Inflatable mathematics

David Vogan

Sophus Lie Days, Cornell, April 27, 2008

Outline

The main idea

Begin with linear algebra: solving systems of linear equations by Gaussian elimination.

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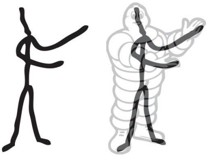

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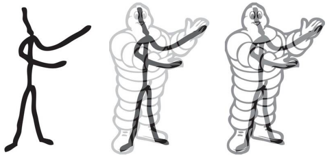

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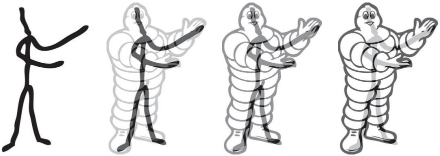

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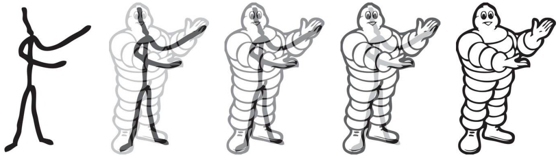

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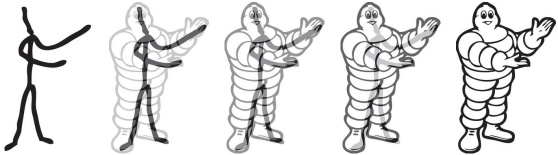

Use same idea for more complicated geometry.

Gaussian elimination: easy cases

System of three equations in three unknowns is

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & c_1 \\ a_{21}\,x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & c_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & c_3 \end{array}$$

I'll assume always the system has just one solution.

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This system is nearly lower triangular, except that the first two equations are interchanged.

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"Typical" system of equations in three unknowns is

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where "typically" $a_{13} \neq 0$. Add multiple of 1st equation to each later eqn to get...

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To say what happens in general, use matrix notation $A\mathbf{x} = \mathbf{c}$. Here $A = (a_{ij})$ is $n \times n$ coeff matrix, and $\mathbf{x} = (x_j)$ is the column vector of n unknowns.

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The new system, after reordering the equations, is lower triangular.

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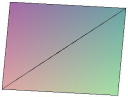

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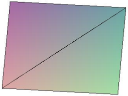

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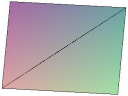

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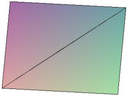

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Two matrices give same flag if and only if differ by

- *multiply row by constant*
- *add multiple of one row to later row.*

Possible flags $L \subset P$

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Moving up $\rightsquigarrow$ more complicated geometry.

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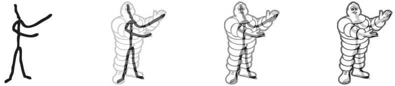

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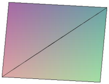

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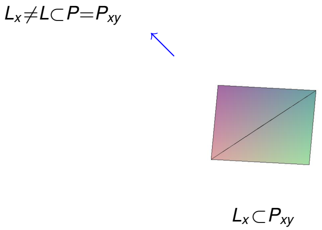

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up one red step: fixed plane ⊃ L ⇝ variable plane ⊃ L.

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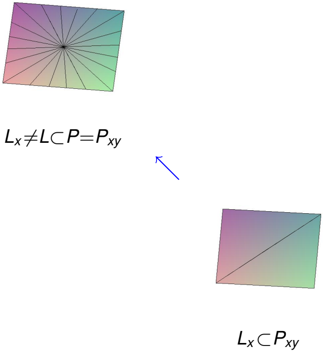

Geometric picture

Moving up ⇝ more complicated geometry.

up one blue step: fixed line ⇝ variable line in a plane.

up one red step: fixed plane ⊃ L ⇝ variable plane ⊃ L.

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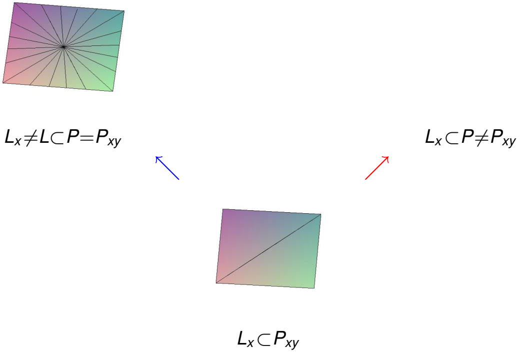

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up one red step: fixed plane $\supset L$ ⇝ variable plane $\supset L$.

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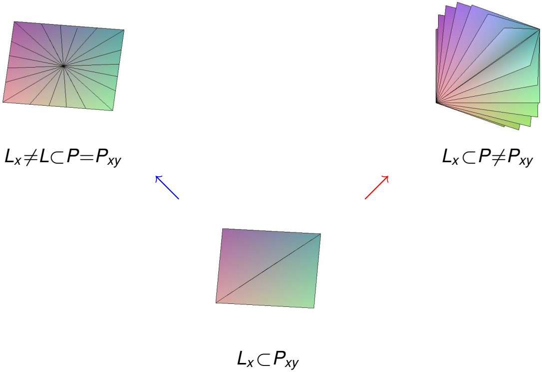

What's a Schubert variety?

Divided flags (in three dimensions) into six "Bruhat cells" by relation with standard flag $L_x \subset P_{xy}$.

Schubert variety is one cell and everything below it:

What's almost true: each Schubert variety "inflated" from a smaller one, replacing each point by a circle.

Fails only at the top. . .

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$L' \subset P', \; L \subset P' \cap P_{xy}$ $\qquad$ $L' \subset P', \; P \supset L' + L_x$

$P_{xy} \not\supset L' \subset P' \not\supset L_x$

$\uparrow L \rightsquigarrow L' \subset P'$ $\qquad$ $\uparrow P \rightsquigarrow P' \supset L'$

$P_{xy} \supset L \subset P' \not\supset L_x$ $\qquad$ $P_{xy} \not\supset L' \subset P \supset L_x$

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$L_x \neq L \subset P_{xy}$ $\qquad$ $L_x \subset P \neq P_{xy}$

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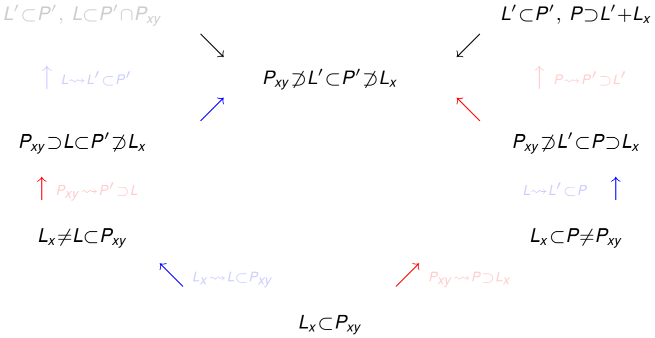

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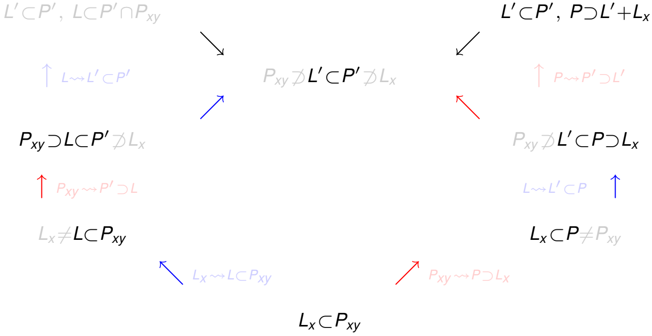

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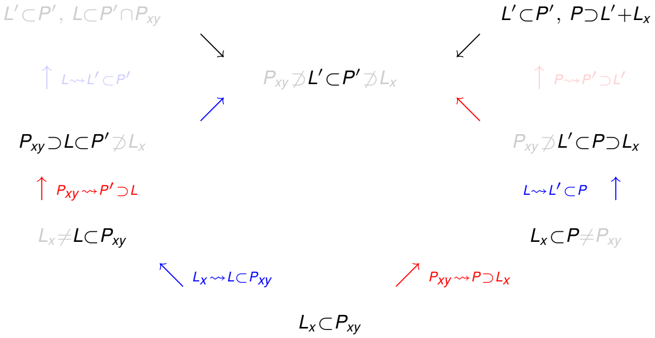

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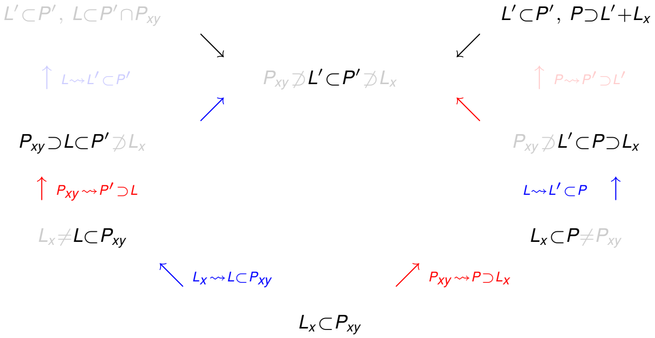

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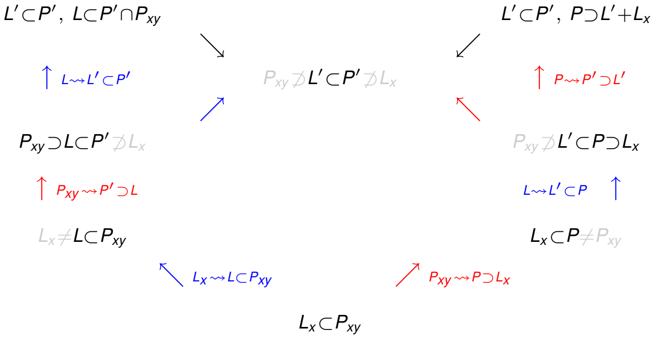

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Mathematics on a need-to-know basis

To compute with Schubert varieties, need only arrangement of blue and red arrows, describing how small Schubert varieties are inflated:

Permutations recorded which rows had pivots in Gaussian elimination. Now they're just symbols.

Rules for making diagram:

1. One entry for each permutation of $\{1, 2, 3\}$.
2. Exchange $1 \dots 2$: blue arrow up.
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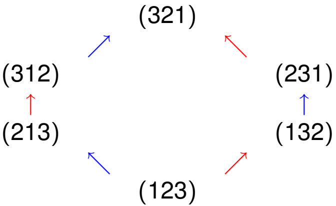

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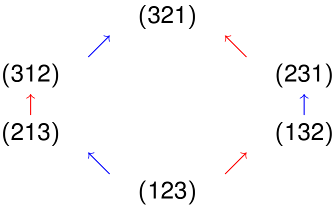

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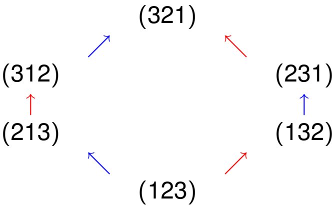

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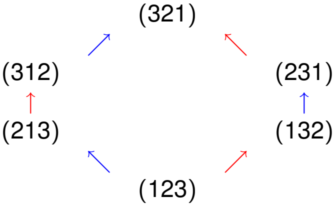

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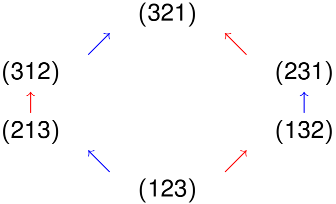

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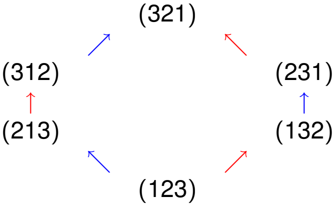

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As many dimensions as you want

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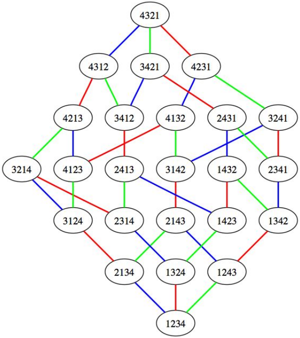

Counting problems in this picture $\leftrightsquigarrow$ geometry of Schubert varieties.

There are lots of counting games to play…

height of a permutation $= \#\{$ pairs (i,j) out of order $\}$.

$\#\{$permutations at height $d\}$ = coefficient of x^d in polynomial

$$(1)(1+x)(1+x+x^2)\cdots(1+x+\cdots x^{n-1}).$$

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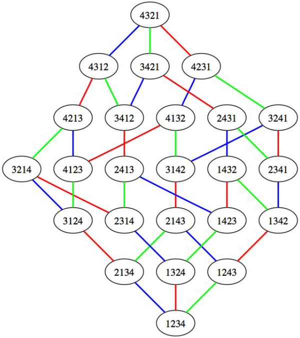

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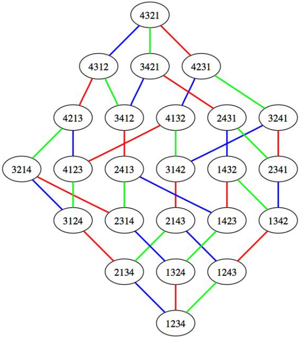

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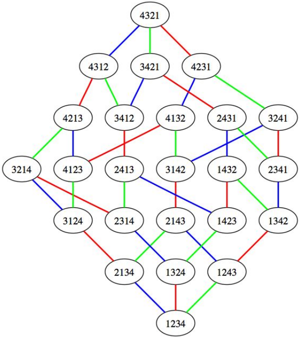

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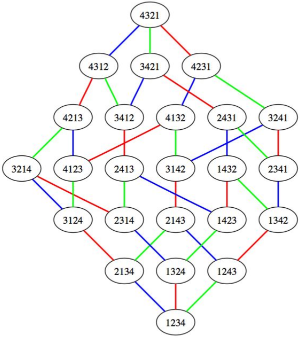

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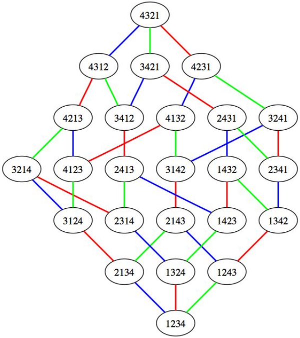

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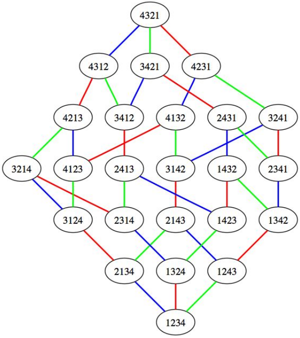

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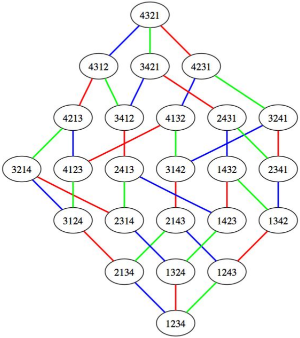

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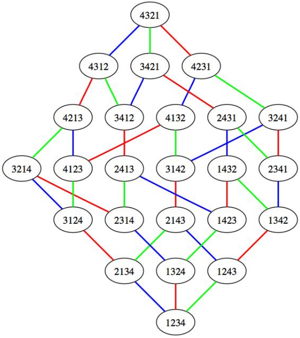

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More complicated groups

Picture just described (with $n!$ vertices) is for invertible $n \times n$ matrices. This is the basic example of a real reductive Lie group. Mathematicians and physicists look at lots of other reductive groups.

Each reductive group has a finite diagram describing how its big Schubert varieties are "inflated" from smaller ones. This one is for a 45-dimensional group called $SO(5,5)$.

For this group there are 251 Schubert varieties, but each arrow still means replace points by circles.

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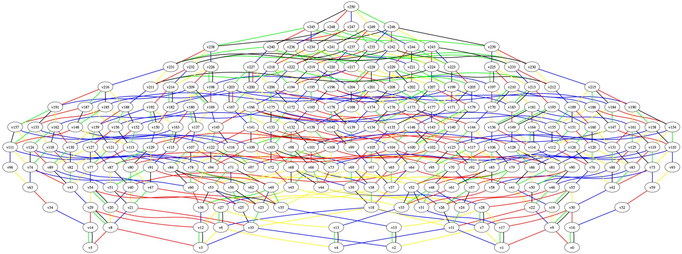

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What do you do with the pretty pictures?

Where we started:

systems of n linear eqns $\overset{\text{Gauss elim}}{\leftrightsquigarrow}$ group $GL(n) \leftrightsquigarrow$ Schubert varieties $\leftrightsquigarrow$ graph with $n!$ vertices, arrows of $n-1$ colors.

Graph tells what cases can happen during Gaussian elimination; how Gaussian elimination changes with the system of equations; even which cases are most common.

Similarly:

math or physics problem $\overset{\text{repn theory}}{\leftrightsquigarrow}$ reductive group $G \leftrightsquigarrow$ Schubert varieties for $G \leftrightsquigarrow$ finite graph for inflating.

1979: David Kazhdan (Harvard) and George Lusztig (MIT) showed how to answer questions about representation theory by calculating in the finite graph.

Defined Kazhdan-Lusztig polynomial $P_{x,y}$ for x and y in the graph. Polynomial in q, non-neg integer coeffs.

Polynomial is non-zero only if y is above x in graph. Calculated by a recursion based on knowing all $P_{x',y'}$ for y' smaller than y.

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How the computation works

Now fixing a reductive group G and its graph of Schubert varieties.

- For each pair (x, y) of graph vertices, want to compute KL polynomial $P_{x,y}$.
- Induction: start with y's on bottom of graph, work up. For each y, start with $x = y$, work down.
- Seek line up x (to x') same color as some line down y (to y').

  If it's there, then $P_{x,y} = P_{x',y}$ (known by induction).

  If not, (x, y) is primitive: no color down from y goes up from x.
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- graph vertex $y \rightsquigarrow$ big Schubert variety F_y.
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  $P_{x,y}$ describes how F_y looks near F_x.
- Pick line down y (line to y'); means $F_y \approx$ inflated from $F_{y'}$.
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- Geometry translates to algebra $P_{x,y} \approx P_{x',y'} + qP_{x,y'}$. Precisely:

$$P_{x,y} = P_{x',y'} + qP_{x,y'} - \sum_{x' \le z < y'} \mu(z, y') q^{(l(y')-l(z)-1)/2} P_{x',z}.$$

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Forming the *Atlas* group

Between 1980 and 2000, increasingly sophisticated computer programs calculated special kinds of Kazhdan-Lusztig polynomials; none dealt with the complications attached to general real reductive groups.

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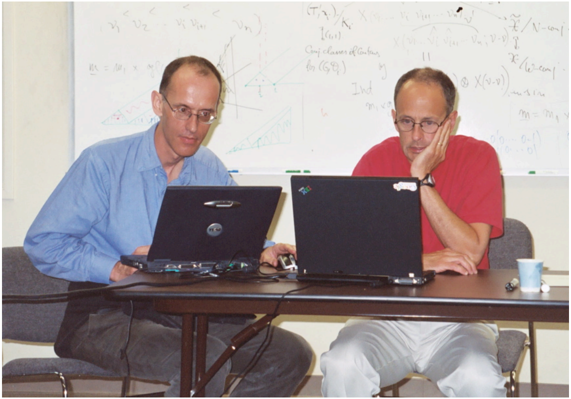

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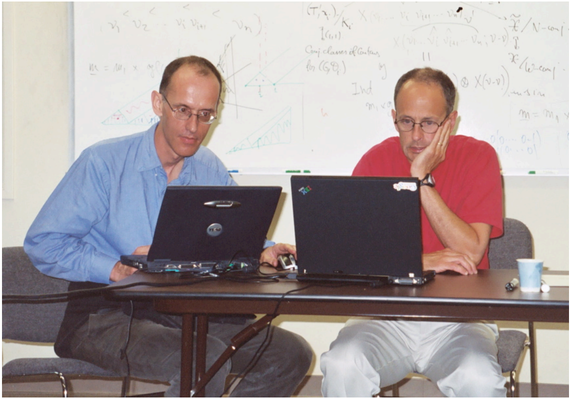

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What's the computer have to do?

TASK	COMPUTER RQMT
Make graph: 453,060 nodes, 8 edges at each	250M RAM, 10 minutes (latest software: thirty seconds)
List primitive pairs of vertices: 6,083,626,944	450M RAM, few seconds
Calculate polynomial for each primitive pair	Fetch few kB from memory, few thousand integer ops ×6 billion
Look for polynomial in store, add if it's new	$4 \frac{\text{bytes}}{\text{coef}} \times 20 \frac{\text{coefs}}{\text{poly}} \times ??\ \text{polys}$ RAM
Write number for poly in table	25G RAM

Big unknown: number of distinct polynomials.
Hoped 400 million polys ⇝ 75G total RAM.
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$$\text{one 150G computation} \xrightarrow{\binom{\text{modular}}{\text{arithmetic}}} \text{four 50G computations}$$

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$$\text{one 150G computation} \xrightarrow{\binom{\text{modular}}{\text{arithmetic}}} \text{four 50G computations}$$

12/03/06 Marc van Leeuwen made Fokko's code modular.

12/19/06 mod 251 computation on `sage`. Took 17 hours:

```
Total elapsed time = 62575s.  Finished at l = 64, y = 453059
d_store.size() = 1181642979, prim_size = 3393819659

VmData:   64435824 kB
```

Writing to disk took two days. Investigating why $\rightsquigarrow$ output bug, so mod 251 answers no good.

The Tribulation (continued)

12/21/06 9 P.M. Started mod 256 computation on `sage`. Computed 452,174 out of 453,060 rows of KL polynomials in 14 hours, then `sage` crashed.

12/22/06 EVENING Restarted mod 256. Finished in just 11 hours

(hip, hip, HURRAH! hip, hip, HURRAH! `pthread_join(cheer[k], NULL);`):

```
Total elapsed time = 40229s.  Finished at l = 64, y = 453059
d_store.size() = 1181642979, prim_size = 3393819659

VmData:  54995416 kB
```

12/23/06 Started mod 255 computation on `sage`, which crashed.

`sage` down til 12/26/06 (regional holiday in Seattle).

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12/27/06 Started computation mod 253. Halfway, sage crashed.

consult experts ⇝ probably not Sasquatch.

Did I mention sage is in Seattle?

Decided not to abuse sage further for a year.

1/3/07 Atlas members one year older ⇝ thirty years wiser as team ⇝ safe to go back to work.

Wrote KL polynomials mod 253 (12 hrs).

Now we had answers mod 253, 255, 256.

Chinese Remainder Theorem (CRT)

gives answer mod 253·255·256 = 16,515,840.

One little computation for each of 13 billion coefficients.

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1/6/07 7 A.M. Output file 7G too big: BUG in output routine.

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"Fokko, look at us. We're spending Sunday alone at work."

Fokko was startled by this remark, but not at a loss for words.
"I don't know about you, but I'm having the time of my life!"

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Fokko du Cloux

December 20, 1954–November 10, 2006