TRANSLATION PRINCIPLE FOR DIRAC INDEX

SALAH MEHDI, PAVLE PANDŽIĆ AND DAVID VOGAN

ABSTRACT. Let G be a finite cover of a closed connected transpose-stable subgroup of $GL(n, \mathbb{R})$ with complexified Lie algebra $\mathfrak{g}$. Let K be a maximal compact subgroup of G, and assume that G and K have equal rank. We prove a translation principle for the Dirac index of virtual $(\mathfrak{g}, K)$-modules. As a byproduct, to each coherent family of such modules, we attach a polynomial on the dual of the compact Cartan subalgebra of $\mathfrak{g}$. This "index polynomial" generates an irreducible representation of the Weyl group contained in the coherent continuation representation. We show that the index polynomial is the exact analogue on the compact Cartan subgroup of King's character polynomial. The character polynomial was defined in [K1] on the maximally split Cartan subgroup, and it was shown to be equal to the Goldie rank polynomial up to a scalar multiple. In the case of representations of Gelfand-Kirillov dimension at most half the dimension of G/K, we also conjecture an explicit relationship between our index polynomial and the multiplicities of the irreducible components occuring in the associated cycle of the corresponding coherent family.

1. INTRODUCTION

Let G be a connected real reductive group; precisely, a finite cover of a closed connected transpose-stable subgroup of $GL(n, \mathbb{R})$ with complexified Lie algebra $\mathfrak{g}$. Let K be a maximal compact subgroup of G. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the corresponding Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k}$ is the complexified Lie algebra of K.

Let $T \subset K$ be a maximal torus, so that $H_c = G^T$ is a maximally compact Cartan subgroup, with Lie algebra $\mathfrak{h}_c$. Let $\Lambda \subset \widehat{H_c} \subset \mathfrak{h}_c^\star$ be the lattice of weights of finite-dimensional $(\mathfrak{g}, K)$-modules. For a fixed $\lambda_0 \in \mathfrak{h}_c^\star$ regular, a family of virtual $(\mathfrak{g}, K)$-modules X_λ, $\lambda \in \lambda_0 + \Lambda$, is called *coherent* if for each λ, X_λ has infinitesimal character λ, and for any finite-dimensional $(\mathfrak{g}, K)$-module F, and for any λ,

$$(1.1a) \qquad X_\lambda \otimes F = \sum_{\mu \in \Delta(F)} X_{\lambda + \mu},$$

where $\Delta(F)$ denotes the multiset of all weights of F. (A more complete discussion appears in Section 4.) The reason for studying coherent families is that if X is any irreducible $(\mathfrak{g}, K)$-module of infinitesimal character λ_0, then there is a *unique*

coherent family with the property that

$$(1.1b) \qquad\qquad\qquad\qquad X_{\lambda_0} = X.$$

For any invariant of Harish-Chandra modules, one can therefore ask how the invariant of X_λ changes with $\lambda \in \lambda_0 + \Lambda$. The nature of this dependence is then a new invariant of X. This idea is facilitated by the fact that

(1.1c) $\qquad\quad$ X_λ is irreducible or zero whenever λ is integrally dominant;

zero is possible only for singular λ. (See for example [V2], sections 7.2 and 7.3.) The notion of "integrally dominant" is recalled in (4.2); we write $(\lambda_0 + \Lambda)^+$ for the cone of integrally dominant elements. We may therefore define

$$(1.1d) \qquad\quad \operatorname{Ann}(X_\lambda) = \text{annihilator in } U(\mathfrak{g}) \text{ of } X_\lambda \qquad (\lambda \in (\lambda_0 + \Lambda)^+).$$

The ideal $\operatorname{Ann}(X_\lambda)$ is primitive if X_λ is irreducible, and equal to $U(\mathfrak{g})$ if $X_\lambda = 0$. Write $\operatorname{rk}(U(\mathfrak{g})/\operatorname{Ann}(X_\lambda))$ for the Goldie rank of the algebra $U(\mathfrak{g})/\operatorname{Ann}(X_\lambda)$. Let $W_\mathfrak{g}$ be the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}_c$. Joseph proved that the $\mathbb{N}$-valued map defined on integrally dominant $\lambda \in \lambda_0 + \Lambda$ by

$$(1.1e) \qquad\qquad\qquad \lambda \mapsto \operatorname{rk}(U(\mathfrak{g})/\operatorname{Ann}(X_\lambda))$$

extends to a $W_\mathfrak{g}$-harmonic polynomial P_X on $\mathfrak{h}_c^\star$ called the *Goldie rank polynomial* for X. The polynomial P_X is homogeneous of degree $\sharp R_\mathfrak{g}^+ - \operatorname{Dim}(X)$, where $\sharp R_\mathfrak{g}^+$ denotes the number of positive $\mathfrak{h}_c$-roots in $\mathfrak{g}$ and $\operatorname{Dim}(X)$ is the Gelfand-Kirillov dimension of X. Moreover, P_X generates an irreducible representation of $W_\mathfrak{g}$. See [J1], [J2] and [J3].

There is an interpretation of the $W_\mathfrak{g}$-representation generated by P_X in terms of the Springer correspondence. For all $\lambda \in (\lambda_0 + \Lambda)^+$ such that $X_\lambda \neq 0$ (so for example for all integrally dominant regular λ), the associated variety $\mathcal{V}(\operatorname{gr}(\operatorname{Ann}(X_\lambda)))$ (defined by the associated graded ideal of $\operatorname{Ann}(X_\lambda)$, in the symmetric algebra $S(\mathfrak{g})$) is the Zariski closure of a single nilpotent $G_\mathbb{C}$-orbit $\mathcal{O}$ in $\mathfrak{g}^\star$, independent of λ. (Here $G_\mathbb{C}$ is a connected complex reductive algebraic group having Lie algebra $\mathfrak{g}$.) Barbasch and Vogan proved that the Springer representation of $W_\mathfrak{g}$ attached to $\mathcal{O}$ coincides with the $W_\mathfrak{g}$-representation generated by the Goldie rank polynomial P_X (see [BV1]).

Here is another algebro-geometric interpretation of P_X. Write

$$(1.1f) \qquad\qquad\qquad \mathcal{O} \cap (\mathfrak{g}/\mathfrak{k})^* = \coprod_{j=1}^{r} \mathcal{O}^j$$

for the decomposition into (finitely many) orbits of $K_\mathbb{C}$. (Here $K_\mathbb{C}$ is the complexification of K.) Then the associated cycle of each X_λ is

$$(1.1g) \qquad\qquad \operatorname{Ass}(X_\lambda) = \coprod_{j=1}^{r} m_X^j(\lambda)\overline{\mathcal{O}^j} \qquad (\lambda \in (\lambda_0 + \Lambda)^+)$$

(see Definition 2.4, Theorem 2.13, and Corollary 5.20 in [V3]). The component multiplicity $m_X^j(\lambda)$ is a function taking nonnegative integer values, and extends to a polynomial function on $\mathfrak{h}_c^*$. We call this polynomial the *multiplicity polynomial* for X on the orbit $\mathcal{O}^j$. The connection with the Goldie rank polynomial is that each m_X^j is a scalar multiple of P_X; this is a consequence of the proof of Theorem 5.7 in [J2].

On the other hand, Goldie rank polynomials can be interpreted in terms of the asymptotics of the global character $\mathrm{ch}_{\mathfrak{g}}(X_\lambda)$ of X_λ on a maximally split Cartan subgroup $H_s \subset G$ with Lie algebra $\mathfrak{h}_{s,0}$. Namely, if $x \in \mathfrak{h}_{s,0}$ is a generic regular element, King proved that the map

$$(1.1\mathrm{h}) \qquad\qquad \lambda \mapsto \lim_{t \to 0} t^{\mathrm{Dim}(X)} \mathrm{ch}_{\mathfrak{g}}(X_\lambda)(\exp tx)$$

on $\lambda_0 + \Lambda$ extends to a polynomial $C_{X,x}$ on $\mathfrak{h}_c^\star$. We call this polynomial *King's character polynomial*. It coincides with the Goldie rank polynomial P_X up to a constant factor depending on x (see [K1]). More precisely, as a consequence of [SV], one can show that there is a formula

$$(1.1\mathrm{i}) \qquad\qquad C_{X,x} = \sum_{j=1}^{r} a_x^j m_X^j;$$

the constants a_x^j are independent of X, and this formula is valid for any $(\mathfrak{g}, K)$-module whose annihilator has associated variety contained in $\overline{\mathcal{O}}$. The polynomial $C_{X,x}$ expresses the dependence on λ of the leading term in the Taylor expansion of the numerator of the character of X_λ on the maximally split Cartan H_s.

In this paper, we assume that G and K have equal rank. Under this assumption, we use Dirac index to obtain the analog of King's asymptotic character formula (1.1h), or equivalently of the Goldie rank polynomial (1.1e), in the case when H_s is replaced by a compact Cartan subgroup T of G. In the course of doing this, we first prove a translation principle for the Dirac index.

To define the notions of Dirac cohomology and index, we first recall that there is a *Dirac operator* $D \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$, where $C(\mathfrak{p})$ is the Clifford algebra of $\mathfrak{p}$ with respect to an invariant non-degenerate symmetric bilinear form B (see Section 2). If S is a spin module for $C(\mathfrak{p})$ then D acts on $Y \otimes S$ for any $(\mathfrak{g}, K)$-module Y. The *Dirac cohomology* $H_D(Y)$ of Y is defined as

$$H_D(Y) = \mathrm{Ker}\, D / \mathrm{Ker}\, D \cap \mathrm{Im}\, D.$$

It is a module for the spin double cover $\widetilde{K}$ of K. Dirac cohomology was introduced by Vogan in the late 1990s (see [V4]) and turned out to be an interesting invariant attached to $(\mathfrak{g}, K)$-modules (see [HP2] for a thorough discussion).

We would now like to study how Dirac cohomology varies over a coherent family. This is however not possible; since Dirac cohomology is not an exact functor, it cannot be defined for virtual $(\mathfrak{g}, K)$-modules. To fix this problem, we will replace Dirac cohomology by the Dirac index. (We note that there is a relationship between Dirac cohomology and translation functors; see [MP], [MPa1], [MPa2], [MPa3].)

Let $\mathfrak{t}$ be the complexified Lie algebra of the compact Cartan subgroup T of G. Then $\mathfrak{t}$ is a Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{k}$. In this case, the spin module S for $\widetilde{K}$ is the direct sum of two pieces S^+ and S^-, and the Dirac cohomology $H_D(Y)$ breaks up accordingly into $H_D(Y)^+$ and $H_D(Y)^-$. If Y is admissible and has infinitesimal character, define the *Dirac index of Y* to be the virtual $\widetilde{K}$-module

$$(1.2\mathrm{a}) \qquad\qquad I(Y) = H_D(Y)^+ - H_D(Y)^-.$$

This definition can be extended to arbitrary finite length modules (not necessarily with infinitesimal character), replacing H_D by the higher Dirac cohomology of [PS]. See Section 3. Then I, considered as a functor from finite length $(\mathfrak{g}, K)$-modules to virtual $\widetilde{K}$-modules, is additive with respect to short exact sequences (see Lemma 3.3

and the discussion below (3.7)), so it makes sense also for virtual $(\mathfrak{g}, K)$-modules. Furthermore, I satisfies the following property (Proposition 4.3): for any finite-dimensional $(\mathfrak{g}, K)$-module F,

$$I(Y \otimes F) = I(Y) \otimes F.$$

Let now $\{X_\lambda\}_{\lambda \in \lambda_0 + \Lambda}$ be a coherent family of virtual $(\mathfrak{g}, K)$-modules.

By a theorem of Huang and Pandžić, the $\mathfrak{k}$-infinitesimal character of any $\widetilde{K}$-type contributing to the Dirac cohomology $H_D(Y)$ of an irreducible $(\mathfrak{g}, K)$-module Y is $W_\mathfrak{g}$-conjugate to the $\mathfrak{g}$-infinitesimal character of Y (see Theorem 2.2). In terms of the virtual representations $\widetilde{E}$ of $\widetilde{K}$ defined in Section 4, the conclusion is that we may write

$$(1.2b) \qquad I(X_{\lambda_0}) = \sum_{w \in W_\mathfrak{g}} a_w \widetilde{E}_{w\lambda_0}$$

with $a_w \in \mathbb{Z}$. Then, for any $\nu \in \Lambda$, we have (Theorem 4.7):

$$(1.2c) \qquad I(X_{\lambda_0+\nu}) = \sum_{w \in W_\mathfrak{g}} a_w \widetilde{E}_{w(\lambda_0+\nu)}$$

with the same coefficients a_w. It follows that $I(X_{\lambda_0}) \neq 0$ implies $I(X_{\lambda_0+\nu}) \neq 0$, provided both λ_0 and $\lambda_0 + \nu$ are regular for $\mathfrak{g}$ (Corollary 4.10).

Combining the translation principle for Dirac index (1.2c) with the Weyl dimension formula for $\mathfrak{k}$, we conclude that the map

$$(1.2d) \qquad \lambda_0 + \Lambda \to \mathbb{Z}, \qquad \lambda \mapsto \dim I(X_\lambda)$$

extends to a $W_\mathfrak{g}$-harmonic polynomial Q_X on $\mathfrak{t}^\star$ (see Section 5). We call the polynomial Q_X the *index polynomial* attached to X and λ_0. If Q_X is nonzero, its degree is equal to the number $R_\mathfrak{k}^+$ of positive $\mathfrak{t}$-roots in $\mathfrak{k}$. More precisely, Q_X belongs to the irreducible representation of $W_\mathfrak{g}$ generated by the Weyl dimension formula for $\mathfrak{k}$ (Proposition 5.2). Furthermore, the coherent continuation representation generated by X must contain a copy of the index polynomial representation (Proposition 5.3). We also prove that the index polynomial vanishes for small representations. Namely, if the Gelfand-Kirillov dimension $\mathrm{Dim}(X)$ is less than the number $\sharp R_\mathfrak{g}^+ - \sharp R_\mathfrak{k}^+$ of positive noncompact $\mathfrak{t}$-roots in $\mathfrak{g}$, then $Q_X = 0$ (Proposition 5.6).

An important feature of the index polynomial is the fact that Q_X is the exact analogue of King's character polynomial (1.1h), but attached to the character on the compact Cartan subgroup instead of the maximally split Cartan subgroup (see Section 6). In fact, Q_X expresses the dependence on λ of the (possibly zero) leading term in the Taylor expansion of the numerator of the character of X_λ on the compact Cartan T: for any $y \in \mathfrak{t}_0$ regular, we have

$$\lim_{t \to 0+} t^{\sharp R_\mathfrak{g}^+ - \sharp R_\mathfrak{k}^+} \operatorname{ch}_\mathfrak{g}(X_\lambda)(\exp ty) = \frac{\prod_{\alpha \in R_\mathfrak{k}^+} \alpha(y)}{\prod_{\alpha \in R_\mathfrak{g}^+} \alpha(y)} Q_X(\lambda).$$

In particular, if G is semisimple of Hermitian type, and if X is the $(\mathfrak{g}, K)$-module of a holomorphic discrete series representation, then the index polynomial Q_X coincides, up to a scalar multiple, with the Goldie rank polynomial P_X (Proposition 6.5). Moreover, if X is the $(\mathfrak{g}, K)$-module of any discrete series representation (for G not necessarily Hermitian), then Q_X and P_X are both divisible by the product of linear factors corresponding to the roots generated by the τ-invariant of X

(Proposition 6.9). Recall that the τ-invariant of the $(\mathfrak{g}, K)$-module X consists of the simple roots α such that the translate of X to the wall defined by α is zero (see Section 4 in [V1] or Chapter 7 in [V2]).

Recall the formula (1.1i) relating King's character polynomial to the multiplicity polynomials for the associated cycle. In Section 7, we conjecture a parallel relationship between the index polynomial Q_X and the multiplicity polynomials. For that, we must assume that the $W_{\mathfrak{g}}$-representation generated by the Weyl dimension formula for $\mathfrak{k}$ corresponds to a nilpotent $G_{\mathbb{C}}$-orbit $\mathcal{O}_K$ via the Springer correspondence. (At the end of Section 7, we list the classical groups for which this assumption is satisfied.) Then we conjecture (Conjecture 7.2): if $\mathcal{V}(\mathrm{gr}(\mathrm{Ann}(X))) \subset \overline{\mathcal{O}_K}$, then

$$(1.2e) \qquad\qquad Q_X = \sum_j c^j m_X^j.$$

Here the point is that the coefficients c^j should be integers independent of X. We check that this conjecture holds in the case when $G = SL(2, \mathbb{R})$ and also when $G = SU(1, n)$ with $n \geq 2$.

In the following we give a few remarks related to the significance of the above conjecture.

Associated varieties are a beautiful and concrete invariant for representations, but they are too crude to distinguish representations well. For example, all holomorphic discrete series have the same associated variety. Goldie rank polynomials and multiplicity functions both offer more information, but the information is somewhat difficult to compute and to interpret precisely. The index polynomial is easier to compute and interpret precisely; it can be computed from knowing the restriction to K, and conversely, it contains fairly concrete information about the restriction to K. In the setting of (1.2e) (that is, for fairly small representations), the conjecture says that the index polynomial should be built from multiplicity polynomials in a very simple way.

The conjecture implies that, for these small representations, the index polynomial must be a multiple of the Goldie rank polynomial. This follows from the fact that each m_X^i is a multiple of P_X, mentioned below (1.1g). The interesting thing about this is that the index polynomial is perfectly well-defined for *larger* representations as well. In some sense it is defining something like "multiplicities" for $\mathcal{O}_K$ even when $\mathcal{O}_K$ is not a leading term.

This is analogous to a result of Barbasch, which says that one can define for any character expansion a number that is the multiplicity of the zero orbit for finite-dimensional representations. In the case of discrete series, this number turns out to be the formal degree (and so is something really interesting). This indicates that the index polynomial is an example of an interesting "lower order term" in a character expansion. We can hope that a fuller understanding of all such lower order terms could be a path to extending the theory of associated varieties to a more complete invariant of representations.

2. SETTING

Let G be a finite cover of a closed connected transpose-stable subgroup of $GL(n, \mathbb{R})$, with Lie algebra $\mathfrak{g}_0$. We denote by Θ the Cartan involution of G corresponding to the usual Cartan involution of $GL(n, \mathbb{R})$ (the transpose inverse). Then $K = G^{\Theta}$ is a maximal compact subgroup of G. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan

decomposition of $\mathfrak{g}_0$, with $\mathfrak{k}_0$ the Lie algebra of K. Let B be the trace form on $\mathfrak{g}_0$. Then B is positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$, and $\mathfrak{p}_0$ is the orthogonal of $\mathfrak{k}_0$ with respect to B. We shall drop the subscript '0' on real vector spaces to denote their complexifications. Thus $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denotes the Cartan decomposition of the complexified Lie algebra of G. The linear extension of B to $\mathfrak{g}$ is again denoted by B. Let $G_\mathbb{C}$ be a connected reductive algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$.

Let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to B and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$.

The Dirac operator D is defined as

$$D = \sum_i b_i \otimes d_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p}),$$

where b_i is any basis of $\mathfrak{p}$ and d_i is the dual basis with respect to B. Then D is independent of the choice of the basis b_i and is K-invariant. Moreover, the square of D is given by the following formula due to Parthasarathy [P]:

$$(2.1) \qquad D^2 = -(\mathrm{Cas}_\mathfrak{g} \otimes 1 + \|\rho_\mathfrak{g}\|^2) + (\mathrm{Cas}_{\mathfrak{k}_\Delta} + \|\rho_\mathfrak{k}\|^2).$$

Here $\mathrm{Cas}_\mathfrak{g}$ is the Casimir element of $U(\mathfrak{g})$ and $\mathrm{Cas}_{\mathfrak{k}_\Delta}$ is the Casimir element of $U(\mathfrak{k}_\Delta)$, where $\mathfrak{k}_\Delta$ is the diagonal copy of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, defined using the obvious embedding $\mathfrak{k} \hookrightarrow U(\mathfrak{g})$ and the usual map $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p}) \to C(\mathfrak{p})$. See [HP2] for details.

If X is a $(\mathfrak{g}, K)$-module, then D acts on $X \otimes S$, where S is a spin module for $C(\mathfrak{p})$. The Dirac cohomology of X is the module

$$H_D(X) = \mathrm{Ker}\, D / \mathrm{Ker}\, D \cap \mathrm{Im}\, D$$

for the spin double cover $\widetilde{K}$ of K. If X is unitary or finite-dimensional, then

$$H_D(X) = \mathrm{Ker}\, D = \mathrm{Ker}\, D^2.$$

The following result of [HP1] was conjectured by Vogan [V3]. Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. We view $\mathfrak{t}^* \subset \mathfrak{h}^*$ by extending functionals on $\mathfrak{t}$ by 0 over $\mathfrak{a}$. Denote by $R_\mathfrak{g}$ (resp. $R_\mathfrak{k}$) the set of $(\mathfrak{g}, \mathfrak{h})$-roots (resp. $(\mathfrak{k}, \mathfrak{t})$-roots). We fix compatible positive root systems $R_\mathfrak{g}^+$ and $R_\mathfrak{k}^+$ for $R_\mathfrak{g}$ and $R_\mathfrak{k}$ respectively. In particular, this determines the half-sums of positive roots $\rho_\mathfrak{g}$ and $\rho_\mathfrak{k}$ as usual. Write $W_\mathfrak{g}$ (resp. $W_\mathfrak{k}$) for the Weyl group associated with $(\mathfrak{g}, \mathfrak{h})$-roots (resp. $(\mathfrak{k}, \mathfrak{t})$-roots).

Theorem 2.2. *Let X be a $(\mathfrak{g}, K)$-module with infinitesimal character corresponding to $\Lambda \in \mathfrak{h}^*$ via the Harish-Chandra isomorphism. Assume that $H_D(X)$ contains the irreducible $\widetilde{K}$-module E_γ with highest weight $\gamma \in \mathfrak{t}^*$. Then Λ is equal to $\gamma + \rho_\mathfrak{k}$ up to conjugation by the Weyl group $W_\mathfrak{g}$. In other words, the $\mathfrak{k}$-infinitesimal character of any $\widetilde{K}$-type contributing to $H_D(X)$ is $W_\mathfrak{g}$-conjugate to the $\mathfrak{g}$-infinitesimal character of X.*

3. Dirac index

Throughout the paper we assume that $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank, i.e., that there is a compact Cartan subalgebra $\mathfrak{h} = \mathfrak{t}$ in $\mathfrak{g}$. In this case, $\mathfrak{p}$ is even-dimensional, so (as long as $\mathfrak{p} \neq \{0\}$) the spin module S for the spin group $\mathrm{Spin}(\mathfrak{p})$ (and therefore for $\widetilde{K}$) is the direct sum of two pieces, which we will call S^+ and S^-. To say which is which, it is enough to choose an $SO(\mathfrak{p})$-orbit of maximal isotropic subspaces of $\mathfrak{p}$. We will sometimes make such a choice by fixing a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ for $\mathfrak{t}$ in $\mathfrak{g}$, and writing $\mathfrak{n} = \mathfrak{n}_\mathfrak{k} + \mathfrak{n}_\mathfrak{p}$ for the corresponding sum of positive root spaces. Then $\mathfrak{n}_\mathfrak{p}$ is a choice of maximal isotropic subspace of $\mathfrak{p}$.

The full spin module may be realized using $\mathfrak{n}_\mathfrak{p}$ as $S \simeq \bigwedge \mathfrak{n}_\mathfrak{p}$, with the $C(\mathfrak{p})$-action defined so that elements of $\mathfrak{n}_\mathfrak{p}$ act by wedging, and elements of the dual isotropic space $\mathfrak{n}_\mathfrak{p}^-$ corresponding to the negative roots act by contracting. (Details may be found for example in [Chev] at the beginning of Chapter 3.) In particular, the action of $C(\mathfrak{p})$ respects parity of degrees: odd elements of $C(\mathfrak{p})$ carry $\bigwedge^{\mathrm{even}} \mathfrak{n}_\mathfrak{p}$ to $\bigwedge^{\mathrm{odd}} \mathfrak{n}_\mathfrak{p}$ and so on. Because $\mathrm{Spin}(\mathfrak{p}) \subset C_{\mathrm{even}}(\mathfrak{p})$, it follows that $\mathrm{Spin}(\mathfrak{p})$ preserves the decomposition

$$S \simeq \bigwedge \mathfrak{n}_\mathfrak{p} = \bigwedge\nolimits^{\mathrm{even}} \mathfrak{n}_\mathfrak{p} \oplus \bigwedge\nolimits^{\mathrm{odd}} \mathfrak{n}_\mathfrak{p} \stackrel{\mathrm{def.}}{=} S^+ \oplus S^-.$$

The group $\widetilde{K}$ acts on S as usual, through the map $\widetilde{K} \to \mathrm{Spin}(\mathfrak{p}) \subset C(\mathfrak{p})$, and hence also the Lie algebra $\mathfrak{k}$ acts, through the map $\alpha : \mathfrak{k} \to \mathfrak{so}(\mathfrak{p}) \hookrightarrow C(\mathfrak{p})$. We call these actions of $\widetilde{K}$ and $\mathfrak{k}$ the spin actions. It should however be noted that although we wrote $S \simeq \bigwedge \mathfrak{n}_\mathfrak{p}$, the $\mathfrak{t}$-weights of S for the spin action are not the weights of $\bigwedge \mathfrak{n}_\mathfrak{p}$, i.e., the sums of distinct roots in $\mathfrak{n}_\mathfrak{p}$, but rather these weights shifted by $-(\rho_\mathfrak{g} - \rho_\mathfrak{k})$. This difference comes from the construction of the map α and the action of $C(\mathfrak{p})$ on S.

In particular, the weights of $S^+ \simeq \bigwedge^{\mathrm{even}} \mathfrak{n}_\mathfrak{p}$ are

$$-\rho_\mathfrak{g} + \rho_\mathfrak{k} + (\text{sum of an even number of distinct roots in } \mathfrak{n}_\mathfrak{p}).$$

Similarly, the weights of $S^- \simeq \bigwedge^{\mathrm{odd}} \mathfrak{n}_\mathfrak{p}$ are

$$-\rho_\mathfrak{g} + \rho_\mathfrak{k} + (\text{sum of an odd number of distinct roots in } \mathfrak{n}_\mathfrak{p}).$$

The Dirac operator D interchanges $X \otimes S^+$ and $X \otimes S^-$ for any $(\mathfrak{g}, K)$-module X. (That is because it is of degree 1 in the Clifford factor.) It follows that the Dirac cohomology $H_D(X)$ also breaks up into even and odd parts, which we denote by $H_D(X)^+$ and $H_D(X)^-$ respectively. If X is of finite length, then $H_D(X)$ is finite-dimensional, as follows from (2.1), which implies that $\mathrm{Ker}\, D^2$ is finite-dimensional for any admissible module X. If X is of finite length and has infinitesimal character, then we define the Dirac index of X as the virtual $\widetilde{K}$-module

$$(3.1) \qquad\qquad I(X) = H_D(X)^+ - H_D(X)^-.$$

The first simple but important fact is the following proposition, which is well known for the case of discrete series or finite-dimensional modules.

Proposition 3.2. *Let X be a finite length $(\mathfrak{g}, K)$-module with infinitesimal character. Then there is an equality of virtual $\widetilde{K}$-modules*

$$X \otimes S^+ - X \otimes S^- = I(X).$$

Proof. By Parthasarathy's formula for D^2 (2.1), $X \otimes S$ breaks into a direct sum of eigenspaces for D^2:

$$X \otimes S = \sum_\lambda (X \otimes S)_\lambda.$$

Since D^2 is even in the Clifford factor, this decomposition is compatible with the decomposition into even and odd parts, i.e.,

$$(X \otimes S)_\lambda = (X \otimes S^+)_\lambda \oplus (X \otimes S^-)_\lambda,$$

for any eigenvalue λ of D^2. Since D commutes with D^2, it preserves each eigenspace. Since D also switches parity, we see that D defines maps

$$D_\lambda : (X \otimes S^\pm)_\lambda \to (X \otimes S^\mp)_\lambda.$$

for each λ. If $\lambda \neq 0$, then D_λ is clearly an isomorphism (with inverse $\frac{1}{\lambda} D_\lambda$), and hence

$$X \otimes S^+ - X \otimes S^- = (X \otimes S^+)_0 - (X \otimes S^-)_0.$$

Since D is a differential on $\operatorname{Ker} D^2$, and the cohomology of this differential is exactly $H_D(X)$, the statement now follows from the Euler-Poincaré principle. $\square$

Corollary 3.3. *Let*

$$0 \to U \to V \to W \to 0$$

be a short exact sequence of finite length $(\mathfrak{g}, K)$-modules, and assume that V has infinitesimal character (so that U and W must have the same infinitesimal character as V). Then there is an equality of virtual $\widetilde{K}$-modules

$$I(V) = I(U) + I(W).$$

Proof. This follows from the formula in Proposition 3.2, since the left hand side of that formula clearly satisfies the additivity property. $\square$

To study the translation principle, we need to deal with modules $X \otimes F$, where X is a finite length $(\mathfrak{g}, K)$-module, and F is a finite-dimensional $(\mathfrak{g}, K)$-module. Therefore, Proposition 3.2 and Corollary 3.3 are not sufficient for our purposes, because they apply only to modules with infinitesimal character. Namely, if X is of finite length and has infinitesimal character, then $X \otimes F$ is of finite length, but it typically cannot be written as a direct sum of modules with infinitesimal character. Rather, some of the summands of $X \otimes F$ only have generalized infinitesimal character. Recall that $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is the generalized infinitesimal character of a $\mathfrak{g}$-module V if there is a positive integer N such that

$$(z - \chi(z))^N = 0 \quad \text{on } V, \qquad \text{for every } z \in Z(\mathfrak{g}),$$

where $Z(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$. Here is an example showing that Proposition 3.2 and Corollary 3.3 can fail for modules with generalized infinitesimal character.

Example 3.4 ([PS], Section 2)**.** Let $G = SU(1,1) \cong SL(2,\mathbb{R})$, so that $K = S(U(1) \times U(1)) \cong U(1)$, and $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. Then there is an indecomposable $(\mathfrak{g}, K)$-module P fitting into the short exact sequence

$$0 \to V_0 \to P \to V_{-2} \to 0,$$

where V_0 is the (reducible) Verma module with highest weight 0, and V_{-2} is the (irreducible) Verma module with highest weight -2. Using calculations similar to [HP2], 9.6.5, one checks that for the index defined by (3.1) the following holds:

(3.5) $$I(P) = -\mathbb{C}_1; \qquad I(V_0) = -\mathbb{C}_1; \qquad I(V_{-2}) = -\mathbb{C}_{-1},$$

where $\mathbb{C}_1$ respectively $\mathbb{C}_{-1}$ is the one-dimensional $\widetilde{K}$-module of weight 1 respectively -1. So Corollary 3.3 fails for P. It follows that Proposition 3.2 must also fail.

A modified version of Dirac cohomology, called "higher Dirac cohomology", has been recently defined by Pandžić and Somberg [PS]. It is defined as $H(X) = \bigoplus_{k \in \mathbb{Z}_+} H^k(V)$, where

$$H^k(V) = \operatorname{Im} D^{2k} \cap \operatorname{Ker} D \big/ \operatorname{Im} D^{2k+1} \cap \operatorname{Ker} D.$$

If we let $H(X)^\pm$ be the even and odd parts of $H(X)$, and define the stable index as

(3.6) $$I(X) = H(X)^+ - H(X)^-,$$

then Proposition 3.2 holds for any module X of finite length, i.e.,

$$(3.7) \qquad\qquad I(X) = X \otimes S^+ - X \otimes S^-$$

([PS], Theorem 3.4). It follows that the index defined in this way is additive with respect to short exact sequences ([PS], Corollary 3.5), and it therefore makes sense for virtual $(\mathfrak{g}, K)$-modules, i.e., it is well defined on the Grothendieck group of the Abelian category of finite length $(\mathfrak{g}, K)$-modules.

Another way to deal with the above mentioned problem is to use the statement of Proposition 3.2, or (3.7), as the definition of the index $I(X)$. In any case, from now on we work with Dirac index $I(X)$, defined for any virtual $(\mathfrak{g}, K)$-module X, and satisfying (3.7).

4. Coherent families

Fix $\lambda_0 \in \mathfrak{t}^*$ regular and let T be a compact Cartan subgroup of G with complexified Lie algebra $\mathfrak{t}$. We denote by $\Lambda \subset \widehat{T} \subset \mathfrak{t}^*$ the lattice of weights of finite-dimensional representations of G (equivalently, of finite-dimensional $(\mathfrak{g}, K)$-modules). A family of virtual $(\mathfrak{g}, K)$-modules X_λ, $\lambda \in \lambda_0 + \Lambda$, is called *coherent* if

(1) X_λ has infinitesimal character λ; and
(2) for any finite-dimensional $(\mathfrak{g}, K)$-module F, and for any $\lambda \in \lambda_0 + \Lambda$,

$$(4.1) \qquad\qquad X_\lambda \otimes F = \sum_{\mu \in \Delta(F)} X_{\lambda + \mu},$$

where $\Delta(F)$ denotes the multiset of all weights of F.

See [V2], Definition 7.2.5. The reason that we may use coherent families based on the compact Cartan T, rather than the maximally split Cartan used in [V2], is our assumption that G is connected.

A virtual $(\mathfrak{g}, K)$-module X with regular infinitesimal character $\lambda_0 \in \mathfrak{h}_c^\star$ can be placed in a unique coherent family as above (see Theorem 7.2.7 in [V2], and the references therein; this is equivalent to (1.1b)). Using this, one can define an action of the integral Weyl group $W(\lambda_0)$ attached to λ_0 on the set $\mathcal{M}(\lambda_0)$ of virtual $(\mathfrak{g}, K)$-modules with infinitesimal character λ_0. Recall that $W(\lambda_0)$ consists of those elements $w \in W_\mathfrak{g}$ for which $\lambda_0 - w\lambda_0$ is a sum of roots. If we write Q for the root lattice, then the condition for w to be in $W(\lambda_0)$ is precisely that w preserves the lattice coset $\lambda_0 + Q$ (see [V2], Section 7.2). Then for $w \in W(\lambda_0)$, we set

$$w \cdot X \stackrel{\text{def.}}{=} X_{w^{-1}(\lambda_0)}.$$

We view $\mathcal{M}(\lambda_0)$ as a lattice (a free $\mathbb{Z}$-module) with basis the (finite) set of irreducible $(\mathfrak{g}, K)$-modules of infinitesimal character λ_0. A decomposition into irreducible components of this $W(\lambda_0)$-representation, known as the *coherent continuation* representation, was obtained by Barbasch and Vogan (see [BV2]). The study of coherent continuation representations is important for deeper understanding of coherent families.

A weight $\lambda \in \lambda_0 + \Lambda$ is called *integrally dominant* if

$$(4.2) \qquad\qquad \langle \alpha^\vee, \lambda \rangle \geq 0 \text{ whenever } \langle \alpha^\vee, \lambda_0 \rangle \in \mathbb{N} \qquad (\alpha \in R_\mathfrak{g}).$$

Recall from the introduction that we write $(\lambda_0 + \Lambda)^+$ for the cone of integrally dominant weights.

The notion of coherent families is closely related with the Jantzen-Zuckerman translation principle. For example, if λ is regular and $\lambda + \nu$ belongs to the same Weyl chamber for integral roots (whose definition is recalled below), then $X_{\lambda+\nu}$ can be obtained from X_λ by a translation functor, i.e., by tensoring with the finite-dimensional module F_ν with extremal weight ν and then taking the component with generalized infinitesimal character $\lambda + \nu$. The following observation is crucial for obtaining the translation principle for Dirac index.

Proposition 4.3. *Suppose X is a virtual $(\mathfrak{g}, K)$-module and F a finite-dimensional $(\mathfrak{g}, K)$-module. Then*
$$I(X \otimes F) = I(X) \otimes F.$$

Proof. By Proposition 3.2 and (3.7),
$$I(X \otimes F) = X \otimes F \otimes S^+ - X \otimes F \otimes S^-,$$
while
$$I(X) \otimes F = (X \otimes S^+ - X \otimes S^-) \otimes F.$$
It is clear that the right hand sides of these expressions are the same. $\qquad\square$

Combining Proposition 4.3 with (4.1), we obtain

Corollary 4.4. *Let X_λ, $\lambda \in \lambda_0 + \Lambda$, be a coherent family of virtual $(\mathfrak{g}, K)$-modules and let F be a finite-dimensional $(\mathfrak{g}, K)$-module. Then*

$$(4.5) \qquad I(X_\lambda) \otimes F = \sum_{\mu \in \Delta(F)} I(X_{\lambda+\mu}).$$

$\qquad\square$

This says that the family $\{I(X_\lambda)\}_{\lambda \in \lambda_0 + \Lambda}$ of virtual $\widetilde{K}$-modules has some coherence properties, but it is not a coherent family for $\widetilde{K}$, as $I(X_\lambda)$ does not have $\mathfrak{k}$-infinitesimal character λ. Also, the identity (4.5) is valid only for a $(\mathfrak{g}, K)$-module F, and not for an arbitrary $\widetilde{K}$-module F.

Using standard reasoning, as in [V2], Section 7.2, we can now analyze the relationship between Dirac index and translation functors. We first define some virtual representations of $\widetilde{K}$. Our choice of positive roots $R_{\mathfrak{k}}^+$ for T in K defines a Weyl denominator function

$$(4.6a) \qquad d_{\mathfrak{k}}(\exp(y)) = \prod_{\alpha \in R_{\mathfrak{k}}^+} (e^{\alpha(y)/2} - e^{-\alpha(y)/2})$$

on an appropriate cover of T. For $\gamma \in \Lambda + \rho_{\mathfrak{g}}$, the Weyl numerator

$$N_\gamma = \sum_{w \in W_{\mathfrak{k}}} \mathrm{sgn}(w) e^{w\gamma}$$

is a function on another double cover of T. According to Weyl's character formula, the quotient

$$(4.6b) \qquad \mathrm{ch}_{\mathfrak{k},\gamma} = N_\gamma / d_{\mathfrak{k}}$$

extends to a class function on all of $\widetilde{K}$. Precisely, $\mathrm{ch}_{\mathfrak{k},\gamma}$ is the character of a virtual genuine representation $\widetilde{E}_\gamma$ of $\widetilde{K}$:

$$(4.6c) \qquad \widetilde{E}_\gamma = \begin{cases} \mathrm{sgn}(x) \, (\text{irr. of highest weight } x\gamma - \rho_{\mathfrak{k}}) & x\gamma \text{ is dom. reg. for } R_{\mathfrak{k}}^+ \\ 0 & \gamma \text{ is singular for } R_{\mathfrak{k}} \end{cases}$$

It is convenient to extend this definition to all of $\mathfrak{t}^*$ by

$$(4.6\mathrm{d}) \qquad\qquad \widetilde{E}_\lambda = 0 \qquad (\lambda \notin \Lambda + \rho_{\mathfrak{g}}).$$

With this definition, the Huang-Pandžić infinitesimal character result clearly guarantees what we wrote in (1.2b):

$$I(X_{\lambda_0}) = \sum_{w \in W_{\mathfrak{g}}} a_w \widetilde{E}_{w\lambda_0}.$$

We could restrict the sum to those w for which $w\lambda_0$ is dominant for $R_{\mathfrak{k}}^+$, and get a unique formula in which a_w is the multiplicity of the $\widetilde{K}$ representation of highest weight $w\lambda_0 - \rho_{\mathfrak{k}}$ in $I(X_{\lambda_0})$. But for the proof of the next theorem, it is more convenient to allow a more general expression.

Theorem 4.7. *Suppose $\lambda_0 \in \mathfrak{t}^*$ is regular for $\mathfrak{g}$. Let X_λ, $\lambda \in \lambda_0 + \Lambda$, be a coherent family of virtual $(\mathfrak{g}, K)$-modules based on $\lambda_0 + \Lambda$. By Theorem 2.2, we can write*

$$(4.8) \qquad\qquad I(X_{\lambda_0}) = \sum_{w \in W_{\mathfrak{g}}} a_w \widetilde{E}_{w\lambda_0},$$

where $\widetilde{E}$ denotes the family of finite-dimensional virtual $\widetilde{K}$-modules defined in (4.6), and a_w are integers.

Then for any $\nu \in \Lambda$,

$$(4.9) \qquad\qquad I(X_{\lambda_0+\nu}) = \sum_{w \in W_{\mathfrak{g}}} a_w \widetilde{E}_{w(\lambda_0+\nu)},$$

with the same coefficients a_w.

Proof. We proceed in three steps.

Step 1: suppose both λ_0 and $\lambda_0 + \nu$ belong to the same integral Weyl chamber, which we can assume to be the dominant one. Let F_ν be the finite-dimensional $(\mathfrak{g}, K)$-module with extremal weight ν. Let us take the components of (4.5), written for $\lambda = \lambda_0$, with $\mathfrak{k}$-infinitesimal characters which are $W_{\mathfrak{g}}$-conjugate to $\lambda_0 + \nu$. By Theorem 2.2, any summand $I(X_{\lambda_0+\mu})$ of the RHS of (4.5) is a combination of virtual modules with $\mathfrak{k}$-infinitesimal characters which are $W_{\mathfrak{g}}$-conjugate to $\lambda_0 + \mu$. By [V2], Lemma 7.2.18 (b), $\lambda_0 + \mu$ can be $W_{\mathfrak{g}}$-conjugate to $\lambda_0 + \nu$ only if $\mu = \nu$. Thus we are picking exactly the summand $I(X_{\lambda_0+\nu})$ of the RHS of (4.5).

We now determine the components of the LHS of (4.5) with $\mathfrak{k}$-infinitesimal characters which are $W_{\mathfrak{g}}$-conjugate to $\lambda_0 + \nu$. Since $\widetilde{E}$ is a coherent family for $\tilde{K}$, and F_ν can be viewed as a finite-dimensional $\tilde{K}$-module, one has

$$\widetilde{E}_{w\lambda_0} \otimes F_\nu = \sum_{\mu \in \Delta(F_\nu)} \widetilde{E}_{w\lambda_0+\mu}.$$

The $\mathfrak{k}$-infinitesimal character of $\widetilde{E}_{w\lambda_0+\mu}$ is $w\lambda_0 + \mu$, so the components we are looking for must satisfy $w\lambda_0 + \mu = u(\lambda_0 + \nu)$, or equivalently

$$\lambda_0 + w^{-1}\mu = w^{-1}u(\lambda_0 + \nu),$$

for some $u \in W_{\mathfrak{g}}$. Using [V2], Lemma 7.2.18 (b) again, we see that $w^{-1}u$ must fix $\lambda_0 + \nu$, and $w^{-1}\mu$ must be equal to ν. So $\mu = w\nu$, and the component $\widetilde{E}_{w\lambda_0+\mu}$ is in fact $\widetilde{E}_{w(\lambda_0+\nu)}$. So (4.9) holds in this case.

Step 2: suppose that λ_0 and $\lambda_0 + \nu$ lie in two neighbouring chambers, with a common wall defined by a root α, and such that $\lambda_0 + \nu = s_\alpha(\lambda_0)$. Assume

further that for any weight μ of F_ν, $\lambda_0 + \mu$ belongs to one of the two chambers. Geometrically this means that λ_0 is close to the wall defined by α and sufficiently far from all other walls and from the origin. We tensor (4.8) with F_ν. By (4.5) and the fact that $\widetilde{E}$ is a coherent family for $\tilde{K}$, we get

$$\sum_{\mu \in \Delta(F_\nu)} I(X_{\lambda_0+\mu}) = \sum_{w \in W_{\mathfrak{g}}} a_w \sum_{\mu \in \Delta(F_\nu)} \widetilde{E}_{w(\lambda_0+\mu)}.$$

By our assumptions, the only $\lambda_0 + \mu$ conjugate to $\lambda_0 + \nu$ via $W_{\mathfrak{g}}$ are $\lambda_0 + \nu$ and λ_0. Picking the corresponding parts from the above equation, we get

$$I(X_{\lambda_0+\nu}) + cI(X_{\lambda_0}) = \sum_{w \in W_{\mathfrak{g}}} a_w \big(c\widetilde{E}_{w\lambda_0} + \widetilde{E}_{w(\lambda_0+\nu)} \big)$$

where c is the multiplicity of the zero weight of F_ν. This implies (4.9), so the theorem is proved in this case.

Step 3: to get from an arbitrary regular λ_0 to an arbitrary $\lambda_0 + \nu$, we first apply Step 1 to get from λ_0 to all elements of $\lambda_0 + \Lambda$ in the same (closed) chamber. Then we apply Step 2 to pass to an element of a neighbouring chamber, then Step 1 again to get to all elements of that chamber, and so on. $\square$

Corollary 4.10. *In the setting of Theorem 4.7, assume that both λ_0 and $\lambda_0 + \nu$ are regular for $\mathfrak{g}$. Assume also that $I(X_{\lambda_0}) \neq 0$, i.e., at least one of the coefficients a_w in (4.8) is nonzero. Then $I(X_{\lambda_0+\nu}) \neq 0$.*

Proof. This follows immediately from Theorem 4.7 and the fact that $\widetilde{E}_{w(\lambda_0+\nu)}$ can not be zero, since $w(\lambda_0 + \nu)$ is regular for $\mathfrak{g}$ and hence also for $\mathfrak{k}$. $\square$

5. Index polynomial and coherent continuation representation

As in the previous section, let $\lambda_0 \in \mathfrak{t}^\star$ be regular. For each $X \in \mathcal{M}(\lambda_0)$, there is a unique coherent family $\{X_\lambda \mid \lambda \in \lambda_0 + \Lambda\}$ such that $X_{\lambda_0} = X$. Define a function $Q_X: \lambda_0 + \Lambda \to \mathbb{Z}$ by setting

$$(5.1) \qquad\qquad Q_X(\lambda) = \dim I(X_\lambda) \quad (\lambda \in \lambda_0 + \Lambda).$$

Notice that Q_X depends on both X *and* on the choice of representative λ_0 for the infinitesimal character of X; replacing λ_0 by $w_1\lambda_0$ translates Q_X by w_1. By Theorem 4.7 and the Weyl dimension formula for $\mathfrak{k}$, Q_X is a polynomial function in λ. (Note that taking dimension is additive with respect to short exact sequences of finite-dimensional modules, so it makes sense for virtual finite-dimensional modules.) We call the function Q_X the *index polynomial* associated with X (or $\{X_\lambda\}$).

Recall that a polynomial on $\mathfrak{t}^*$ is called $W_{\mathfrak{g}}$-harmonic, if it is annihilated by any $W_{\mathfrak{g}}$-invariant constant coefficient differential operator on $\mathfrak{t}^*$ without constant term (see [V1], Lemma 4.3.)

Proposition 5.2. *For any $(\mathfrak{g}, K)$-module X as above, the index polynomial Q_X is $W_{\mathfrak{g}}$-harmonic. If $Q_X \neq 0$, then it is homogeneous of degree equal to the number of positive roots for $\mathfrak{k}$; more precisely, it belongs to the irreducible representation of $W_{\mathfrak{g}}$ generated by the Weyl dimension formula for $\mathfrak{k}$.*

Proof. The last statement follows from (4.9); the rest of the proposition is an immediate consequence. $\square$

Recall the natural representation of $W(\lambda_0)$ (or indeed of all $W_{\mathfrak{g}}$) on the vector space $S(\mathfrak{t})$ of polynomial functions on $\mathfrak{t}^*$,

$$(w \cdot P)(\lambda) = P(w^{-1}\lambda).$$

The (irreducible) representation of $W(\lambda_0)$ generated by the dimension formula for $\mathfrak{k}$ is called the *index polynomial representation*.

Proposition 5.3. *The map*

$$\mathcal{M}(\lambda_0) \to S(\mathfrak{t}), \qquad X \mapsto Q_X$$

intertwines the coherent continuation representation of $W(\lambda_0)$ with the action on polynomials. In particular, if $Q_X \neq 0$, then the coherent continuation representation generated by X must contain a copy of the index polynomial representation.

Proof. Let $\{X_\lambda\}$ be the coherent family corresponding to X. Then for a fixed $w \in W(\lambda_0)$, the coherent family corresponding to $w \cdot X$ is $\lambda_0 + \nu \mapsto X_{w^{-1}(\lambda_0+\nu)}$ (see [V2], Lemma 7.2.29 and its proof). It follows that

$$
\begin{aligned}
(w \cdot Q_X)(\lambda) &= Q_X(w^{-1} \cdot \lambda) \\
&= \dim I(X_{w^{-1}\lambda}) \\
&= Q_{w \cdot X}(\lambda),
\end{aligned}
$$

i.e., the map $X \mapsto Q_X$ is $W(\lambda_0)$-equivariant. The rest of the proposition is now clear. $\square$

Example 5.4. Let F be a finite-dimensional $(\mathfrak{g}, K)$-module. The corresponding coherent family is $\{F_\lambda\}$ from [V2], Example 7.2.12. In particular, every F_λ is finite-dimensional up to sign, or 0. By Proposition 3.2 and (3.7), for any F_λ,

$$\dim I(F_\lambda) = \dim(F_\lambda \otimes S^+ - F_\lambda \otimes S^-) = \dim F_\lambda(\dim S^+ - \dim S^-) = 0,$$

since S^+ and S^- have the same dimension (as long as $\mathfrak{p} \neq 0$). It follows that

$$Q_F(\lambda) = 0.$$

(Note that the index itself is a nonzero virtual module, but its dimension is zero. This may be a little surprising at first, but it is quite possible for virtual modules.) This means that in this case Proposition 5.3 gives no information about the coherent continuation representation (which is in this case a copy of the sign representation of $W_{\mathfrak{g}}$ spanned by F).

Example 5.5. Let $G = SL(2, \mathbb{R})$, so that weights correspond to integers. Let $\lambda_0 = n_0$ be a positive integer. There are four irreducible $(\mathfrak{g}, K)$-modules with infinitesimal character n_0: the finite-dimensional module F, the holomorphic discrete series D^+ of lowest weight $n_0 + 1$, the antiholomorphic discrete series D^- of highest weight $-n_0 - 1$, and the irreducible principal series representation P.

The coherent continuation representation at n_0 consists of three trivial representations, spanned by $F + D^+ + D^-$, $D^+ - D^-$ and P, and one sign representation, spanned by F. The index polynomial representation is the trivial representation spanned by the constant polynomials. The map $X \mapsto Q_X$ sends P, F and $F + D^+ + D^-$ to zero, and $D^+ - D^-$ to the constant polynomial 2.

The conclusion of Example 5.4 about the index polynomials of finite-dimensional representations being zero can be generalized as follows.

Proposition 5.6. *Let X be a $(\mathfrak{g}, K)$-module as above, with Gelfand-Kirillov dimension $\mathrm{Dim}(X)$. If $\mathrm{Dim}(X) < \sharp R_{\mathfrak{g}}^+ - \sharp R_{\mathfrak{k}}^+$, then $Q_X = 0$.*

Proof. We need to recall the setting of [BV3], Section 2, in particular their Theorem 2.6.(b) (taken from [J2]). Namely, to any irreducible representation σ of $W_{\mathfrak{g}}$ one can associate its degree, i.e., the minimal integer d such that σ occurs in the $W_{\mathfrak{g}}$-representation $S^d(\mathfrak{t})$. Theorem 2.6.(b) of [BV3] says that the degree of any σ occurring in the coherent continuation representation attached to X must be at least equal to $\sharp R_{\mathfrak{g}}^+ - \mathrm{Dim}(X)$. By assumption, the degree of Q_X, $\sharp R_{\mathfrak{k}}^+$, is smaller than $\sharp R_{\mathfrak{g}}^+ - \mathrm{Dim}(X)$. On the other hand, by Proposition 5.3 the index polynomial representation has to occur in the coherent continuation representation. It follows that Q_X must be zero. $\square$

Example 5.7. Wallach modules for $Sp(2n, \mathbb{R})$, $SO^*(2n)$ and $U(p, q)$, studied in [HPP], all have nonzero index, but their index polynomials are zero. This can also be checked explicitly from the results of [HPP], at least in low-dimensional cases.

The situation here is like in Example 5.4; the nonzero Dirac index has zero dimension. In particular the conclusion $Q_X = 0$ in Proposition 5.6 does not imply that $I(X) = 0$.

We note that in the proof of Proposition 5.6, we are applying the results of [BV3] to $(\mathfrak{g}, K)$-modules, although they are stated in [BV3] for highest weight modules. This is indeed possible by results of Casian [C1]. We explain this in more detail. Let $\mathcal{B}$ be the flag variety of $\mathfrak{g}$ consisting of all the Borel subalgebras of $\mathfrak{g}$. For a point $x \in \mathcal{B}$, write $\mathfrak{b}_x = \mathfrak{h}_x + \mathfrak{n}_x$ for the corresponding Borel subalgebra, with nilradical $\mathfrak{n}_x$, and Cartan subalgebra $\mathfrak{h}_x$.

Define a functor $\Gamma_{\mathfrak{b}_x}$ from the category of $\mathfrak{g}$-modules into the category of $\mathfrak{g}$-modules which are $\mathfrak{b}_x$-locally finite, by

$$\Gamma_{\mathfrak{b}_x} M = \big\{ \mathfrak{b}_x - \text{locally finite vectors in } M \big\}.$$

Write $\Gamma_{\mathfrak{b}_x}^q$, $q \geq 0$, for its right derived functors. Instead of considering the various $\mathfrak{b}_x$, $x \in \mathcal{B}$, it is convenient to fix a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ of $\mathfrak{g}$ and twist the module M. By a twist of M we mean that if π is the $\mathfrak{g}$-action on M and σ is an automorphism of $\mathfrak{g}$ then the twist of π by σ is the $\mathfrak{g}$-action $\pi \circ \sigma$ on M. Then Casian's generalized Jacquet functors $J_{\mathfrak{b}_x}^q$ are functors from the category of $\mathfrak{g}$-modules into the category of $\mathfrak{g}$-modules which are $\mathfrak{b}$-locally finite, given by

$$J_{\mathfrak{b}_x}^q M = \Big\{ \Gamma_{\mathfrak{b}_x}^q \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}) \Big\}^0$$

where the superscript '0' means that the $\mathfrak{g}$-action is twisted by some inner automorphism of $\mathfrak{g}$, to make it $\mathfrak{b}$-locally finite instead of $\mathfrak{b}_x$-locally finite. In case $\mathfrak{b}_x$ is the Borel subalgebra corresponding to an Iwasawa decomposition of G, $J_{\mathfrak{b}_x}^0$ is the usual Jacquet functor of [BB], while the $J_{\mathfrak{b}_x}^q$ vanish for $q > 0$.

The functors $J_{\mathfrak{b}_x}^q$ make sense on the level of virtual $(\mathfrak{g}, K)$-modules and induce an injective map

$$X \mapsto \sum_{x \in \mathcal{B}/K} \sum_q (-1)^q J_{\mathfrak{b}_x}^q X$$

from virtual $(\mathfrak{g}, K)$-modules into virtual $\mathfrak{g}$-modules which are $\mathfrak{b}$-locally finite. Note that the above sum is well defined, since the $J_{\mathfrak{b}_x}^q$ depend only on the K-orbit of x in $\mathcal{B}$.

An important feature of the functors $J^q_{\mathfrak{b}_x}$ is the fact that they satisfy the following identity relating the $\mathfrak{n}_x$-homology of X with the $\mathfrak{n}$-cohomology of the modules $J^q_{\mathfrak{b}_x} X$ (see page 6 in [C1]):

$$\sum_{p,q \geq 0} (-1)^{p+q} \operatorname{tr}_{\mathfrak{h}} H^p(\mathfrak{n}, J^q_{\mathfrak{b}_x} X) = \sum_q (-1)^q \operatorname{tr}_{\mathfrak{h}} H_q(\mathfrak{n}_x, X)^0.$$

Here the superscript '0' is the appropriate twist interchanging $\mathfrak{h}_x$ with $\mathfrak{h}$, and $\operatorname{tr}_{\mathfrak{h}}$ denotes the formal trace of the $\mathfrak{h}$-action. More precisely, if Z is a locally finite $\mathfrak{h}$-module with finite-dimensional weight components Z_μ, $\mu \in \mathfrak{h}^*$, then

$$\operatorname{tr}_{\mathfrak{h}} Z = \sum_{\mu \in \mathfrak{h}^*} \dim Z_\mu \, e^\mu.$$

Using this and Osborne's character formula, the global character of X on an arbitrary θ-stable Cartan subgroup can be read off from the characters of the $J^q_{\mathfrak{b}_x} X$ (see [C1] and [C2]). In particular, we deduce that if τ is an irreducible representation of the Weyl group $W_{\mathfrak{g}}$ occuring in the coherent representation attached to X then τ occurs in the coherent continuation representation attached to $J^q_{\mathfrak{b}_x} X$ for some $q \geq 0$ and some Borel subalgebra $\mathfrak{b}_x$. Moreover, from the definitions, one has $\operatorname{Dim}(X) \geq \operatorname{Dim}(J^q_{\mathfrak{b}_x} X)$. Applying the results in [BV3] to the module $J^q_{\mathfrak{b}_x} X$, we deduce that:

$$d^o(\tau) \geq \sharp R^+_{\mathfrak{g}} - \operatorname{Dim}(J^q_{\mathfrak{b}_x} X) \geq \sharp R^+_{\mathfrak{g}} - \operatorname{Dim}(X),$$

where $d^o(\tau)$ is the degree of τ.

6. Index polynomials and Goldie rank polynomials

Recall that H_s denotes a maximally split Cartan subgroup of G with complexified Lie algebra $\mathfrak{h}_s$. As in Section 4, we let X be a module with regular infinitesimal character $\lambda_0 \in \mathfrak{h}^*_s$, and $\{X_\lambda\}_{\lambda \in \lambda_0 + \Lambda}$ the corresponding coherent family on H_s. With notation from (1.1d) and (1.1e), Joseph proved that the mapping

$$\lambda \mapsto P_X(\lambda) = \operatorname{rk}(U(\mathfrak{g})/\operatorname{Ann}(X_\lambda)),$$

extends to a $W_{\mathfrak{g}}$-harmonic polynomial on $\mathfrak{h}^*_s$, homogeneous of degree $\sharp R^+_{\mathfrak{g}} - \operatorname{Dim}(X)$, where $\operatorname{Dim}(X)$ is the Gelfand-Kirillov dimension of X (see [J1], [J2] and [J3]). He also found relations between the Goldie rank polynomial P_X and Springer representations; and (less directly) Kazhdan-Lusztig polynomials (see [J4] and [J5]).

Recall from (1.1h) King's analytic interpretation of the Goldie rank polynomial: that for $x \in \mathfrak{h}_{s,0}$ regular, the expression

$$(6.1) \qquad \lim_{t \to 0+} t^d \operatorname{ch}_{\mathfrak{g}}(X_\lambda)(\exp tx)$$

is zero if d is an integer bigger than $\operatorname{Dim}(X)$; and if $d = \operatorname{Dim}(X)$, it is (for generic x) a nonzero polynomial $C_{X,x}$ in λ called the character polynomial. Up to a constant, this character polynomial is equal to the Goldie rank polynomial attached to X. In other words, the Goldie rank polynomial expresses the dependence on λ of the leading term in the Taylor expansion of the numerator of the character of X_λ on the maximally split Cartan H_s. For more details, see [K1] and also [J2], Corollary 3.6.

The next theorem shows that the index polynomial we studied in Section 5 is the exact analogue of King's character polynomial, but attached to the character on the compact Cartan subgroup instead of the maximally split Cartan subgroup.

Theorem 6.2. *Let X be a $(\mathfrak{g}, K)$-module with regular infinitesimal character and let X_λ be the corresponding coherent family on the compact Cartan subgroup. Write $r_\mathfrak{g}$ (resp. $r_\mathfrak{k}$) for the number of positive $\mathfrak{t}$-roots for $\mathfrak{g}$ (resp. $\mathfrak{k}$). Suppose $y \in \mathfrak{t}_0$ is any regular element. Then the limit*

$$(6.3) \qquad \lim_{t \to 0+} t^d \, ch_\mathfrak{g}(X_\lambda)(\exp ty)$$

is zero if d is an integer bigger than $r_\mathfrak{g} - r_\mathfrak{k}$. If $d = r_\mathfrak{g} - r_\mathfrak{k}$, then the limit (6.3) is equal to

$$\frac{\prod_{\alpha \in R_\mathfrak{k}^+} \alpha(y)}{\prod_{\alpha \in R_\mathfrak{g}^+} \alpha(y)} Q_X(\lambda),$$

where Q_X is the index polynomial attached to X as in (5.1). In other words, the index polynomial, up to an explicit constant, expresses the dependence on λ of the (possibly zero) leading term in the Taylor expansion of the numerator of the character of X_λ on the compact Cartan T.

Proof. The restriction to K of any G-representation has a well defined distribution character, known as the K-character. The restriction of this K-character to the set of elliptic G-regular elements in K is a function, equal to the function giving the G-character (see [HC], and also [AS], (4.4) and the appendix). Therefore Proposition 3.2 and (3.7) imply

$$\mathrm{ch}_\mathfrak{g}(X_\lambda)(\exp ty) = \frac{\mathrm{ch}_\mathfrak{k}(I(X_\lambda))}{\mathrm{ch}_\mathfrak{k}(S^+ - S^-)}(\exp ty).$$

Also, it is clear that

$$\lim_{t \to 0+} \mathrm{ch}_\mathfrak{k}(I(X_\lambda))(\exp ty) = \mathrm{ch}_\mathfrak{k}(I(X_\lambda))(e) = \dim I(X_\lambda) = Q_X(\lambda).$$

Therefore the limit (6.3) is equal to

$$\lim_{t \to 0+} t^d \frac{\mathrm{ch}_\mathfrak{k}(I(X_\lambda))(\exp ty)}{\mathrm{ch}_\mathfrak{k}(S^+ - S^-)(\exp ty)} = Q_X(\lambda) \lim_{t \to 0+} \frac{t^d}{\mathrm{ch}_\mathfrak{k}(S^+ - S^-)(\exp ty)}.$$

On the other hand, it is well known and easy to check that

$$\mathrm{ch}_\mathfrak{k}(S^+ - S^-) = \frac{d_\mathfrak{g}}{d_\mathfrak{k}},$$

where $d_\mathfrak{g}$ (resp. $d_\mathfrak{k}$) denotes the Weyl denominator for $\mathfrak{g}$ (resp. $\mathfrak{k}$). It is immediate from the product formula (4.6a) that we know that

$$d_\mathfrak{g}(\exp ty) = t^{r_\mathfrak{g}} \prod_{\alpha \in R_\mathfrak{g}^+} \alpha(y) + \text{ higher order terms in } t$$

and similarly

$$d_\mathfrak{k}(\exp ty) = t^{r_\mathfrak{k}} \prod_{\alpha \in R_\mathfrak{k}^+} \alpha(y) + \text{ higher order terms in } t.$$

So we see that

$$\lim_{t \to 0+} \frac{t^d}{\mathrm{ch}_\mathfrak{k}(S^+ - S^-)(\exp ty)} = \lim_{t \to 0+} t^{d - r_\mathfrak{g} + r_\mathfrak{k}} \frac{\prod_{\alpha \in R_\mathfrak{k}^+} \alpha(y)}{\prod_{\alpha \in R_\mathfrak{g}^+} \alpha(y)}.$$

The theorem follows.

$\square$

We are now going to consider some examples (of discrete series representations) where we compare the index polynomial and the Goldie rank polynomial. To do so, we identify the compact Cartan subalgebra with the maximally split one using a Cayley transform.

Recall that if X is a discrete series representation with Harish-Chandra parameter λ, then

$$I(X) = \pm H_D(X) = \pm E_\lambda,$$

where E_λ denotes the $\widetilde{K}$-type with infinitesimal character λ. (The sign depends on the relation between the positive system defined by λ and the fixed one used in Section 3 to define the index. See [HP1], Proposition 5.4, or [HP2], Corollary 7.4.5.) The index polynomial Q_X is then given by the Weyl dimension formula for this $\widetilde{K}$-type, i.e., by

$$(6.4) \qquad\qquad Q_X(\lambda) = \prod_{\alpha \in R_{\mathfrak{k}}^+} \frac{\langle \lambda, \alpha \rangle}{\langle \rho_{\mathfrak{k}}, \alpha \rangle}.$$

Comparing this with [K2], Proposition 3.1, we get:

Proposition 6.5. *Suppose G is linear, semisimple and of Hermitian type. Let X be the $(\mathfrak{g}, K)$-module of a holomorphic discrete series representation. Then the index polynomial Q_X coincides with the Goldie rank polynomial P_X up to a scalar multiple.*

Of course, Q_X is not always equal to P_X, since the degrees of these two polynomials are different in most cases.

In the following we consider the example of discrete series representations for $SU(n, 1)$. The choice is dictated by the existence of explicit formulas for the Goldie rank polynomials computed in [K2].

The discrete series representations for $SU(n, 1)$ with a fixed infinitesimal character can be parametrized by integers $i \in [0, n]$. To see how this works, we introduce some notation. First, we take for K the group $S(U(n) \times U(1)) \cong U(n)$. The compact Cartan subalgebra $\mathfrak{t}$ consists of diagonal matrices, and we identify it with $\mathbb{C}^{n+1}$ in the usual way. We make the usual choice for the dominant $\mathfrak{k}$-chamber C: it consists of those $\lambda \in \mathbb{C}^{n+1}$ for which

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Then C is the union of $n+1$ $\mathfrak{g}$-chambers $D_0, \ldots, D_n$, where D_0 consists of $\lambda \in C$ such that $\lambda_{n+1} \leq \lambda_n$, D_n consists of $\lambda \in C$ such that $\lambda_{n+1} \geq \lambda_1$, and for $1 \leq i \leq n-1$,

$$D_i = \{\lambda \in C \,\big|\, \lambda_{n-i} \geq \lambda_{n+1} \geq \lambda_{n-i+1}\}.$$

Now for $i \in [0, n]$, and for $\lambda \in D_i$, which is regular for $\mathfrak{g}$ and analytically integral for K, we denote by $X_\lambda(i)$ the discrete series representation with Harish-Chandra parameter λ. We use the same notation for the corresponding $(\mathfrak{g}, K)$-module. For $i = 0$, $X_\lambda(i)$ is holomorphic and this case is settled by Proposition 6.5; the result is that both the index polynomial and the Goldie rank polynomial are proportional to the Vandermonde determinant

$$(6.6) \qquad\qquad V(\lambda_1, \ldots, \lambda_n) = \prod_{1 \leq p < q \leq n} (\lambda_p - \lambda_q).$$

The case $i = n$ of antiholomorphic discrete series representations is analogous. For $1 \le i \le n - 1$, the index polynomial of $X_\lambda(i)$ is still given by (6.6). On the other hand, the character polynomial is up to a constant multiple given by the formula (6.5) of [K2], as the sum of two determinants. We note that King's expression can be simplified and that the character polynomial of $X_\lambda(i)$ is in fact equal to

$$(6.7) \qquad \begin{vmatrix} \lambda_1^{n-2} & \cdots & \lambda_{n-i}^{n-2} & \lambda_{n-i+1}^{n-2} & \cdots & \lambda_n^{n-2} \\ \lambda_1^{n-3} & \cdots & \lambda_{n-i}^{n-3} & \lambda_{n-i+1}^{n-3} & \cdots & \lambda_n^{n-3} \\ \vdots & & \vdots & \vdots & & \vdots \\ \lambda_1 & \cdots & \lambda_{n-i} & \lambda_{n-i+1} & \cdots & \lambda_n \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{vmatrix}$$

up to a constant multiple.

For $i = 1$, (6.7) reduces to the Vandermonde determinant $V(\lambda_1, \ldots, \lambda_{n-1})$. Similarly, for $i = n-1$, we get $V(\lambda_2, \ldots, \lambda_n)$. In these cases, the Goldie rank polynomial divides the index polynomial.

For $2 \le i \le n-2$, the Goldie rank polynomial is more complicated. For example, if $n = 4$ and $i = 2$, (6.7) becomes

$$-(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4),$$

and this does not divide the index polynomial. For $n = 5$ and $i = 2$, (6.7) becomes

$$- (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5)$$
$$(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 - \lambda_1 \lambda_4 - \lambda_1 \lambda_5 + \lambda_2 \lambda_3 - \lambda_2 \lambda_4 - \lambda_2 \lambda_5 - \lambda_3 \lambda_4 - \lambda_3 \lambda_5 + \lambda_4^2 + \lambda_4 \lambda_5 + \lambda_5^2),$$

and one can check that the quadratic factor is irreducible.

More generally, for any $n \ge 4$ and $2 \le i \le n - 2$, the Goldie rank polynomial (6.7) is divisible by $(\lambda_p - \lambda_q)$ whenever $1 \le p < q \le n - i$ or $n - i + 1 \le p < q \le n$. This is proved by subtracting the qth column from the pth column. On the other hand, if $1 \le p \le n - i < q \le n$, we claim that (6.7) is not divisible by $(\lambda_p - \lambda_q)$. Indeed, we can substitute $\lambda_q = \lambda_p$ into (6.7) and subtract the qth column from the pth column. After this we develop the determinant with respect to the pth column. The resulting sum of two determinants is equal to the Vandermonde determinant $V(\lambda_1, \ldots, \lambda_{p-1}, \lambda_{p+1}, \ldots, \lambda_n)$, and this is not identically zero.

This proves that for $X = X_\lambda(i)$ the greatest common divisor of P_X and Q_X is

$$(6.8) \qquad \prod_{1 \le p < q \le n-i} (\lambda_p - \lambda_q) \prod_{n-i+1 \le r < s \le n} (\lambda_r - \lambda_s).$$

Comparing with the simple roots Ψ_i corresponding to the chamber D_i described on p. 294 of [K2], we see that the linear factors of (6.8) correspond to roots generated by the compact part of Ψ_i. On the other hand, the set of compact roots in Ψ_i is equal to the τ-invariant of $X_\lambda(i)$, as proved in [HS], Proposition 3.6 (see also [K1], Remark 4.5). Recall that the τ-invariant of a $(\mathfrak{g}, K)$-module X consists of the simple roots α such that the translate of X to the wall defined by α is 0; see [V1], Section 4.

In particular, we have checked a special case of the following proposition.

Proposition 6.9. *Assume that G is a real reductive Lie group in the Harish-Chandra class and that G and K have equal rank. Let X be the discrete series representation of G with Harish-Chandra parameter λ. Then the index polynomial Q_X and the Goldie rank polynomial P_X are both divisible by the product of linear factors corresponding to the roots generated by the τ-invariant of X.*

Proof. The τ-invariant of X is still given as above, as the compact part of the simple roots corresponding to λ. In particular, the roots generated by the τ-invariant are all compact, and the corresponding factors divide Q_X, which is given by (6.4).

On the other hand, by [V1], Proposition 4.9, the Goldie rank polynomial is always divisible by the factors corresponding to roots generated by the τ-invariant. We note that the result in [V1] is about the Bernstein degree polynomial, which is up to a constant factor equal to the Goldie rank polynomial by [J2], Theorem 5.7.   □

Note that for $G = SU(n, 1)$, the result we obtained is stronger than the conclusion of Proposition 6.9. Namely, we proved that the product of linear factors corresponding to the roots generated by the τ-invariant of X is in fact the greatest common divisor R of P_X and Q_X. We note that it is easy to calculate the degrees of all the polynomials involved. Namely, if $2 \le i \le n-2$, the degree of R is $\binom{i}{2} + \binom{n-i}{2}$. Since $\text{Dim}(X) = 2n - 1$ (see [K2]), and $\sharp R_{\mathfrak{g}}^+ = \binom{n+1}{2}$, the degree of P_X is $\binom{n-1}{2}$. It follows that the degree of P_X/R is $i(n-i) - (n-1)$. On the other hand, since the degree of Q_X is $\sharp R_{\mathfrak{k}}^+ = \binom{n}{2}$, the degree of Q_X/R is $i(n-i)$.

7. Index polynomials and nilpotent orbits

Assume again that we are in the setting (1.1) of the introduction, so that $Y = Y_{\lambda_0}$ is an irreducible $(\mathfrak{g}, K)$-module. (We use a different letter from the X in the introduction as a reminder that we will soon be imposing some much stronger additional hypotheses on Y.) Recall from (1.1g) the expression

$$\text{(7.1a)} \qquad \text{Ass}(Y_\lambda) = \coprod_{j=1}^{r} m_Y^j(\lambda)\overline{\mathcal{O}^j} \qquad (\lambda \in (\lambda_0 + \Lambda)^+),$$

and the fact that each m_Y^j extends to a polynomial function on $\mathfrak{t}^*$, which is a multiple of the Goldie rank polynomial:

$$\text{(7.1b)} \qquad m_Y^j = a_Y^j P_Y,$$

with a_Y^j a nonnegative rational number depending on Y. On the other hand, the Weyl dimension formula for $\mathfrak{k}$ defines a polynomial on the dual of the compact Cartan subalgebra $\mathfrak{t}^*$ in $\mathfrak{g}$, with degree equal to the cardinality $\sharp R_{\mathfrak{k}}^+$ of positive roots for $\mathfrak{k}$. Write σ_K for the representation of the Weyl group $W_{\mathfrak{g}}$ generated by this polynomial. Suppose that σ_K is a Springer representation, i.e., it is associated with a nilpotent $G_{\mathbb{C}}$-orbit $\mathcal{O}_K$:

$$\text{(7.1c)} \qquad \sigma_K \overset{\text{Springer}}{\longleftrightarrow} \mathcal{O}_K \subset \mathfrak{g}^*.$$

Here $G_{\mathbb{C}}$ denotes a connected complex reductive algebraic group having Lie algebra $\mathfrak{g}$. Assume also that there is a Harish-Chandra module Y of regular infinitesimal character λ_0 such that

$$\text{(7.1d)} \qquad \mathcal{V}(\text{gr}(\text{Ann}(Y))) = \overline{\mathcal{O}_K}.$$

Recall from the discussion before (1.1f) that $\mathcal{V}(\mathrm{gr}(\mathrm{Ann}(Y)))$ is the variety associated with the graded ideal of $\mathrm{Ann}(Y)$ in the symmetric algebra $S(\mathfrak{g})$.

Our assumptions force the degree of the Goldie rank polynomial P_Y attached to Y to be

$$\sharp R_{\mathfrak{g}}^+ - \mathrm{Dim}(Y) = \sharp R_{\mathfrak{g}}^+ - \frac{1}{2}\dim \mathcal{O}_K = \frac{1}{2}(\dim \mathcal{N} - \dim \mathcal{O}_K) = \sharp R_{\mathfrak{k}}^+,$$

where $\mathcal{N}$ denotes the cone of nilpotent elements in $\mathfrak{g}^\star$. In other words, the Goldie rank polynomial P_Y has the same degree as the index polynomial Q_Y.

We conjecture that for representations attached to $\mathcal{O}_K$, the index polynomial admits an expression analogous to (1.1i).

Conjecture 7.2. *Assume that the $W_{\mathfrak{g}}$-representation σ_K generated by the Weyl dimension formula for $\mathfrak{k}$ corresponds to a nilpotent $G_{\mathbb{C}}$-orbit $\mathcal{O}_K$ via the Springer correspondence. Then for each $K_{\mathbb{C}}$-orbit $\mathcal{O}_K^j$ on $\mathcal{O}_K \cap (\mathfrak{g}/\mathfrak{k})^\star$, there exists an integer c_j such that for any Harish-Chandra module Y for G satisfying $\mathcal{V}(\mathrm{gr}(\mathrm{Ann}(Y))) \subset \overline{\mathcal{O}_K}$, we have*

$$Q_Y = \sum_j c_j m_Y^j.$$

Here Q_Y is the index polynomial attached to Y as in Section 5.

Example 7.3. Consider $G = SL(2,\mathbb{R})$ with $K = SO(2)$. Then σ_K is the trivial representation of $W_{\mathfrak{g}} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{O}_K$ is the principal nilpotent orbit. $\mathcal{O}_K$ has two real forms $\mathcal{O}_K^1$ and $\mathcal{O}_K^2$. One checks from our computations in Example 5.5 and from the table below that $c_1 = 1$ and $c_2 = -1$. This shows that the conjecture is true in the case when $G = SL(2,\mathbb{R})$.

Y	$\mathcal{V}(Y)$	Q_Y
finite-dimensional modules	$\{0\}$	0
holomorphic discrete series	$\mathcal{O}_K^1$	1
antiholomorphic discrete series	$\mathcal{O}_K^2$	-1
principal series	$\mathcal{O}_K^1 \cup \mathcal{O}_K^2$	0

Here $\mathcal{V}(Y) \subset \mathcal{V}(\mathrm{gr}(\mathrm{Ann}(Y)))$ is the associated variety of Y.

Example 7.4. Let $n > 1$ and let $G = SU(1,n)$ with $K = U(n)$. Then $\mathcal{O}_K$ is the minimal nilpotent orbit of dimension $2n$. It has two real forms $\mathcal{O}_K^1$ and $\mathcal{O}_K^2$. The holomorphic and antiholomorphic discrete series representations Y_λ^1 and Y_λ^2 all have Gelfand-Kirillov dimension equal to n. By [Ch], Corollary 2.13, the respective associated cycles are equal to

$$\mathrm{Ass}(Y_\lambda^i) = m_{Y^i}^i(\lambda)\mathcal{O}_K^i, \qquad i = 1, 2,$$

with the multiplicity $m_{Y^i}^i(\lambda)$ equal to the dimension of the lowest K-type of Y_λ^i. The index of the holomorphic discrete series representations is the lowest K-type shifted by a one dimensional representation of K with weight $\rho(\mathfrak{p}^-)$, so it has the same dimension as the lowest K-type. The situation for the antiholomorphic discrete series representations is analogous, but there is a minus sign. Hence

$$m_{Y^i}^i(\lambda) = (-1)^{i-1}Q_{Y^i}(\lambda), \qquad i = 1, 2.$$

This already forces the coefficients c_1 and c_2 from Conjecture 7.2 to be 1 and -1 respectively.

Since $\mathcal{O}_K$ is the minimal orbit, it follows that for infinite-dimensional Y,

$$\mathcal{V}(\mathrm{gr}(\mathrm{Ann}(Y))) \subseteq \overline{O_K} \quad \Rightarrow \quad \mathcal{V}(\mathrm{gr}(\mathrm{Ann}(Y))) = \overline{O_K}.$$

If $\mathcal{V}(\mathrm{gr}(\mathrm{Ann}(Y))) = \overline{O_K}$ and Y is irreducible, then $\mathcal{V}(Y)$ must be either $\overline{\mathcal{O}_K^1}$ or $\overline{\mathcal{O}_K^2}$. This follows from minimality of $\mathcal{O}_K$ and from [V3], Theorem 1.3. Namely, the codimension of the boundary of $\mathcal{O}_K^i$ in $\overline{\mathcal{O}_K^i}$ is $n \geq 2$.

On the other hand, by [KO], Lemma 3.5, $\mathcal{V}(Y) = \overline{\mathcal{O}_K^i}$ implies Y is holomorphic if $i = 1$, respectively antiholomorphic if $i = 2$. Let us assume $i = 1$; the other case is analogous.

It is possible to write Y as a $\mathbb{Z}$-linear combination of generalized Verma modules; see for example [HPZ], Proposition 3.6. So we see that it is enough to check the conjecture assuming Y is a generalized Verma module. In this case, one easily computes that $I(Y)$ is the lowest K-type of Y shifted by the one dimensional $\widetilde{K}$-module with weight $\rho(\mathfrak{p}^-)$; see [HPZ], Lemma 3.2. So the index polynomial is the dimension of the lowest K-type. By [NOT], Proposition 2.1, this is exactly the same as the multiplicity m_Y^1 of $\overline{\mathcal{O}_K^1}$ in the associated cycle. This proves the conjecture in this case (with $c_1 = 1$).

We finish the paper by mentioning that neither of the two assumptions (7.1c) and (7.1d) above is automatically fulfilled. For example, one can check that (7.1c) is fulfilled if $G = SU(p, q)$, if $G = SO_e(2p, 2q + 1)$ with $q \geq p - 1$, if $G = Sp(2n, \mathbb{R})$, if $G = SO_e(2p, 2q)$, or if $G = SO^*(2n)$, and that (7.1c) is not fullfilled if $G = SO_e(2p, 2q + 1)$ with $q \leq p - 2$, or if $G = Sp(2p, 2q)$. Furthermore, one can show that both (7.1c) and (7.1d) are fulfilled if there is a discrete series representation of G, such that the integral roots for its Harish-Chandra parameter are precisely the compact roots.

References

[AS]   Atiyah, M. and Schmid, W., *A geometric construction of the discrete series for semisimple Lie groups.* Inv. Math. **42** (1977), 1–62.

[BV1]   Barbasch, D. and Vogan, D., *The local structure of characters.* J. Funct. Anal. **37** (1980), no. 1, 27–55.

[BV2]   Barbasch, D. and Vogan, D., *Weyl group representations and nilpotent orbits.* Representation theory of reductive groups (Park City, Utah, 1982), 21–33, Progr. Math., 40, Birkhäuser, Boston, MA, 1983.

[BV3]   Barbasch, D. and Vogan, D., *Primitive ideals and orbital integrals in complex exceptional groups.* J. Algebra **80** (1983), 350–382.

[BB]   Beilinson, A. and Bernstein, J., *Localisation de $\mathfrak{g}$-modules.* C. R. Acad. Sci. Paris **292** (1981), 15–18.

[Car]   Carter, R. W., *Finite groups of Lie type.* Wiley Classics Library. John Wiley & Sons Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[C1]   Casian, L., *Asymptotics of matrix coefficients, perverse sheaves and Jacquet modules.* Math. Ann. **275** (1986), no. 1, 5–23.

[C2]   Casian, L., *Primitive ideals and representations.* J. Algebra **101** (1986), no. 2, 497–515.

[Ch]   Chang, J.-T., *Characteristic cycles of holomorphic discrete series.* Trans. Amer. Math. Soc. **334** (1992), no. 1, 213-227.

[Chev] Chevalley, C., *The algebraic theory of spinors*, Columbia University Press, New York, 1954.

[CM]   Collingwood, D. H. and McGovern W. M., *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993.

[HC]   Harish-Chandra, *The characters of semisimple Lie groups*, Trans. Amer. Math. Soc. **83** (1956), 98–163.

[atlas] du Cloux, F. and van Leeuwen, M. Software for structure and representations of real reductive groups, version 0.5.5. Available from `http://www.liegroups.org`.

[HS]   Hecht, H. and Schmid, W., *A proof of Blattner's conjecture.* Invent. Math. **31** (1975), 129–154.

[HP1]   Huang, J.-S. and Pandžić, P., *Dirac cohomology, unitary representations and a proof of a conjecture of Vogan.* J. Amer. Math. Soc. **15** (2002), 185–202.

[HP2]   Huang, J.-S. and Pandžić, P., *Dirac Operators in Representation Theory.* Mathematics: Theory and Applications, Birkhäuser, 2006.

[HPP]   Huang, J.-S., Pandžić, P. and Protsak, V., *Dirac cohomology of Wallach representations.* Pacific J. Math. **250** (2011), no. 1, 163–190.

[HPZ]   Huang, J.-S., Pandžić, P. and Zhu, F., *Dirac cohomology, K-characters and branching laws.* Amer. J. Math. **135** (2013), no.5, 1253–1269.

[J1]   Joseph, A., *Goldie rank in the enveloping algebra of a semisimple Lie algebra I.* J. Algebra **65** (1980), 269–283.

[J2]   Joseph, A., *Goldie rank in the enveloping algebra of a semisimple Lie algebra II.* J. Algebra **65** (1980), 284–306.

[J3]   Joseph, A., *Goldie rank in the enveloping algebra of a semisimple Lie algebra III.* J. Algebra **73** (1981), 295–326.

[J4]   Joseph, A., *On the variety of a highest weight module.* J. Algebra **88** (1984), 238–278.

[J5]   Joseph, A., *Orbital varieties, Goldie rank polynomials and unitary highest weight modules.* In Algebraic and analytic methods in representation theory (Soenderborg, 1994), 53–98, Perspect. Math. **17**, Academic Press.

[K1]   King, D., *The character polynomial of the annihilator of an irreducible Harish-Chandra module.* Amer. J. Math **103** (1980), 1195–1240.

[K2]   King, D., *Character polynomials of discrete series representations.* Noncommutative harmonic analysis and Lie groups (Marseille, 1980), 267–302, Lecture Notes in Math., 880, Springer, Berlin-New York, 1981.

[KO]   Kobayashi,T., and Oshima, Y., *Classification of symmetric pairs with discretely decomposable restrictions of* $(\mathfrak{g}, K)$*-modules*, J. Reine Angew. Math., DOI: 10.1515/crelle-2013-0045, July 2013.

[L]   Lusztig, G., *A class of irreducible representations of a Weyl group. Indag. Math* **41** *(1979), 323–335.*

[M]   Macdonald, I. G., *Some irreducible representations of Weyl groups. Bull. London Math. Soc.* **4** *(1972), 148–150.*

[MP]   Mehdi, S. and Pandžić, P., *Dirac cohomology and translation functors.* J. Algebra **375** (2013), 328–336.

[MPa1]   Mehdi, S. and Parthasarathy, R., *A product for harmonic spinors on reductive homogeneous spaces*, J. Lie Theory **18** (2008), no. 1, 33–44.

[MPa2]   Mehdi, S. and Parthasarathy, R., *Representation theoretic harmonic spinors for coherent families.* INSA Platinum Jubilee Special Issue, Indian J. Pure Appl. Math. **41** (2010), 133–144.

[MPa3]   Mehdi, S. and Parthasarathy, R., *Cubic Dirac cohomology for generalized Enright-Varadarajan modules*, J. Lie Theory **21** (2011), 861–884.

[NOT]   K. Nishiyama, H. Ochiai, K. Taniguchi, *Bernstein degree and associated cycles of Harish-Chandra modules – Hermitian symmetric case. Nilpotent orbits, associated cycles and Whittaker models for highest weight representations.* Astérisque **273** (2001), 13–80.

[PS]   Pandžić, P. and Somberg, P., *Higher Dirac cohomology of modules with generalized infinitesimal character*, preprint, 2013.

[P]   Parthasarathy, R., *Dirac operator and the discrete series.* Ann. of Math. **96** (1972), 1–30.

[SV]   Schmid, W. and Vilonen, K., *On the geometry of nilpotent orbits.* Asian J. Math. **3** (1999), 223–274.

[V1]   Vogan, D., *Gelfand-Kirillov dimension for Harish-Chandra modules.* Invent. Math. **48** (1978), 75–98.

[V2]   Vogan, D., *Representations of real reductive Lie groups.* Birkhäuser, Boston, MA, 1981.

[V3]   Vogan, D., *Associated varieties and unipotent representations.* In Harmonic analysis on reductive groups (Brunswick, ME, 1989), Progr. Math., vol. 101, Birkhäuser, Boston, MA, 1991, pp. 315–388.

[V4]   Vogan, D., *Dirac operators and unitary representations.* 3 talks at MIT Lie groups seminar, Fall 1997.

Institut Elie Cartan de Lorraine, CNRS - UMR 7502, Université de Lorraine, Metz, F-57045, France.
  E-mail address: salah.mehdi@univ-lorraine.fr

Department of Mathematics, University of Zagreb, Zagreb, Croatia.
  E-mail address: pandzic@math.hr

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.
  E-mail address: dav@math.mit.edu