Branching to maximal compact subgroups

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From *K* to *G* and back again

Summary

# Branching to maximal compact subgroups

David Vogan

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### Outline

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Helgason's theorem and algebraic geometry

Interpreting the branching law: Zuckerman's theorem

Relating representations of K and G

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#### $G \operatorname{cplx} \supset G(\mathbb{R}) \operatorname{real} \supset K(\mathbb{R}) \operatorname{maxl} \operatorname{compact}$

Want to study representations  $(\pi, \mathcal{H}_{\pi})$  of  $G(\mathbb{R})$ , bu these are complicated and difficult. Reps of  $K(\mathbb{R})$  are easy, so try two things: understand  $\pi|_{K(\mathbb{R})}$ ; and use understanding to answer questions about  $\pi$ .

# Sample question: how often does trivial representation of $K(\mathbb{R})$ appear in $\pi|_{K(\mathbb{R})}$ ?

Answer: multiplicity zero unless  $\pi$  is (quotient of) spherical principal series, then one.

Application:  $\pi$  can appear in functions on  $G(\mathbb{R})/K(\mathbb{R})$  only if  $\pi$  spherical; then exactly once.

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Write  $\theta$  = Cartan involution of  $G(\mathbb{R})$  and G;

 $K(\mathbb{R}) = G(\mathbb{R})^{\theta}$  (real groups),  $K = G^{\theta}$  (complex algebraic groups).

Iwasawa decomp  $G(\mathbb{R}) = K(\mathbb{R})A(\mathbb{R})_0N(\mathbb{R}).$ 

Here  $A = \max \left| \operatorname{cplx} \operatorname{torus} w \right|$  acts by inverse.

 $L(\mathbb{R}) = \text{centralizer of } A \text{ in } G(\mathbb{R})$  $P(\mathbb{R}) = L(\mathbb{R})N(\mathbb{R}).$ 

Group  $P(\mathbb{R})$  is minimal parabolic subgroup of  $G(\mathbb{R})$ .

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Theorem (Helgason)

1. Rep of hint is so v has  $K(\mathbb{R})$ -lined we  $c := v \delta := 1$  in 2. the g v is a highest wit c := v is domewon int wit.

Says:  $K(\mathbb{R})$ -fixed vecs  $\leftrightarrow M(\mathbb{R})N(\mathbb{R})$ -fixed vecs. Reason:  $M(\mathbb{R})N(\mathbb{R})$  = deformation of  $K(\mathbb{R})$ .

Conjugate  $K(\mathbb{R})$  by elts of  $A(\mathbb{R})_0$ ,  $\rightsquigarrow$  limiting subgroup.

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highest weight  $= \delta \otimes \nu, \, \delta \in M(\mathbb{R}), \, \nu \in A(\mathbb{R})_0.$ 

Theorem (Helgason)

1. Rep of him b is v has  $\mathcal{K}(\mathbb{R})$  direct we consider the i = 1 the 2. The size highest  $w_i^* \longrightarrow v$  is domesion interval.

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#### Theme: complexify, use algebraic geometry.

Helgason's theorem concerns compact  $K(\mathbb{R})$ , minimal parabolic  $P(\mathbb{R})$ .

Theme says complexify, considering algebraic groups  $K = G^{\theta}$  and P = LN parabolic in *G*.

Continuous reps of  $K(\mathbb{R}) \iff$  algebraic reps of K.

Theme says consider projective algebraic variety

 $\mathcal{P} =$ subgps of G conjugate to P,

a partial flag variety.

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#### Proposition

 $K \cdot P$  is open in  $\mathcal{P}$ :  $K/M \simeq K \cdot P \subset \mathcal{P} \simeq G/P$ . Here M = cplx pts of  $M(\mathbb{R}) =$  cent in K of A.

Follows immediately from Iwasawa decomposition.

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- $\{a|g:secs:of \delta \otimes u \circ on \mathcal{P}\} \hookrightarrow \{a|g:secs:of \delta \circ on K/M\}$
- $5 = \{A \mid g \mid ep \mid of \mid G\} |_{C} \longrightarrow Lod_{M}^{2} \{\{h \mid ghest \mid wt\} |_{M}\}$

Picture:  $\mathcal{P} = K/M \bigcup \{\text{divisors}\}.$ Section on  $K/M \rightarrow$  pole order on each divisor. Section extends to  $\mathcal{P} \iff$  no pole on any divisor. Branching to maximal compact subgroups

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### Repn of G = alg secs of vector bdle on $\mathcal{P}$

 $\mathcal{P} = K/M \bigcup \{ \text{divisors } D_1, \ldots, D_r \}$ 

Divisors correspond to simple restricted roots of A.

Bdle on  $\mathcal{P} \rightsquigarrow I_0(\delta) = \operatorname{Ind}_M^K(\delta) = \operatorname{secs} \operatorname{on} K/M$ . Bdle on  $\mathcal{P} \rightsquigarrow I_i(\tau_i) = \operatorname{secs}$  with pole along divisor

Bundle  $\tau_j$  depends on  $\delta$  and on character  $\nu$  of A.

Sections on  $\mathcal{P} \approx I_0(\delta) - \sum_{j=1}^r I_j(\tau_j)$ .

Branching law: describes restr to K of rep of G.

As  $\nu$  tends to infinity, *G* representation grows toward  $I_0(\delta)$ .

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Sections on  $\mathcal{P} \approx I_0(\delta) - \sum_{j=1}^r I_j(\tau_j)$ .

Branching law: describes restr to K of rep of G.

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As  $\nu$  tends to infinity, G representation grows toward  $I_0(\delta)$ .

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Repn of G = alg secs of vector bdle on  $\mathcal{P}$ 

 $\mathcal{P} = K/M \bigcup \{ \text{divisors } D_1, \dots, D_r \}$ 

Divisors correspond to simple restricted roots of A.

Bdle on  $\mathcal{P} \rightsquigarrow I_0(\delta) = \operatorname{Ind}_M^K(\delta) = \operatorname{secs} \operatorname{on} K/M.$ 

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What do the terms on the right mean? Classical picture:

$$I_0(\delta) = \operatorname{Ind}_{M(\mathbb{R})}^{K(\mathbb{R})}(\delta) = \left(\operatorname{Ind}_{P(\mathbb{R})}^G(\mathbb{R})(\delta \otimes \nu \otimes 1\right)|_{K(\mathbb{R})},$$

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### Proposition (Wolf, Beilinson-Bernstein)

- K acts on B with finitely many orbits.
- Unique open orbit ---- Borel subgp of Iwasawa P.
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### Bundle $\xi \iff$ alg char of $H \cap K$ .

Bundle must be "positive" (as in Borel-Weil).

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Theorem (Zuckerman) If F any fin-diml irr rep of G (cplx reductive), then

 $F|_{\mathcal{K}} = \sum_{Z \subset \mathcal{B}} (-1)^{\operatorname{codim}(Z)} I(\tau(Z, F)).$ 

Sum is over orbits of K (complexified max compact) on flag variety B

1st term (codim 0)  $\rightsquigarrow$  princ series  $\leftrightarrow M$  rep  $F^N$ . Next terms (codim 1)  $\rightsquigarrow$  poles on divisors  $\mathcal{P} - K/M$ .

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$$F|_{\mathcal{K}} = \sum_{Z \subset \mathcal{B}} (-1)^{\operatorname{codim}(Z)} I(\tau(Z, F)).$$

Sum is over orbits of K (complexified max compact) on flag variety  $\mathcal{B}$ .

1st term (codim 0)  $\rightsquigarrow$  princ series  $\leftrightarrow M$  rep  $F^N$ . Next terms (codim 1)  $\rightsquigarrow$  poles on divisors  $\mathcal{P} - K/M$ .

Higher terms correct for double counting.

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*M* representation on highest weight for *G* is a - b. Helgason's thm: triv of *K* appears  $\Leftrightarrow a = b$ 

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# Helgason thm when fin diml restr to K is spherical led us to Zuckerman thm descr of fin diml restr to K. What's the next step?

Zuckerman formula (fin diml) = (alt sum of std reps) suggests (any irr rep)  $\stackrel{?}{=}$  (integer comb of std reps). Leads to Kazhdan-Lusztig theory, not dull.

But orig Helgason thm suggests instead looking for formulas (irr of K)  $\stackrel{?}{=}$  (alt sum of std reps).

Application: invert the matrix above to get branching laws (std rep for  $G(\mathbb{R})$ ) = (sum of irrs of K).

Won't write theorem for general G (painful notation); pass directly to examples...

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 $G(\mathbb{R}) = SL(2,\mathbb{R}), K(\mathbb{R}) = SO(2).$ 

 $I^{ps} = (I^{ps} - I^{+}(0) - I^{-}(0)) + (I^{+}(0) - I^{+}(2)) + (I^{-}(0) - I^{-}(-2)) + \cdots$ =  $\tau_{0} + \tau_{2} + \tau_{-2} + \cdots$  $I^{+}(m) = (I^{+}(m) - I^{+}(m+2)) + (I^{+}(m+2) - I^{+}(m+4)) + \cdots$ =  $\tau_{m+1} + \tau_{m+3} + \cdots$  ( $m \in \mathbb{N}$ ) Branching to maximal compact subgroups

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 $I^{ps} = (I^{ps} - I^{+}(0) - I^{-}(0)) + (I^{+}(0) - I^{+}(2)) + (I^{-}(0) - I^{-}(-2)) + \cdots$ =  $\tau_{0} + \tau_{2} + \tau_{-2} + \cdots$  $I^{+}(m) = (I^{+}(m) - I^{+}(m+2)) + (I^{+}(m+2) - I^{+}(m+4)) + \cdots$ =  $\tau_{m+1} + \tau_{m+3} + \cdots$  ( $m \in \mathbb{N}$ ) Branching to maximal compact subgroups

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 $G(\mathbb{R}) = SL(2,\mathbb{R}), K(\mathbb{R}) = SO(2).$ Chars of  $K \rightsquigarrow \tau_k$   $(k \in \mathbb{Z})$ . Princ series  $I^{ps} = sph princ series restr to K$ . Hol disc series  $I^+(m)$  ( $m \in \mathbb{N}$  HC param). Antihol disc series  $I^{-}(m)$  ( $m \in -\mathbb{N}$  HC param).

 $T^{*} = (I^{**} - I^{*}(0) - I^{*}(0)) + (I^{*}(0) - I^{*}(2)) + (I^{*}(0) - I^{*}(-2)) + \cdots$   $= \tau_{0} + \tau_{2} + \tau_{-2} + \cdots$   $I^{+}(m) = (I^{+}(m) - I^{+}(m+2)) + (I^{+}(m+2) - I^{+}(m+4)) + \cdots$   $= \tau_{m+1} + \tau_{m+3} + \cdots \qquad (m \in \mathbb{N})$ 

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 $I^{DS} = (I^{ps} - I^{+}(0) - I^{-}(0)) + (I^{+}(0) - I^{+}(2)) + (I^{-}(0) - I^{-}(-2)) + \cdots$ =  $\tau_{0} + \tau_{2} + \tau_{-2} + \cdots$  $I^{+}(m) = (I^{+}(m) - I^{+}(m+2)) + (I^{+}(m+2) - I^{+}(m+4)) + \cdots$ =  $\tau_{m+1} + \tau_{m+3} + \cdots$  ( $m \in \mathbb{N}$ ) Branching to maximal compact subgroups

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 $I^{+} = (I^{p_3} - I^+(0) - I^-(0)) + (I^+(0) - I^+(2)) + (I^-(0) - I^-(-2)) + \cdots$ =  $\tau_0 + \tau_2 + \tau_{-2} + \cdots$  $I^+(m) = (I^+(m) - I^+(m+2)) + (I^+(m+2) - I^+(m+4)) + \cdots$ =  $\tau_{m+1} + \tau_{m+3} + \cdots$  ( $m \in \mathbb{N}$ ) Branching to maximal compact subgroups

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- Helgason's theorem on spherical fin-diml reps connects Borel-Weil picture of fin-diml reps. to inf-diml reps.
- Zuckerman's theorem extends this to description of fin-diml rep as alt sum of "standard" inf-diml reps.
- Variation on this theme writes any fin-diml of K as alt sum of standard inf-diml reps.
- Inverting these formulas writes standard inf-diml as sum of irrs of K.

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