbb{C}}^-, \wedge^b \mathfrak{p}_{\mathbb{C}}^-\right). \tag{2.12)(d}$$

This bigrading is related to the Hodge structure on the cohomology of U/K and $\Gamma\backslash G/K$. Inserting this description in Theorem 2.10, we find

$$H^m(\mathfrak{g}, K; \mathbb{C}) \simeq \mathrm{Hom}_K(\wedge^m \mathfrak{p}_{\mathbb{C}}, \mathbb{C}) \simeq \bigoplus_{a+b=m} \mathrm{Hom}_K\left(\wedge^a \mathfrak{p}_{\mathbb{C}}^+, \wedge^b \mathfrak{p}_{\mathbb{C}}^+\right). \tag{2.12)(e}$$

To continue, we need to understand $\wedge^a \mathfrak{p}_{\mathbb{C}}^+$ as a representation of $K = U(p) \times U(q)$; or, equivalently, as a representation of $K_{\mathbb{C}} = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$.

For that, we consider the parabolic subgroup of $GL(n,\mathbb{C})$

$$Q = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in GL(p,\mathbb{C}), B \in M_{p\times q}(\mathbb{C}), C \in GL(q,\mathbb{C}) \right\}. \tag{2.12)(f}$$

Then Q has a Levi decomposition $Q = LU$, with $L = K_{\mathbb{C}}$ and $Lie(U) = \mathfrak{p}_{\mathbb{C}}^+$. Because U is abelian, Corollary 2.7 implies

$$\left(\wedge^a \mathfrak{p}_{\mathbb{C}}^+\right)^* = H^a(\mathfrak{u}; \mathbb{C}). \tag{2.12)(g}$$

The last cohomology group is computed by Kostant's version of the Bott-Borel-Weil theorem:

THEOREM 2.13 ([**7**]). *Suppose $Q = LU$ is a parabolic subgroup of a complex reductive Lie group G, and that F is an irreducible finite-dimensional representation of G. Then $H^*(\mathfrak{u}; F)$ is a sum of inequivalent irreducible representations of L, parametrized by the quotient of Weyl groups $W(G)/W(L)$. The number of summands in degree a is the number of elements of $W(G)$ of length a which are minimal representatives for their $W(L)$ cosets.*

The statement is explained more completely in [**7**]; a special case is discussed in section 2 of [**14**]. In order to apply Kostant's theorem to our present situation (with $F = \mathbb{C}$), we just need to compute the Weyl group elements in question. Here $W(G) = S_n$, the symmetric group of all permutations of $\{1, \dots, n\}$, and $W(L)$ is the natural subgroup $S_p \times S_q$. A permutation σ is minimal in its $W(L)$ coset if and only if

$$\sigma(1) < \cdots < \sigma(p), \qquad \sigma(p+1) < \cdots < \sigma(p+q). \tag{2.14)(a}$$

Suppose that is the case; we want to know the length of σ. For k between 1 and p, define integers d_k between 0 and q by the requirements

$$\begin{cases} d_k = 0 & & \text{if } \sigma(k) < \sigma(p+1); \\ d_k = d & (0 < d < q) & \text{if } \sigma(p+d) < \sigma(k) < \sigma(p+d+1); \text{ and} \\ d_k = q & & \text{if } \sigma(p+q) < \sigma(k). \end{cases} \tag{2.14)(b}$$

Then it is easy to check that

$$0 \le d_1 \le d_2 \le \cdots \le d_p \le q, \qquad \sum d_k = \ell(\sigma). \tag{2.14)(c}$$

Conversely, each sequence $\{d_k\}$ satisfying the inequalities in (2.14)(c) corresponds to a unique permutation σ as (2.14)(a). Combining these calculations, Theorem 2.13, and (2.12)(g), we get

COROLLARY 2.15. *The exterior algebra $\wedge\mathfrak{p}^+_{\mathbb{C}}$ is a direct sum of inequivalent representations of $K_{\mathbb{C}}$. The number of representations appearing in degree a is equal to the number of sequences of integers*

$$0 \le d_1 \le d_2 \le \cdots \le d_p \le q, \qquad \sum d_k = a.$$

The total number appearing in all degrees is $\binom{n}{p}$.

Applying the formula in (2.12)(e) now gives

COROLLARY 2.16. *Suppose $G = U(p,q)$ and $K = U(p) \times U(q)$. Then the cohomology $H^*(\mathfrak{g}, K; \mathbb{C})$ is non-zero only in even degrees. More precisely, the dimension of $H^{2a}(\mathfrak{g}, K; \mathbb{C})$ is equal to the number of sequences of integers*

$$0 \le d_1 \le d_2 \le \cdots \le d_p \le q, \qquad \sum d_k = a.$$

The total dimension of the cohomology (and the Euler characteristic) is equal to $\binom{n}{p}$.

The formula of Corollary 2.16 shows that the cohomology occurs in degrees ranging from 0 to $2pq$, and that it has dimension 1 in those extreme degrees. This is consistent with Theorem 2.10, since U/K is a compact complex manifold of dimension pq.

Corollary 2.16 and Theorem 2.10 together compute completely the cohomology groups appearing in Theorem 2.4.

3. Cohomology of irreducible representations: the discrete series

We saw in Corollary 2.16 that the cohomology of the trivial representation is quite complicated. It is therefore natural to fear that the cohomology of something as complicated as a discrete series representation will be completely incomprehensible. This is not the case, and that fact is significant. The point is that discrete series representations are in many senses among the "atoms" of the representation theory of reductive groups. The trivial representation (in the Langlands classification, or in the theory of Eisenstein series) appears as a residue from the reduciblity of a certain principal series representation; it can be properly understood only in the context of a fairly complete understanding of that reducibility, and of all the other pieces involved in it. Once this point of view is thoroughly grasped, what is amazing is that one can give any kind of closed formula for the cohomology of the trivial representation, and that such formulas were given twenty years before the invention of intersection cohomology.

For this section, we will assume that G is a connected reductive group having a compact Cartan subgroup

$$T \subset K \subset G. \tag{3.1)(a}$$

We follow roughly the notation of [**14**], section 5. We fix therefore a system of positive roots Φ^+ for T in $\mathfrak{g}_{\mathbb{C}}$, and write

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha. \tag{3.1)(b}$$

We will use the trivial weight $0 \in \Lambda$ for T; this has the required property that $0 + \rho$ is dominant and regular for Φ^+. We define

$$\pi(\Phi^+) = \text{discrete series representation with character } \Theta_\rho. \tag{3.1)(c}$$

This is the representation with Harish-Chandra parameter ρ. (Wigner's result mentioned after Problem 2.1 guarantees that discrete series representations with other Harish-Chandra parameters cannot have non-vanishing cohomology; this fact can also be deduced from a calculation like the one given for Theorem 3.2 below.) We will write

$$X(\Phi^+) = \text{Harish-Chandra module of } \pi(\Phi^+). \tag{3.1)(d}$$

Finally, recall from [**14**], section 3 that Φ^+ is the disjoint union of the compact and noncompact positive roots:

$$\Phi^+ = \Phi_c^+ \cup \Phi_n^+. \tag{3.1)(e}$$

Define

$$R = |\Phi_n^+| = \frac{1}{2} \dim G/K. \tag{3.1)(f}$$

THEOREM 3.2. *With notation as in (3.1), the cohomology of the discrete series representation is given by*

$$H^p(\mathfrak{g}, K; X(\Phi^+)) = \begin{cases} 0 & \text{if } p \neq R \\ \mathbb{C} & \text{if } p = R. \end{cases}$$

PROOF. We try to compute the $X(\Phi^+)$-valued p-forms for $(\mathfrak{g}, K)$ (Definition 1.10). Suppose μ is the highest weight of a representation of K occurring in both $\wedge^p \mathfrak{p}_{\mathbb{C}}$ and in $X(\Phi^+)$. The first requirement means that μ must be a sum of p distinct noncompact roots, so that

$$\mu = \beta_1 + \cdots + \beta_r - \beta_1' - \cdots - \beta_s'.$$

Here $\{\beta_1, \ldots, \beta_r\}$ and $\{\beta_1', \ldots, \beta_s'\}$ are subsets of Φ_n^+, and $r + s = p$. On the other hand, the Corollary to Theorem 1 of section 5 in [**14**] says that the second requirement means μ is of the form

$$\mu = 2\rho_n + \sum_{\gamma\in\Phi^+} c_\gamma \gamma,$$

with c_γ a non-negative integer. Consequently

$$\beta_1 + \cdots + \beta_r - \beta_1' - \cdots - \beta_s' = 2\rho_n + \sum_{\gamma\in\Phi^+} c_\gamma\gamma.$$

Each positive root has strictly positive inner product with ρ. Taking the inner product of both sides with ρ, we conclude that

$$r = |\Phi_n^+|, \quad s = 0, \quad c_\gamma = 0.$$

In particular, $p = r + s = |\Phi_n^+|$, and $\mu = 2\rho_n$.

It follows first of all that $\Omega^p = 0$ for $p \neq R$. For $p = R$, the only representation of K common to $\wedge^R \mathfrak{p}_{\mathbb{C}}$ and $X(\Phi^+)$ is the one of highest weight $2\rho_n$. This has multiplicity one in $X(\Phi^+)$ by [**14**], and multiplicity one in $\wedge^R \mathfrak{p}_{\mathbb{C}}$ by an easy computation. So $\dim \Omega^R = 1$. Since all the other forms are zero, the differentials in the complex must be zero; and the theorem follows. Q.E.D.

If G/K is Hermitian symmetric, the "Hodge type" of the cohomology class of $X(\Phi^+)$ is equal to (a, b), where

$$a = |\Phi_n^+ \cap (\text{roots in } \mathfrak{p}_{\mathbb{C}}^+)|, \quad b = |\Phi_n^+ \cap (\text{roots in } \mathfrak{p}_{\mathbb{C}}^-)|.$$

4. Introduction to cohomologically induced representations

In this section we will introduce a family of representations "interpolating" between the trivial representation and the discrete series representations $X(\Phi^+)$. We work with a connected real reductive group G in Harish-Chandra's class ([**4**], section 3). (Allowing G to be disconnected but still in Harish-Chandra's class complicates the notation slightly, but does not introduce any essential new difficulties.) We fix a maximal compact subgroup $K \subset G$, and write θ for the corresponding Cartan involution. Just as in Definition 2.8, the Cartan decomposition is written $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.

DEFINITION 4.1. A *θ-stable parabolic subalgebra of* $\mathfrak{g}$ is a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}_{\mathbb{C}}$ such that

1. $\theta\mathfrak{q} = \mathfrak{q}$, and
2. $\overline{\mathfrak{q}} \cap \mathfrak{q} = \mathfrak{l}_{\mathbb{C}}$ is a Levi subalgebra of $\mathfrak{q}$.

Here the bar refers to complex conjugation with respect to the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$. Necessarily the Levi subalgebra $\mathfrak{l}_{\mathbb{C}}$ is defined over $\mathbb{R}$; the real subalgebra $\mathfrak{l}$ is θ-stable, and is in fact the normalizer of $\mathfrak{q}$ in $\mathfrak{g}$. We define the *Levi subgroup of* $\mathfrak{q}$ by

$$L = \{g \in G \mid \mathrm{Ad}(g)(\mathfrak{q}) \subset \mathfrak{q}\}.$$

Notice that we refer to $\mathfrak{q}$ as a θ-stable parabolic subalgebra of $\mathfrak{g}$ even though it is actually a subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

PROPOSITION 4.2 ([**6**], Chapter V). *Suppose* $\mathfrak{q}$ *is a* θ*-stable parabolic subalgebra of* $\mathfrak{g}$ *with Levi subgroup* L. *Then*

1. *L is a connected real reductive group of the same rank as G.*
2. *L is preserved by θ, and the restriction of θ to L is a Cartan involution.*
3. *L contains a maximal torus $T \subset K$.*

We will be interested in θ-stable parabolics up to conjugation by K. Proposition 4.2 shows that we may therefore study those containing a fixed maximal torus in K. Here is a construction that gives all of them.

CONSTRUCTION 4.3. Fix a maximal torus $T \subset K$. Recall that the centralizer H of T in G is a Cartan subgroup. It has Cartan decomposition $H = TA$, with $\mathfrak{a}$ the centralizer of T in $\mathfrak{p}$. Write $\Phi_c \subset i\mathfrak{t}_0^*$ for the set of roots of T in $\mathfrak{k}_{\mathbb{C}}$, so that

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} + \sum_{\alpha \in \Phi_c} \mathfrak{k}_{\mathbb{C},\alpha}.$$

Similarly, write $\Phi_n \subset i\mathfrak{t}_0^*$ for the set of non-zero weights of T on $\mathfrak{p}_{\mathbb{C}}$, so that

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} + \sum_{\beta \in \Phi_n} \mathfrak{p}_{\mathbb{C},\beta}.$$

We write $\Phi = \Phi_c \cup \Phi_n$, a subset of $i\mathfrak{t}_0^*$ with multiplicities. Actually it is convenient to abuse notation slightly to allow an element of Φ to remember whether it came from Φ_c or Φ_n. A formal way to do this is to regard an element of Φ as a character of the group generated by T and θ; θ acts by $+1$ on elements of Φ_c, and by -1 on elements of Φ_n.

Now fix a system of positive roots Φ_c^+ for T in $\mathfrak{k}_{\mathbb{C}}$. Fix a weight $\lambda \in i\mathfrak{t}_0^*$ that is dominant for K; that is, so that

$$\langle \lambda, \alpha \rangle \geq 0 \qquad (\alpha \in \Phi_c^+).$$

We define the *θ-stable parabolic associated to* λ by

$$\mathfrak{q}(\lambda) = \mathfrak{h}_{\mathbb{C}} + \sum_{\substack{\gamma \in \Phi \\ \langle \lambda, \gamma \rangle \geq 0}} \mathfrak{g}_{\mathbb{C},\gamma}.$$

The corresponding Levi subalgebra is

$$\mathfrak{l}(\lambda)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \sum_{\substack{\gamma \in \Phi \\ \langle \lambda, \gamma \rangle = 0}} \mathfrak{g}_{\mathbb{C},\gamma}.$$

The Levi subgroup $L(\lambda)$ may be described as follows. Extend λ to a complex-linear functional on all of $\mathfrak{g}$, by making it zero on each weight space $\mathfrak{g}_{\mathbb{C},\gamma}$ (for $\gamma \in \Phi$). Then λ takes purely imaginary values on $\mathfrak{g}_0$. The group $L(\lambda)$ is just the stabilizer of λ in the coadjoint action:

$$L(\lambda) = \{ g \in G \mid \mathrm{Ad}^*(g)(\lambda) = \lambda \}.$$

PROPOSITION 4.4. *Every θ-stable parabolic subalgebra of $\mathfrak{g}$ arises by Construction 4.3. In particular,*

1. *there are only finitely many K-conjugacy classes of θ-stable parabolic subalgebras; and*
2. *the Levi subgroups of θ-stable parabolic subalgebras are precisely the isotropy groups for the coadjoint action of G at elements of $\mathfrak{k}^*$.*

This is a fairly easy consequence of Proposition 4.2. The coadjoint orbits passing through $\mathfrak{k}^*$ are called *elliptic*; so the homogeneous spaces $G/L(\lambda)$ are precisely the elliptic coadjoint orbits.

EXAMPLE 4.5. Suppose again that $G = U(p,q)$, $K = U(p)\times U(q)$. The Cartan involution is conjugation by the diagonal matrix whose first p entries are $+1$ and whose last q entries are -1. Write $n = p+q$, so that $G_{\mathbb{C}} \simeq GL(n,\mathbb{C})$ as in Example 2.11. Suppose we are given an r-tuple of pairs (p_i, q_i) of non-negative integers, so that

$$\sum p_i = p, \qquad \sum q_i = q, \qquad p_i + q_i \neq 0.$$

(These conditions are slightly weaker than the ones in Theorem 2.4.) We can rearrange the coordinates in $\mathbb{C}^n$ so that our Hermitian form has p_1 plus signs, followed by q_1 minus signs, followed by p_2 plus signs, and so on:

$$|z_1|^2 + \cdots |z_{p_1}|^2 - |z_{p_1+1}|^2 - \cdots |z_{p_1+q_1}|^2 + \cdots$$

In this new realization, the Cartan involution is still conjugation by a diagonal matrix with entries ± 1. Now let $\mathfrak{q}$ be the standard parabolic subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ with blocks of sizes $p_1+q_1,\ p_2+q_2, \ldots$ along the diagonal. Then $\mathfrak{q}$ is a θ-stable parabolic subalgebra. The corresponding Levi subgroup consists of diagonal blocks; it is

$$L = U(p_1,q_1)\times\cdots\times U(p_r,q_r).$$

It is not difficult to see that these are all the θ-stable parabolic subalgebras in $\mathfrak{g}$, up to conjugation by K; and in fact no two of these are conjugate.

Here is the main theorem.

THEOREM 4.6. *Suppose G is a connected real reductive Lie group in Harish-Chandra's class, and $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$ with Levi subgroup L (Definition 4.1). Write $\mathfrak{u}$ for the nil radical of $\mathfrak{q}$, and define*

$$R = \dim \mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}.$$

1. *Attached to $\mathfrak{q}$ there is an irreducible unitary representation $\pi(\mathfrak{q})$ of G. Up to equivalence, $\pi(\mathfrak{q})$ depends only on the K-conjugacy class of $\mathfrak{q}$.*
2. *Write $X(\mathfrak{q})$ for the Harish-Chandra module of $\pi(\mathfrak{q})$. Then*

$$H^p(\mathfrak{g},K;X(\mathfrak{q})) \simeq H^{p-R}(\mathfrak{l}, L\cap K;\mathbb{C}).$$

3. *Suppose π is an irreducible unitary representation of G with Harish-Chandra module X, and that $H^*(\mathfrak{g},K;X)\neq 0$. Then there is a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ so that $\pi\simeq\pi(\mathfrak{q})$.*

We will say a little bit about the proof of this theorem in sections 5 and 6. Here are some remarks. In the setting of Construction 4.3, a θ-stable Borel subalgebra containing $\mathfrak{t}$ is the same as a choice Φ^+ of a system of positive roots for Φ. When in addition $\operatorname{rank} G = \operatorname{rank} K$, we have already defined a representation $\pi(\Phi^+)$

attached to such a positive system: it is a discrete series representation. In this case $L = T = L \cap K$, so the formula in Theorem 4.6 for the cohomology agrees with the formula in Theorem 3.2.

If $\mathfrak{q} = \mathfrak{g}_{\mathbb{C}}$, then $L = G$. We take $\pi(\mathfrak{g}_{\mathbb{C}})$ to be the trivial representation of G; then the formula in Theorem 4.6 for the cohomology is a tautology.

If $G = U(p,q)$, then Theorem 4.6 can be combined with Example 4.5 and Theorem 2.10 to give something very close to Theorem 2.4. The differences arise because the list of representations in Theorem 4.6 has a few repetitions. These have been edited out of the list in Theorem 2.4. (To get inequivalent representations in Theorem 4.6, one should impose the additional requirement on $\mathfrak{q}$ that L have no non-abelian compact simple factors. This is done in Theorem 2.4 by the last two conditions on the p_i and q_i.)

The representations $\pi(\mathfrak{q})$ were first constructed in general (as possibly non-unitary representations) by Parthasarathy in [**12**]. It seems very likely that he was aware of their connection with Lie algebra cohomology. At any rate the calculation of cohomology in Theorem 4.6 is (as we will see in the next section) not very difficult. Part 3 of the theorem was proved in [**17**], using powerful partial results of Kumaresan from [**10**]. The last part of the theorem, that $\pi(\mathfrak{q})$ is actually unitary, was proved in [**16**].

5. Cohomologically induced representations: characterization and cohomology

In this section we will give a characterization of the representations $\pi(\mathfrak{q})$ in Theorem 4.6, and use it to compute their cohomology. The main ingredient is a certain representation of K constructed from the θ-stable parabolic $\mathfrak{q}$. In order to describe this representation, it is helpful to have a slight reformulation of Construction 4.3. In the notation of that construction, the bilinear form defines an isomorphism $i\mathfrak{t}_0^* \simeq i\mathfrak{t}_0$. Let $H_\lambda \in i\mathfrak{t}_0$ be the element corresponding to λ. Explicitly, this means

$$\gamma(H_\lambda) = \langle \lambda, \gamma \rangle \qquad (\gamma \in i\mathfrak{t}_0^*) \tag{5.1)(a}$$

The θ-stable parabolic associated to λ (Construction 4.3) is then

$$\mathfrak{q}(\lambda) = \mathfrak{q}(H_\lambda) = \mathfrak{h}_{\mathbb{C}} + \sum_{\substack{\gamma \in \Phi \\ \gamma(H_\lambda) \geq 0}} \mathfrak{g}_{\mathbb{C},\gamma} \tag{5.1)(b}$$

Similarly, its Levi subgroup is

$$L(\lambda) = L(H_\lambda) = \{ g \in G \mid \mathrm{Ad}(g)(H_\lambda) = H_\lambda \}. \tag{5.1)(c}$$

Define

$$2\rho(\mathfrak{u} \cap \mathfrak{p}) = \sum_{\substack{\gamma \in \Phi_n \\ \gamma(H_\lambda) > 0}} \gamma \quad \in \quad \widehat{T} \subset i\mathfrak{t}_0^*, \tag{5.1)(d}$$

the sum of the roots of T in $\mathfrak{u} \cap \mathfrak{p}$.

PROPOSITION 5.2. *In the setting of (5.1), write $R = \dim \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$ as in Theorem 4.6. The largest eigenvalue of $\mathrm{Ad}(H_\lambda)$ on $\wedge \mathfrak{p}_{\mathbb{C}}$ is equal to $2\rho(\mathfrak{u} \cap \mathfrak{p})(H_\lambda)$. The corresponding eigenspace is isomorphic to*

$$\wedge^R(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}) \otimes \wedge(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}).$$

The adjoint action of $\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}}$ *is trivial on this space.*

PROOF. The triangular decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}} + \mathfrak{l}_{\mathbb{C}}\cap\mathfrak{p}_{\mathbb{C}} + \overline{\mathfrak{u}}\cap\mathfrak{p}_{\mathbb{C}}$ gives rise to a decomposition of the exterior algebra

$$\wedge\mathfrak{p}_{\mathbb{C}} = (\wedge(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}))\otimes(\wedge(\mathfrak{l}_{\mathbb{C}}\cap\mathfrak{p}_{\mathbb{C}}))\otimes(\wedge(\overline{\mathfrak{u}}\cap\mathfrak{p}_{\mathbb{C}}))$$

Any weight of T appearing is a sum of weights from the three factors. According to (5.1), $\mathrm{Ad}(H_\lambda)$ has positive eigenvalues on the first factor, zero eigenvalues on the second, and negative eigenvalues on the third. This proves everything but the last claim. For that, (5.1) implies also that $\mathrm{Ad}(\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}})$ acts to raise the eigenvalues of $\mathrm{Ad}(H_\lambda)$. Q.E.D.

For the next result, we need to fix a set of positive roots of T in $\mathfrak{l}_{\mathbb{C}}\cap\mathfrak{k}_{\mathbb{C}}$; this allows us to speak of highest weights for representations of $L\cap K$. Adjoining to this the set of roots of T in $\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}}$ gives a set of positive roots of T in $\mathfrak{k}_{\mathbb{C}}$, and so allows us to speak of highest weights for representations of K.

COROLLARY 5.3. *Let* μ_L *be the highest weight of a representation* δ_L *of* $L\cap K$ *appearing in* $\wedge(\mathfrak{l}_{\mathbb{C}}\cap\mathfrak{p}_{\mathbb{C}})$.

1. *There is a unique representation* δ *of* K *of highest weight* $\mu = \mu_L + 2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})$.
2. *There is a natural isomorphism*

$$\mathrm{Hom}_K(V_\delta, \wedge^p\mathfrak{p}_{\mathbb{C}}) \simeq \mathrm{Hom}_{L\cap K}(V_{\delta_L}, \wedge^{p-R}(\mathfrak{l}_{\mathbb{C}}\cap\mathfrak{p}_{\mathbb{C}})).$$

3. *Suppose* γ *is a non-empty sum of roots in* $\mathfrak{u}$. *Then the representation of* K *of highest weight* $\mu+\gamma$ *does not occur in* $\wedge\mathfrak{p}_{\mathbb{C}}$.

PROOF. Suppose τ_L is any irreducible representation of $L\cap K$ of highest weight γ, and W is a representation of K. Then the Cartan-Weyl theory tells us that there is at most one representation (τ, V_τ) of K of highest weight γ; and

$$\mathrm{Hom}_K(V_\tau, W) \simeq \mathrm{Hom}_{L\cap K}(V_{\tau_L}, W^{\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}}}) \subset \mathrm{Hom}_{L\cap K}(V_{\tau_L}, W). \qquad (5.4)(a)$$

If τ does not exist, then the same formula is true with $V_\tau = 0$. We apply (5.4)(a) to $\tau_L = \delta_L\otimes\wedge^R(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})$. Evidently the element H_λ of $\mathfrak{l}_{\mathbb{C}}\cap\mathfrak{k}_{\mathbb{C}}$ acts on τ_L by the scalar $2\rho(\mathfrak{u}\cap\mathfrak{p})(H_\lambda)$. Proposition 5.2 therefore allows us to conclude that

$$\mathrm{Hom}_{L\cap K}(V_{\tau_L}, \wedge^p\mathfrak{p}_{\mathbb{C}}) \simeq \mathrm{Hom}_{L\cap K}(V_{\delta_L}, \wedge^{p-R}(\mathfrak{l}_{\mathbb{C}}\cap\mathfrak{p}_{\mathbb{C}})). \qquad (5.4)(b)$$

Furthermore any $L\cap K$-map on the left must automatically take values in the $\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}}$-invariants. Now (5.4) gives 2. of the corollary. The right side of (5.4)(b) is non-zero (for some p) by the assumption on δ_L; so V_τ cannot be zero, and 1. follows. For 3., we apply (5.4) again with τ_L equal to the representation of $L\cap K$ of highest weight $\mu+\gamma$. By (5.1), H_λ acts on τ_L by the scalar

$$\mu(H_\lambda) + \gamma(H_\lambda) > \mu(H_\lambda) = 2\rho(\mathfrak{u}\cap\mathfrak{p})(H_\lambda).$$

This eigenvalue does not occur in $\wedge\mathfrak{p}_{\mathbb{C}}$, so (5.4)(a) implies that V_τ cannot occur in $\wedge\mathfrak{p}_{\mathbb{C}}$. Q.E.D.

COROLLARY 5.5. *In the setting of (5.1), there is a unique irreducible representation* $\delta(\mathfrak{q})$ *of* K *of highest weight* $2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})$. *We have*

$$\mathrm{Hom}_K(V_{\delta(\mathfrak{q})}, \wedge^p\mathfrak{p}_{\mathbb{C}}) \simeq \mathrm{Hom}_{L\cap K}(\mathbb{C}, \wedge^{p-R}(\mathfrak{l}_{\mathbb{C}}\cap\mathfrak{p}_{\mathbb{C}})).$$

This is just Corollary 5.3 with δ_L equal to the trivial representation of $L\cap K$. Here is a characterization of the representations in Theorem 4.6.

THEOREM 5.6 ([**17**], Proposition 6.1). *Suppose* $\mathfrak{q}$ *is a* θ*-stable parabolic subalgebra of* $\mathfrak{g}$, *and* $\delta(\mathfrak{q})$ *is the representation of* K *described in Corollary 5.5. Then there is a unique irreducible unitary representation* $\pi(\mathfrak{q})$ *of* G *with the following properties:*

1. *The restriction of* $\pi(\mathfrak{q})$ *to* K *contains* $\delta(\mathfrak{q})$ *exactly once.*
2. *Every representation of* K *appearing in* $\pi(\mathfrak{q})$ *has highest weight* $2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})+\gamma$, *with* γ *a sum of roots of* T *in* $\mathfrak{u}$.
3. *The Casimir operator (a central element of the universal enveloping algebra) acts by 0 in* $\pi(\mathfrak{q})$.

Only the uniqueness part of this statement is proved in [**17**]; the existence appears in [**16**]. We will discuss the construction of $\pi(\mathfrak{q})$ in section 6. Assuming that we have constructed this representation, let us see how to calculate the Lie algebra cohomology. As in Theorem 4.6, we write $X(\mathfrak{q})$ for the Harish-Chandra module. According to Definition 1.10, this is calculated by a complex

$$\Omega^p(\mathfrak{g}, K; X(\mathfrak{q})) = \mathrm{Hom}_K(\wedge^p\mathfrak{p}_{\mathbb{C}}, X(\mathfrak{q})). \tag{5.7)(a}$$

According to Corollary 5.3 and Theorem 5.6, the only representation of K occurring in both $X(\mathfrak{q})$ and $\wedge\mathfrak{p}_{\mathbb{C}}$ is $\delta(\mathfrak{q})$. Corollary 5.5 then gives

$$\Omega^p(\mathfrak{g}, K; X(\mathfrak{q})) \simeq \mathrm{Hom}_K(\wedge^p\mathfrak{p}_{\mathbb{C}}, V_{\delta(\mathfrak{q})}) \simeq \mathrm{Hom}_{L\cap K}(\wedge^{p-R}(\mathfrak{l}_{\mathbb{C}}\cap\mathfrak{p}_{\mathbb{C}}), \mathbb{C}). \tag{5.7)(b}$$

Consequently

$$\Omega^p(\mathfrak{g}, K; X(\mathfrak{q})) \simeq \Omega^{p-R}(\mathfrak{l}, L\cap K; \mathbb{C}). \tag{5.7)(c}$$

We have seen in Proposition 2.6 that the differential in the second complex is zero. The same is true of the first:

PROPOSITION 5.8 ([**2**], Proposition II.3.1). *Suppose that* X *is the Harish-Chandra module of a unitary representation of* G, *and that the Casimir operator acts by zero on* X. *Then the differential in* $\Omega^p(\mathfrak{g}, K; X)$ *is zero; so*

$$H^p(\mathfrak{g}, K; X) \simeq \mathrm{Hom}_K(\wedge^p\mathfrak{p}_{\mathbb{C}}, X).$$

In light of Proposition 5.8, the formula (5.7)(c) immediately implies the cohomology formula in Theorem 4.6.

6. Cohomologically induced representations: construction

In this section we will say a little about the construction of a unitary representation $\pi(\mathfrak{q})$ satisfying the conditions in Theorem 5.6. There are a number of ways to construct a Harish-Chandra module satisfying conditions (1)–(3) of Theorem 5.6, beginning with Parthasathy's method in [**12**]. The only method known for constructing a *unitary* representation is algebraic in nature, and is based on ideas of Zuckerman. It is the subject of [**6**]; we will say almost nothing about it. Instead we will discuss a more analytic construction suggested by Kostant in [**8**], and elaborated by Schmid in [**13**]. The tools are those of complex analysis, so we begin with some general remarks about that.

PROPOSITION 6.1. *Suppose G is a Lie group and H is a closed subgroup. Write $\mathfrak{h} \subset \mathfrak{g}$ for their Lie algebras. Then G-invariant complex structures on the homogeneous space G/H are in one-to-one correspondence with complex Lie subalgebras $\mathfrak{q} \subset \mathfrak{g}_{\mathbb{C}}$, having the following two properties.*

1. *We have $\mathfrak{q} \cap \overline{\mathfrak{q}} = \mathfrak{h}_{\mathbb{C}}$, and $\mathfrak{q} + \overline{\mathfrak{q}} = \mathfrak{g}_{\mathbb{C}}$.*
2. *The complexified adjoint action of H on $\mathfrak{g}_{\mathbb{C}}$ preserves $\mathfrak{q}$.*

SKETCH OF PROOF. This is well-known and (almost) elementary. Suppose we are given a $\mathfrak{q}$ satisfying these two conditions. The first condition (together with the fact that $\mathfrak{q}$ is a complex subspace of $\mathfrak{g}_{\mathbb{C}}$) means that $\mathfrak{q}$ defines a complex structure on the tangent space $\mathfrak{g}/\mathfrak{h}$ to G/H at eH. Next, we use the action of G to move this complex structure to all the other tangent spaces; the second condition guarantees that this is well-defined. In this way we get a G-invariant almost complex structure on G/H. The fact that $\mathfrak{q}$ is a Lie subalgebra means that this almost complex structure is integrable. By the Newlander-Nirenberg theorem (this is the not-so-elementary part of the argument) an integrable almost complex structure is a complex structure. The converse is similar (but entirely elementary). Q.E.D.

Notice that $\mathfrak{q}$ and H are almost a pair in the sense of Definition 1.6. The only change is that $\mathfrak{q}$ is a complex Lie algebra instead of a real one. (We could define a *complex pair* accordingly, but we will spare the reader.) In any case it is more or less clear what a $(\mathfrak{q}, H)$-module ought to be, by analogy with Definition 1.7; we simply require the representation of $\mathfrak{q}$ to be complex-linear instead of real-linear.

It is well-known that the G-equivariant complex vector bundles on G/H are parametrized naturally by the finite-dimensional complex representations of H. Here is the analogous result for holomorphic bundles.

PROPOSITION 6.2. *Suppose G is a Lie group and H is a closed subgroup. Suppose that we are given a G-invariant complex structures on the homogeneous space G/H corresponding to the complex Lie algebra $\mathfrak{q} \subset \mathfrak{g}_{\mathbb{C}}$ (Proposition 6.1). Then the G-equivariant holomorphic vector bundles on G/H are naturally parametrized by the finite-dimensional $(\mathfrak{q}, H)$-modules (Definition 1.7). This parametrization sends a vector bundle $\mathcal{V}$ to the fiber $V = \mathcal{V}_{eH}$.*

We omit the proof. If V is a finite-dimensional $(\mathfrak{q}, H)$-module, then the corresponding holomorphic vector bundle on G/H is written $\mathcal{V} = G \times_{\mathfrak{q},H} V$.

If $\mathcal{V}$ is a holomorphic vector bundle on a complex manifold X, then one can define Dolbeault cohomology groups $H^{0,p}(X, \mathcal{V})$. (The definition uses a certain differential $\overline{\partial}$ on $(0,p)$-forms with values in $\mathcal{V}$. It is formally quite similar to the de Rham d on ordinary forms.) For $p = 0$, the Dolbeault cohomology is the space of all holomorphic sections of $\mathcal{V}$. If X is a Stein manifold, the higher cohomology groups are all zero. The Dolbeault theorem asserts that $H^{0,p}(X, \mathcal{V})$ is isomorphic to the pth Čech cohomology of X with coefficients in the sheaf $\mathcal{O}(\mathcal{V})$ of germs of holomorphic sections of $\mathcal{V}$.

If now $\mathcal{V}$ is a G-equivariant holomorphic vector bundle on G/H, then there is a natural action of G on the Dolbeault complex, and so on the cohomology groups $H^{0,p}(G/H, \mathcal{V})$. In this way we get a representation of G on $H^{0,p}(G/H, \mathcal{V})$. The representations we want to discuss are of this form.

Suppose now that we are in the setting of Definition 4.1, so that $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$ with Levi subgroup L. By Definition 4.1 and Proposition 6.1, $\mathfrak{q}$ defines a G-invariant holomorphic structure on G/L. It is not difficult to see that $\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}$ defines a K-invariant holomorphic structure on $K/L \cap K$, and the natural inclusion

$$K/L \cap K \to G/L \tag{6.3)(a}$$

is a holomorphic embedding. We now introduce a holomorphic line bundle on G/L. Write $\mathfrak{u}$ for the nil radical of $\mathfrak{q}$, so that we have a Levi decomposition

$$\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}. \tag{6.3)(b}$$

This decomposition is invariant under L. Under $L \cap K$ we have a further decomposition

$$\mathfrak{u} = \mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}} \ \oplus \ \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}. \tag{6.3)(c}$$

We write

$$R = \dim \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}, \quad S = \dim \mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}. \tag{6.3)(d}$$

Then one sees easily that

$$S = \dim_{\mathbb{C}} K/L \cap K \qquad R + S = \dim_{\mathbb{C}} G/L. \tag{6.3)(e}$$

EXAMPLE 6.4. This example has G disconnected, and so does not quite meet our hypotheses; but it is nevertheless attractive. Let G be the general linear group $GL(2n, \mathbb{R})$, and let X be the Grassmann variety of n-dimensional complex planes in $\mathbb{C}^{2n}$. This is a compact complex manifold of complex dimension n^2; indeed it is a projective algebraic variety. The complex group $G_{\mathbb{C}} = GL(2n, \mathbb{C})$ acts transitively on X. The isotropy group at the standard copy of $\mathbb{C}^n \subset \mathbb{C}^{2n}$ is

$$Q = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A, C \in GL(n, \mathbb{C}), B \in M_{n \times n}(\mathbb{C}) \right\};$$

so $X \simeq G_{\mathbb{C}}/Q$.

Now G acts on X, but the action is not transitive. Here is a way to understand the orbits. Suppose V is an n-plane in $\mathbb{C}^{2n}$. Then $\overline{V}$ (the set of vectors obtained from V by conjugating coordinate by coordinate) is another n-plane; so $V \cap \overline{V} = W_{\mathbb{C}}$ is a subspace of $\mathbb{C}^{2n}$ defined over $\mathbb{R}$; that is, it is the complexification of a subspace W of $\mathbb{R}^{2n}$. Similarly, $V + \overline{V} = U_{\mathbb{C}}$ is the complexification of a subspace $U \supset W$ of $\mathbb{R}^{2n}$. Write d for the dimension of W; evidently $0 \le d \le n$. The spaces U, V, and W have the following properties.

$$W \subset U \subset \mathbb{R}^{2n}, \qquad \dim W = d, \qquad \dim U = 2n - d; \tag{6.4)(a}$$

$$W_{\mathbb{C}} \subset V \subset U_{\mathbb{C}}; \tag{6.4)(b}$$

$$V/W_{\mathbb{C}} \text{ defines a complex structure } J_{U/W} \text{ on } U/W. \tag{6.4)(c}$$

(Explicitly, $V/W_{\mathbb{C}}$ is the $+i$ eigenspace of the complexification of $J_{U/W}$.)

Conversely, suppose $W \subset U$ are subspaces of $\mathbb{R}^{2n}$, of dimensions d and $2n - d$ respectively; and suppose we are given a complex structure $J_{U/W}$ on U/W. Then the complex structure corresponds to a complex subspace $V' \subset (U/W)_{\mathbb{C}}$ of dimension $n - d$. The preimage V of V' in $W_{\mathbb{C}}$ is an n-dimensional subspace, and it gives rise to W and U by the construction above. In this way we find a bijection between the collection of n-planes in $\mathbb{C}^{2n}$, and the collection of triples $(W, U, J_{U/W})$ satisfying (6.4)(a)–(c).

In terms of this description, it is easy to understand the orbits of $G = GL(2n, \mathbb{R})$ acting on X. The dimension d of W is obviously constant on orbits. Write X_d for the set of all triples $(W, U, J_{U/W})$ as above with $\dim W = d$. It is easy to see that G is transitive on pairs of subspaces $W \subset U$ of dimensions d and $2n - d$; and that the isotropy group at (W, U) maps onto $GL(W/U)$. This last group acts transitively on the complex structures on W/U, so we conclude that G acts transitively on X_d. In particular, there are exactly $n+1$ orbits. Only one of these is open; it is X_0, which is just the space of all complex structures on $\mathbb{R}^{2n}$. As a base point in X_0 we may take some standard complex structure $\mathbb{R}^{2n} \simeq \mathbb{C}^n$; the isotropy group is evidently $GL(n, \mathbb{C})$, so that

$$GL(2n, \mathbb{R})/GL(n, \mathbb{C}) \simeq X_0 = \{\text{complex structures on } \mathbb{R}^{2n}\} \subset X \simeq G_{\mathbb{C}}/Q.$$

Because the standard complex structure J on $\mathbb{R}^{2n}$ is given by a skew-symmetric matrix (consisting of n diagonal blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$), the group $L = GL(n, \mathbb{C})$ is the Levi factor of a θ-stable parabolic subalgebra. Consequently X_0 is one of the spaces considered in (6.3). The compact subvariety $K/L \cap K = O(2n)/U(n)$ is easy to identify in this case: it consists of all complex structures J on $\mathbb{R}^{2n}$ which preserve the inner product. We compute $S = \dim_{\mathbb{C}} K/L \cap K = (n^2 - n)/2$, $R = (n^2 + n)/2$.

We turn now to a consideration of Dolbeault cohomology groups on the spaces G/L. As we indicated before (6.3), the higher cohomology groups vanish in the case of a Stein manifold. Now a compact complex submanifold of a Stein manifold is necessarily finite; but G/L has the compact complex submanifold $K/L \cap K$, which has complex dimension S. Schmid and Wolf have shown that G/L comes as close to being a Stein manifold as this subvariety will allow. Here is a precise statement.

THEOREM 6.5 ([**15**]). *G/L is $S+1$-complete in the sense of Andreotti and Grauert.*

What this means is that G/L admits an exhaustion function (a non-negative smooth function ϕ with $\phi^{-1}([0, N])$ compact for all N) such that the Levi form of ϕ has at most S non-positive eigenvalues at each point of G/L. The Levi form is a Hermitian form on the holomorphic tangent bundle constructed from second partial derivatives of ϕ. In holomorphic local coordinates, its matrix is $\partial^2\phi/\partial z_i \partial \overline{z}_j$.

COROLLARY 6.6 ([**1**], page 250). *If $\mathcal{S}$ is any coherent sheaf on G/L, then $H^p(G/L, \mathcal{S}) = 0$ for $p > S$. In particular, the Dolbeault cohomology $H^{0,p}(G/L, \mathcal{V})$ with coefficients in a holomorphic vector bundle $\mathcal{V}$ vanishes for $p > S$.*

We can now introduce the line bundle on G/L that we will be working with.

DEFINITION 6.7. Suppose $\mathfrak{q}$ is a θ-stable parabolic subalgebra for G, with Levi factor L. Use the notation of (6.3). Consider the one-dimensional $(\mathfrak{q}, L)$-module

$$L_{2\rho(\mathfrak{u})} = \wedge^{R+S}(\mathfrak{g}_{\mathbb{C}}/\mathfrak{q})^* \simeq \wedge^{R+S}\mathfrak{u} \simeq \wedge^{R+S}\mathfrak{q}/\mathfrak{l}_{\mathbb{C}} \qquad (6.7)(a)$$

The first description exhibits $L_{2\rho(\mathfrak{u})}$ as the fiber at eL of the top exterior power of the holomorphic cotangent bundle of G/L. The corresponding holomorphic line bundle

$$\mathcal{L}_{2\rho(\mathfrak{u})} = G \times_{\mathfrak{q}, L} L_{2\rho(\mathfrak{u})} \qquad (6.7)(b)$$

on G/L is therefore the canonical bundle. As a $(\mathfrak{q}\cap\mathfrak{k}_{\mathbb{C}}, L\cap K)$-module, there is a factorization

$$L_{2\rho(\mathfrak{u})} = \wedge^R(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})\otimes\wedge^S(\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}}); \tag{6.7)(c}$$

the factors are denoted $L_{2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})}$ and $L_{2\rho(\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}})}$ respectively. They induce holomorphic line bundles

$$\mathcal{L}_{2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})} = K\times_{\mathfrak{q}\cap\mathfrak{k}_{\mathbb{C}},L\cap K} L_{2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})} \tag{6.7)(d}$$

and similarly $\mathcal{L}_{2\rho(\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}})}$ on $K/L\cap K$. This last is the canonical bundle for $K/L\cap K$.

Finally, we define $\widetilde{\pi}(\mathfrak{q})$ to be the representation of G on the Dolbeault cohomology space

$$\widetilde{\mathcal{H}}(\mathfrak{q}) = H^{0,S}(G/L, \mathcal{L}_{2\rho(\mathfrak{u})}). \tag{6.7)(e}$$

Notice that Corollary 6.6 guarantees that this is the highest degree in which the cohomology can be non-zero.

The definition needs some remarks. First, the representation space is usually infinite-dimensional. We therefore need a topology on it to make any sensible statements. The natural topology comes from the Dolbeault complex. The $(0,S)$-forms with values in $\mathcal{L}_{2\rho(\mathfrak{u})}$ carry a natural C^∞ topology, and the closed forms constitute a closed subspace. The exact forms, however, do not obviously constitute a closed subspace; so the quotient topology on $H^{0,S}$ is not obviously Hausdorff. Wong has shown in [**19**] that the exact forms actually are closed, so that the topology is Hausdorff.

Theorem 4.6 asks for a unitary representation on a Hilbert space. The space $\widetilde{\mathcal{H}}(\mathfrak{q})$ is not a Hilbert space unless it is finite-dimensional; so $\widetilde{\pi}(\mathfrak{q})$ cannot be exactly the representation we are looking for. Wong also shows in [**19**] that $\widetilde{\pi}(\mathfrak{q})$ is infinitesimally equivalent to a representation constructed algebraically by Zuckerman; and this representation was already known from [**16**] to be unitary. We will have no more to say about the details of this (successful) approach to proving Theorem 4.6, concentrating instead on ideas of Schmid for analyzing $\widetilde{\pi}(\mathfrak{q})$. These ideas are taken from his dissertation, which was published in [**13**]. We choose them because they are easier to understand, and because they motivate many arguments in the algebraic theory.

Theorem 5.6 suggests that we ought to find some connection between $\widetilde{\pi}(\mathfrak{q})$ and the representation $\delta(\mathfrak{q})$ of K (Corollary 5.5). The first step is provided by the following result.

LEMMA 6.8. *In the setting of Definition 6.7, the representation of K on the Dolbeault cohomology of the line bundle $\mathcal{L}_{2\rho(\mathfrak{u})}$ is the irreducible representation $\delta(\mathfrak{q})$ described in Corollary 5.5:*

$$V_{\delta(\mathfrak{q})} \simeq H^{0,S}(K/L\cap K, \mathcal{L}_{2\rho(\mathfrak{u})}).$$

PROOF. Write W for the cohomology group in the lemma. All such cohomology groups (with coefficients in irreducible equivariant vector bundles) are computed by the Bott-Borel-Weil theorem. But in this case we can manage with even less. Recall that S is the complex dimension of $K/L\cap K$, and that the line bundle factors as $\mathcal{L}_{2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})}\otimes\mathcal{L}_{2\rho(\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}})}$. The second factor is the canonical bundle of $K/L\cap K$. The Serre Duality Theorem provides an isomorphism

$$W^* \simeq H^{0,0}(K/L\cap K, \mathcal{L}^*_{2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})})$$

We observed after Proposition 6.2 that the group on the right is the space of holomorphic sections of the line bundle. According to the Borel-Weil theorem, W^* is therefore the irreducible representation of K of lowest weight $-2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})$. By Corollary 5.5, $W^* \simeq V^*_{\delta(\mathfrak{q})}$. The lemma follows. Q.E.D.

To go further, we need some additional notation. In the setting of Definition 6.7, let us write $\mathcal{O}^G$ for the sheaf of germs of holomorphic sections of $\mathcal{L}_{2\rho(\mathfrak{u})}$ on G/L, and $\mathcal{O}^K$ for the corresponding sheaf on $K/L\cap K$. We may also regard $\mathcal{O}^K$ as a sheaf on G/L supported on $K/L\cap K$. According to Definition 6.7, Lemma 6.8, and the remarks after Proposition 6.2, the Čech cohomology groups of these sheaves in degree S are

$$H^S(G/L,\mathcal{O}^G) \simeq \widetilde{\mathcal{H}}(\mathfrak{q}) \tag{6.9)(a}$$

$$H^S(G/L,\mathcal{O}^K) \simeq H^S(K/L\cap K,\mathcal{O}^K) \simeq V(\mathfrak{q}). \tag{6.9)(b}$$

So we are looking for a connection between the sheaves $\mathcal{O}^G$ and $\mathcal{O}^K$ on G/L. This is provided by the restriction map: any holomorphic germ on G/L has a restriction to $K/L\cap K$. The restriction map is surjective (on sheaves of germs), since any holomorphic germ on $K/L\cap K$ has an extension to a germ on G/L. Its kernel is the sheaf $\mathcal{V}^1$ of germs of holomorphic sections of $\mathcal{L}_{2\rho(\mathfrak{u})}$ on G/L that vanish on $K/L\cap K$. We therefore have a short exact sequence of sheaves on G/L

$$0 \to \mathcal{V}^1 \to \mathcal{O}^G \to\to \mathcal{O}^K \to 0. \tag{6.9)(c}$$

These are all coherent sheaves, so the vanishing theorem of Corollary 6.6 applies. The long exact sequence in sheaf cohomology attached to (6.9)(c) therefore ends in degree S; in light of (6.9)(a) and (6.9)(b), the last terms are

$$\cdots \to H^S(G/L,\mathcal{V}^1) \to \widetilde{\mathcal{H}}(\mathfrak{q}) \to V(\mathfrak{q}) \to 0. \tag{6.9)(d}$$

As an immediate consequence, we deduce that

$$\delta(\mathfrak{q}) \text{ occurs in } \widetilde{\pi}(\mathfrak{q}). \tag{6.9)(e}$$

This is a (small) step in the direction of Theorem 5.6. To continue, we need to understand the representations of K appearing in the cohomology of $\mathcal{V}^1$. Schmid's method for doing so is to introduce the sheaves

$$\mathcal{V}^n = \text{ germs of sections of } \mathcal{L}_{2\rho(\mathfrak{u})} \text{ vanishing to } n\text{th order on } K/L\cap K \tag{6.9)(f}$$

on G/L. So for example

$$\mathcal{V}^0 = \mathcal{O}^G, \qquad \mathcal{V}^0/\mathcal{V}^1 = \mathcal{O}^K.$$

The next result is a generalization of Lemma 6.8.

LEMMA 6.10([**13**], (4.3)). *Suppose we are in the setting of Definition 6.7; use the notation of (6.9). Then for all $n \geq 0$, the quotient sheaf $\mathcal{V}^n/\mathcal{V}^{n+1}$ is supported on $K/L\cap K$. It may be described as follows. Write $\mathcal{N}$ for the holomorphic normal bundle of $K/L\cap K$ in G/L, and $\mathcal{N}^*$ for the dual bundle. Explicitly,*

$$\mathcal{N}^* \simeq K \times_{\mathfrak{q}\cap\mathfrak{k}_{\mathbb{C}},L\cap K} (\mathfrak{g}_{\mathbb{C}}/(\mathfrak{q}+\mathfrak{k}_{\mathbb{C}}))^* \simeq K \times_{\mathfrak{q}\cap\mathfrak{k}_{\mathbb{C}},L\cap K} (\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}).$$

Write $S^n(\mathcal{N}^*)$ for the nth symmetric power of $\mathcal{N}^*$, and $\mathcal{O}(\mathcal{W})$ for the sheaf of germs of holomorphic sections of a vector bundle $\mathcal{W}$. Then

$$\mathcal{V}^n/\mathcal{V}^{n+1} \simeq \mathcal{O}(S^n(\mathcal{N}^*)\otimes\mathcal{L}_{2\rho(\mathfrak{u})}).$$

In particular, every cohomology group of $\mathcal{V}^n/\mathcal{V}^{n+1}$ is a finite-dimensional representation of K.

If δ is an irreducible representation of K appearing in $H^S(G/L, \mathcal{V}^n/\mathcal{V}^{n+1})$, then the highest weight of δ must be of the form $2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}) + \gamma$, with γ a sum of n roots of T in $\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}$.

The first part of the lemma amounts to a coordinate-free treatment of Taylor expansions; it can be done with $K/L\cap K \subset G/L$ replaced by any closed complex submanifold of a complex manifold. The second part is a generalization of Lemma 6.8, and can be proved in a similar way. We omit the details.

COROLLARY 6.11. *Suppose we are in the setting of Definition 6.7; use the notation of (6.9). The quotient sheaf $\mathcal{O}^G/\mathcal{V}^{n+1}$ is supported on $K/L\cap K$, and has finite-dimensional cohomology sheaves. Consider the short exact sequence*

$$0 \to \mathcal{V}^{n+1} \to \mathcal{O}^G \to \mathcal{O}^G/\mathcal{V}^{n+1} \to 0.$$

The corresponding long exact sequence in cohomology ends in degree S, and the last terms are

$$\cdots \to H^S(G/L, \mathcal{V}^{n+1}) \to \widetilde{\mathcal{H}}(\mathfrak{q}) \to H^S(K/L\cap K, \mathcal{O}^G/\mathcal{V}^{n+1}) \to 0.$$

Any irreducible representation of K appearing in this last group must have highest weight $2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}) + \gamma$, with γ a sum of at most n roots of T in $\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}$.

This follows from Lemma 6.10 just as we deduced (6.9)(d) above.

Let us see where we stand. For each non-negative integer n, we define a subspace of $\widetilde{\mathcal{H}}(\mathfrak{q})$ by

$$\begin{aligned}\widetilde{\mathcal{H}}(\mathfrak{q})^n &= \text{kernel of the map } \widetilde{\mathcal{H}}(\mathfrak{q}) \to H^S(K/L\cap K, \mathcal{O}^G/\mathcal{V}^{n+1}) \\ &= \text{image of the map } H^S(G/L, \mathcal{V}^{n+1}) \to \widetilde{\mathcal{H}}(\mathfrak{q})\end{aligned} \qquad (6.12)(a)$$

Because of the first description, $\widetilde{\mathcal{H}}(\mathfrak{q})^n$ is a closed K-invariant subspace of $\widetilde{\mathcal{H}}(\mathfrak{q})$. It is also clear from the definitions that there are containments

$$\widetilde{\mathcal{H}}(\mathfrak{q})^n \subset \widetilde{\mathcal{H}}(\mathfrak{q})^m \qquad (n \geq m). \qquad (6.12)(b)$$

The Lie algebra $\mathfrak{g}$ acts on Dolbeault cocycles by first-order differential operators. It is plausible to think that such operators should decrease order of vanishing along a subvariety by at most one. This is true, and is proved in [**13**], Lemma 6.8:

$$\widetilde{\pi}(\mathfrak{q})(X)\widetilde{\mathcal{H}}(\mathfrak{q})^{n+1} \subset \widetilde{\mathcal{H}}(\mathfrak{q})^n \qquad (X \in \mathfrak{g}). \qquad (6.12)(c)$$

Now define

$$\widetilde{\mathcal{H}}(\mathfrak{q})^\infty = \bigcap_n \widetilde{\mathcal{H}}(\mathfrak{q})^n. \qquad (6.12)(d)$$

This is a closed, K-invariant, $\mathfrak{g}$-invariant subspace of $\widetilde{\mathcal{H}}(\mathfrak{q})$. Here is what one can prove fairly easily using these ideas.

THEOREM 6.13. *Suppose we are in the setting of Definition 6.7; use the notation of (6.12). Then the $(\mathfrak{g}, K)$-module of K-finite vectors in $\widetilde{\mathcal{H}}(\mathfrak{q})/\widetilde{\mathcal{H}}(\mathfrak{q})^\infty$ satisfies the three conditions in Theorem 5.6. More precisely:*

1. *The restriction to K contains $\delta(\mathfrak{q})$ exactly once.*
2. *Every representation of K appearing has highest weight $2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}) + \gamma$, with γ a sum of roots of T in $\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}$.*

3. *The Casimir operator acts by 0 (even on all of $\widetilde{\mathcal{H}}(\mathfrak{q})$).*

This is all more or less clear from Corollary 6.11 and (6.12), except for the assertion about the Casimir operator. That is a routine calculation analogous to the calculation of infinitesimal characters for induced representations (see for example [**5**], Proposition 8.22). (The Casimir acts on cohomology classes by differentiation on the left. Since it is central, we may as well differentiate on the right. But cohomology classes satisfy some differential equations on the right, and these allow us to show that the Casimir action is zero.) We omit the details.

In this way we can construct at least a non-unitary representation satisfying the requirements of Theorem 5.6. We conclude with a few more remarks about its relationship to $\widetilde{\pi}(\mathfrak{q})$. Suppose first that $S = 0$, so that $\widetilde{\mathcal{H}}(\mathfrak{q})$ is the space of holomorphic sections of a line bundle on G/L. The subspace $\widetilde{\mathcal{H}}(\mathfrak{q})^n$ consists of sections vanishing to order n at the point $K/L \cap K$. Since a non-zero holomorphic function cannot vanish to infinite order at a point, we see that $\widetilde{\mathcal{H}}(\mathfrak{q})^\infty = 0$.

In general (when $S \neq 0$), $\widetilde{\mathcal{H}}(\mathfrak{q})^n$ may be identified with Čech cohomology classes admitting representatives involving holomorphic functions that vanish to order n along $K/L \cap K$. It follows that $\widetilde{\mathcal{H}}(\mathfrak{q})^\infty$ corresponds to Čech cohomology classes admitting for every n representatives involving holomorphic functions that vanish to order n along $K/L \cap K$. Of course we will have to choose different representatives for different values of n, but there is no general argument to rule out the existence of non-zero classes. On the other hand, Schmid's beautiful analysis of $\widetilde{\mathcal{H}}(\mathfrak{q})/\widetilde{\mathcal{H}}(\mathfrak{q})^\infty$ (roughly outlined in Lemma 6.10, Corollary 6.11, and Theorem 6.13) certainly gives reason to hope that $\widetilde{\mathcal{H}}(\mathfrak{q})^\infty = 0$. This is true, and is part of the result of Wong already mentioned:

THEOREM 6.14 ([**19**]). *Suppose we are in the setting of Definition 6.7; use the notation of (6.12). Then $\widetilde{\mathcal{H}}(\mathfrak{q})^\infty = 0$. Consequently $\widetilde{\mathcal{H}}(\mathfrak{q})$ is a smooth Fréchet representation of G, whose $(\mathfrak{g}, K)$-module satisfies the conditions (1)–(3) of Theorems 5.6 or 6.13.*

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