Character formulas for unipotent representations

David Vogan

Massachusetts Institute of Technology

Characters of unipotent representations

Symmetry in Geometry and Analysis

June 7–11, 2022

in honor of Toshiyuki Kobayashi

Outline

Introduction

Unipotent representations

Translation families

Families of translation families

About the representation of $W(\lambda_0)$

Slides at `http://www-math.mit.edu/~dav/paper.html`

Not exactly an apology...

I retained the announced title

Characters of unipotent representations.

But this talk is really about more basic questions:

1. What is a unipotent representation?
2. Why should I care?
3. (Having understood the answers to (1) and (2)) how can I devote all of my mathematical energy to unipotent representations?

The tools I will discuss are certainly relevant to character theory, but I won't say how.

See, I told you it wasn't an apology.

Gelfand's abstract harmonic analysis

Topological grp G acts on X, have questions about X.

Step 1. Attach to X Hilbert space $\mathcal{H}$ (e.g. $L^2(X)$).
Questions about $X \rightsquigarrow$ questions about $\mathcal{H}$.

Step 2. Find finest G-eqvt decomp $\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha$.
Questions about $\mathcal{H} \rightsquigarrow$ questions about each $\mathcal{H}_\alpha$.

Each $\mathcal{H}_\alpha$ is irreducible unitary representation of G: indecomposable action of G on a Hilbert space.

Step 3. Understand $\widehat{G}_u =$ all irreducible unitary representations of G: unitary dual problem.

Step 4. Answers about irr reps $\rightsquigarrow$ answers about X.

Toshi's work addresses many parts of Gelfand's program in many ways.

Today: **Step 3** for reductive Lie group G.

What does $\widehat{G}_u$ look like (part one)?

Most irr unitary reps of reductive $G \leftsquigarrow$ proper Levi subgroups $L \subset G$ by induction.

Two ways this happens...

Real parabolic induction:

1. L = centralizer of hyperbolic Lie algebra element X.
2. $X \rightsquigarrow P = LU$ real parabolic subgroup.
3. $\pi_L \in \widehat{L}_u \rightsquigarrow \pi_G = \mathrm{Ind}_P^G(\pi_L)$.
4. Think of $\pi_L \in$ family $\{\pi_L \otimes \chi_L \mid \chi_L$ unitary one-diml of $L\}$.
5. π_G always finite direct sum of irr unitary reps.
6. usually (almost all twists χ_L) π_G is irreducible.

Unitary 1-diml reps of L = union of real vec spaces.

So this part of unitary dual is finitely many pieces

$$\widehat{G}_u \supset \text{reps of } L' \times (\text{real vector space}).$$

What does $\widehat{G}_u$ look like (part two)?

Here is the second way that irr unitary reps of reductive G arise from proper Levi subgroups $L \subset G$: Cohomological parabolic induction:

1. $L =$ centralizer of elliptic Lie algebra element Z.
2. $Z \rightsquigarrow \mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subset \mathfrak{g}_{\mathbb{C}}$ θ-stable parabolic subalg.
3. Think of $\pi_L \in$ family $\{\pi_L \otimes \chi_L \mid \chi_L$ unitary one-diml of $L\}$.
4. $\pi_L \in \widehat{L}_u \rightsquigarrow \pi_G = \mathcal{L}_{\mathfrak{q}}^{\mathfrak{g}}(\pi_L)$ virtual G rep.
5. if $\pi_L \otimes \chi_L$ appropriately positive, then $\pi_G = \mathcal{L}_{\mathfrak{q}}^{\mathfrak{g}}(\pi_L)$ is finite direct sum of irr unitary reps.
6. usually (most pos twists χ_L) π_G is irreducible.

In this case unitary 1-diml of $L =$ union of lattices.
So this part of unitary dual is finitely many pieces

$$\widehat{G}_u \supset \text{reps of } L' \times (\text{cone in a lattice}).$$

This is most of $\widehat{G}_u$...

You may know about the irreducible unitary representations of $SL(2,\mathbb{R})$, which were classified by Valentine Bargmann in the 1940s. Here's the list:

Spherical princ series $\pi_{\text{even}}(i\nu) \simeq \pi_{\text{even}}(-i\nu) \quad (\nu \in \mathbb{R})$.

Nonspherical princ series $\pi_{\text{odd}}(i\nu) \simeq \pi_{\text{odd}}(-i\nu) \quad (\nu \in \mathbb{R})$.

The nonspherical representation $\pi_{\text{odd}}(0)$ is a direct sum of two irreducible representations $\pi^+(0)$ and $\pi^-(0)$.

Holomorphic discrete series $\pi^+(n) \quad (n \in \{1, 2, 3, \ldots\}$.

Antiholomorphic discrete series $\pi^-(n) \quad (n \in \{1, 2, 3, \ldots\}$.

These four families (two real vector spaces, two cones in a lattice) are most of $\widehat{G}_u$. What remains are

Complementary series $\pi_{\text{even}}(t) \quad (0 < t < 1)$, and

Trivial representation $\overline{\pi}_{\text{even}}(1)$.

Unipotent representations

Unitary representations for any real reductive G:

1. finite # pieces (unitary dual of smaller group) $\times \mathbb{R}^a$: unitarily induced.
2. finite # pieces (unitary dual of smaller group) $\times \mathbb{N}^b$: cohomologically induced.
3. finite # small polygons: deformations of unipotent representations

So everything is described by structure theory/recursion in terms of unipotent representations.

The most fundamental problem in unitary representation theory is describing unipotent representations.

Idea originates in work of Dan Barbasch in the 1980s.

What's a unipotent representation?

So far we have a very small list of examples:

1. trivial representation of any real reductive G
2. any rep of infl char zero of any real reductive G

Here are a few more:

3. metaplectic reps of $Sp(2n, \mathbb{C})$; more generally
4. ladder representations of various simple G.

How to characterize unip reps? Look for more?

Two key properties:

1. rep is small as possible among similar reps
2. infl char small as possible among similar reps.

Example: trivial rep smallest among fin-diml reps.

Example: zero is smallest infl char among all reps.

What's a family of similar representations?

First example: some principal series reps.

$G = SL(2,\mathbb{R})$. For each integer n, have a rep

$$\Theta_p(n) = \text{induced from } \chi_n\begin{pmatrix} t & x \\ 0 & t^{-1}\end{pmatrix} = t^n$$

infinitesimal character of $\Theta_p(n) = n$

$\Theta_p(n)|_{SO(2)} =$ chars of $SO(2) \equiv n \pmod 2$

So all reps $\Theta_p(n)$ are approximately same size

$\Theta_p(0)$ has smallest infl char.

Conclusion: $\Theta_p(0)$ is unique unipotent one.

What's a family of similar representations?

Second example: finite-diml reps.

$G = SL(2,\mathbb{R})$. For each integer n, have a virtual rep

$$\Theta_f(n) = \text{rep with character } \frac{t - t^{-n}}{t - t^{-1}}$$

infinitesimal character of $\Theta_f(n) = n$

$$\Theta_f(n) = \begin{cases} \text{irr of dimension } n & (n > 0) \\ \text{minus irr of dimension } -n & (n < 0) \\ \text{zero representation} & (n = 0) \end{cases}$$

$$\Theta_f(n)|_{SO(2)} = \begin{cases} -n+1, -n+3, \ldots, n-1 & (n > 0) \\ \text{minus } (-n+1, -n+3, \ldots, n-1) & (n < 0) \\ \text{zero} & n = 0 \end{cases}$$

So rep $\Theta_f(1) =$ trivial rep is smallest, and

$\Theta_f(1)$ has smallest infl char (among nonzero reps)

Conclusion: $\Theta_f(1)$ is unique unipotent one.

What's a family of similar representations?

Third example: discrete series reps.

$G = SL(2,\mathbb{R})$. For each integer n, have a virtual rep

$$\Theta_h(n) = \text{rep with char } -\frac{\sin(n\theta)}{sin(\theta)} \quad \text{on compact Cartan}$$

infinitesimal character of $\Theta_h(n) = n$

$$\Theta_h(n) = \begin{cases} \text{hol disc ser of HC param } n & (n > 0) \\ \text{disc ser plus irr } -n\text{-diml} & (n < 0) \\ \text{hol limit of disc ser} & (n = 0) \end{cases}$$

$$\Theta_h(n)|_{SO(2)} = \quad n+1,\ n+3,\ n+5\ldots$$

So all reps $\Theta_h(n) =$ are similar in size, and

$\Theta_h(0)$ has smallest infl char

Conclusion: $\Theta_h(0)$ is unique unipotent one.

Where we are

Would like to realize each irreducible representation π_0 of G as one point $\pi_0 = \Theta(\lambda_0)$ in a nice family $\lambda \mapsto \Theta(\lambda)$ of virtual representations.

To look for unipotent representations, minimize infinitesimal character over the family Θ.

Next: construction of nice families of representations.

Translation families: background

G real reductive, $\mathfrak{g} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C} \supset \mathfrak{h}$ Cartan subalg.

Structure of $G(\mathbb{C}) \rightsquigarrow$ dual lattices $X_*(H) \subset \mathfrak{h}$, $X^*(H) \subset \mathfrak{h}^*$.

$W = W(\mathfrak{g}, \mathfrak{h}) \subset \mathrm{Aut}(X^*)$ Weyl grp, finite reflection grp.

Theorem (Cartan-Weyl).

1. Restriction to $H(\mathbb{C})$ of any algebraic rep F of $G(\mathbb{C})$ is a W-invariant multiset $\Delta(F) \subset X^*(H)$.
2. If F irreducible, then $\Delta(F)$ contains (with mult one) a unique W-orbit $W \cdot \mu(F)$ of largest weights.
3. $F \mapsto \mu(F)$ is bijection (irr alg reps of $G(\mathbb{C})$) $\leftrightarrow (X^*/W)$.

Theorem (Harish-Chandra).

1. Center $\mathfrak{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$ is isomorphic to
$$S(h)^W = W\text{-invariant poly functions on } \mathfrak{h}^*.$$
2. Homomorphisms $\mathfrak{Z}(\mathfrak{g}) \to \mathbb{C} \leftrightsquigarrow \mathfrak{h}^*/W$.
3. Action of $\mathfrak{Z}(\mathfrak{g})$ on any irr $\mathfrak{g}$-module $X \leftrightsquigarrow \lambda(X) \in \mathfrak{h}^*$.

(W-orbit of) $\lambda(X)$ is the *infinitesimal character* of X.

Transl fams: def by Jantzen/Zuckerman

Here's a general definition of nice family of similar reps.

Definition (Jantzen, Schmid, Zuckerman). Suppose $H \subset G$ is a Cartan in a real reductive group, and $X^* = X^*(H) \subset \mathfrak{h}^*$ is the character lattice. A translation family is a map

$$\Theta \colon \lambda_0 + X^* \to \text{virtual reps of } G,$$

with the following properties:

1. (each irr constituent of) $\Theta(\lambda)$ has infl char λ;
2. if F is a finite-diml algebraic rep of G, then

$$\Theta(\lambda) \otimes F = \sum_{\mu \in \Delta(F)} \Theta(\lambda + \mu).$$

So Θ is a family indexed by infl chars in $\lambda_0 + X^* \subset \mathfrak{h}^*$.

Change λ in Θ $\leftrightsquigarrow$ tensor with fin diml reps of G.

Theorem (Jantzen, Schmid, Zuckerman) Suppose π_0 is a finite length virtual rep of infl char λ_0.

1. $\exists$ translation fam Θ on $\lambda_0 + X^*$ with $\Theta(\lambda_0) = \pi_0$.
2. If λ_0 is *regular* (meaning $W^{\lambda_0} = 1$) then Θ is unique.

Families of translation families (part one)

$H \subset G$, $\lambda_0 \in \mathfrak{h}^*$ infl char, $X^* \subset \mathfrak{h}^*$ char lattice.

Write $\widehat{G}(\lambda_0) =$ (finite) set of irr reps of G of infl char λ_0.

Recall that a trans fam based on $\lambda_0 + X^*$ is a function from $\lambda_0 + X^*$ to virtual reps of G.

Since virtual reps can be added and subtracted,

$$\mathcal{F}(\lambda_0 + X^*) = \text{all trans fams based on } \lambda_0 + X^*$$

is an abelian group: add and subtract values of Θ.

Jantzen-Schmid-Zuckerman uniqueness thm $\Longrightarrow$

Corollary Suppose $\lambda_1 \in \lambda_0 + X^*$ is regular. Then

$$\text{evaluation at } \lambda_1 \colon \mathcal{F}(\lambda_0 + X^*) \to \mathbb{Z}\widehat{G}(\lambda_1)$$

is an isom. So $\mathcal{F}(\lambda_0 + X^*)$ is free$/\mathbb{Z}$, rank $= \#\widehat{G}(\lambda_1)$.

The finite-rank $\mathbb{Z}$ module $\mathcal{F}(\lambda_0 + X^*)$ is the family of translation families in the slide title.

Families of translation families (part two)

$$\mathcal{F}(\lambda_0 + X^*) = \text{all trans fams based on } \lambda_0 + X^*,$$

free abelian group, natural basis indexed by $\widehat{G}(\lambda_1)$.

What other structure does this abelian group carry?

Weyl group $W = W(G(\mathbb{C}), H(\mathbb{C}))$ acts on $\mathfrak{h}^*$ preserving X^*.

But W may not preserve $\lambda_0 + X^*$. Integral Weyl grp for λ_0 is

$W(\lambda_0) =_{\text{def}} \{w \in W \mid w \cdot \lambda_0 \in \lambda_0 + (\text{lattice of roots of } H \text{ in } G)\}$;

the group $W(\lambda)$ is same for all $\lambda \in \lambda_0 + X^*$.

$W(\lambda_0)$ preserves the coset $\lambda_0 + X^*$.

Therefore $W(\lambda_0)$ acts on $\mathcal{F}(\lambda_0 + X^*)$ by

$$(w \cdot \Theta)(\lambda) = \Theta(w^{-1} \cdot \lambda) \qquad (\lambda \in \lambda_0 + X^*).$$

This integral representation of the integral Weyl group is the key to character theory for $\widehat{G}(\lambda_0)$.

The τ invariant

We fix an infl char $\lambda_0 \in \mathfrak{h}^*$, with integral Weyl group

$$W(\lambda_0) = \{w \in W \mid w\lambda_0 - \lambda_0 \in (\text{root lattice})\}.$$

The integral root system is

$$R(\lambda_0) = \{\alpha \in R(G,H) \mid \langle \alpha^\vee, \lambda_0 \rangle \in \mathbb{Z}\}.$$

Fix also a positive system $R^+(\lambda_0) \subset R(\lambda_0)$ making λ_0 weakly dominant, and $\lambda_1 \in \lambda_0 + X^*$ strictly dominant.

$\Pi(\lambda_0) = \text{simple of } R^+(\lambda_0), \qquad S(\lambda_0) = \{s_\alpha \mid \alpha \in \Pi(\lambda_0)\} \subset W(\lambda_0).$

Wkly dom elts of $\lambda_0 + X^*$ are a fund domain for $W(\lambda_0)$.

Trans fam Θ is irreducible (with respect to $R^+(\lambda_0)$) if $\Theta(\lambda)$ is irr for all dom reg $\lambda \in \lambda_0 + X^*$.

Irr fams are a basis for $\mathcal{F}(\lambda_0 + X^*)$, identified with $\widehat{G}(\lambda_1)$.

Definition (Borho-Jantzen-Duflo). The τ-invariant of an irr Θ is

$$\tau(\Theta) = \{s \in S(\lambda_0) \mid s \cdot \Theta = -\Theta\}.$$

Theorem Suppose $E \subset S(\lambda_0) \rightsquigarrow W(E) \subset W(\lambda_0)$ Levi.

$$[\mathrm{sgn}(W(E)) : \mathcal{F}(\lambda_0 + X^*)] = \#\{\text{irr } \Theta \mid E \subset \tau(\Theta)\}$$
$$[\mathrm{triv}(W(E)) : \mathcal{F}(\lambda_0 + X^*)] = \#\{\text{irr } \Theta \mid E \cap \tau(\Theta) = \emptyset\}.$$

Cones and cells of irreducibles

Continue with pos int roots $R^+(\lambda_0)$ making λ_0 wkly dom.

For $\pi \in \widehat{G}(\lambda_0)$, write $\Theta_\pi =$ unique irr fam with $\Theta_\pi(\lambda_0) = \pi$.

The cone over π is

$$\overline{C}(\pi) = \{\text{all irr constituents of all } \Theta_\pi(\lambda) \mid \lambda \in \lambda_0 + X^*\}$$
$$= \{\pi' \in \widehat{G} \mid \pi' \text{ is an irr const of } \pi \otimes F, \text{ some irr alg rep } F \text{ of } G(\mathbb{C})\}.$$

Write $\pi' \leq_\Theta \pi$ if $\pi' \in \overline{C}(\pi)$, a partial preorder on $\widehat{G}$.

$\pi' \leq_\Theta \pi \implies \mathcal{AV}(\pi') \subset \mathcal{AV}(\pi)$.

The cell of π is

$$C(\pi) = \{\text{all irr } \pi' \text{ with } \pi' \leq_\Theta \pi \leq_\Theta \pi'\}$$
$$= \{\pi' \in \widehat{G} \mid \pi' \text{ is an irr const of } \pi \otimes F, \text{ and } \pi \text{ an irr const of } \pi' \otimes E, \text{ some irr alg reps } E, F \text{ of } G(\mathbb{C})\}.$$

Write $\pi' \sim_\Theta \pi$ if $\pi' \in \overline{C}(\pi)$, an equivalence relation on $\widehat{G}$.

More about the $W(\lambda_0)$ representation

Continue with pos int roots $R^+(\lambda_0)$ making λ_0 wkly dom.

Definition (Kazhdan-Lusztig) Make irr transl families a directed $W(\lambda_0)$-graph with edge of weight m from Θ_π to $\Theta_{\pi'}$ whenever

1. $\tau(\pi) \not\subset \tau(\pi')$, and
2. $\dim \mathrm{Ext}^1(\pi, \pi') = m$.

An edge from Θ_π to $\Theta_{\pi'}$ implies $\pi' \leq_\Theta \pi$.

Conversely $\pi' \leq_\Theta \pi \implies \exists$ directed path Θ_π to $\Theta_{\pi'}$.

Theorem (Lusztig-Vogan) Say Θ_π irr transl fam on $\lambda_0 + X^*$, and $s \in S(\lambda_0)$ is a simple reflection. Then

$$s \cdot \Theta_\pi = \begin{cases} -\Theta_\pi & (s \in \tau(\pi)) \\ \Theta_\pi \quad + \displaystyle\sum_{\substack{\pi' \xleftarrow{m} \pi \\ s \in \tau(\pi')}} m \cdot \Theta_{\pi'} & (s \notin \tau(\pi)) \end{cases}$$

Corollary The $W(\lambda_0)$ graph determines the $W(\lambda_0)$ representation on translation families. Each cone $\overline{\mathcal{C}}(\pi)$ spans a $W(\lambda_0)$ subrepresentation, so the cell $\mathcal{C}(\pi)$ carries a natural quotient representation $\Sigma(\pi)$ of $W(\lambda_0)$.

What does the cell representation tell you?

Continue with pos int roots $R^+(\lambda_0)$ making λ_0 wkly dom.

Smallest (weakly dom) infl char in $\lambda_1 \in \lambda_0 + X^*$ is typically very singular: that is, fixed by large set S_1 of simple reflections.

Proposition Cell $\mathcal{C}(\pi)$ contains some irr Θ_{π_1} nonzero at λ_1 $\iff$ $[\mathrm{triv}(W(S_1)) : \Sigma(\pi)] > i0$.

So $\Sigma(\pi)$ determines smallest infl char in $\mathcal{C}(\pi)$.

Theorem (Joseph, Lusztig).

1. Irr $W(\lambda_0)$ reps in $\Sigma(\pi)$ are in a Lusztig family in $\widehat{W(\lambda_0)}$.
2. Family has a unique special rep $\sigma_0(\pi) \in \widehat{W(\lambda_0)}$.
3. $\sigma_0(\pi)$ is Springer for a special nilpotent orbit $\mathcal{O}_0(\pi) \subset \mathfrak{g}(\lambda)^*$.
4. Lusztig's truncated induction of $\sigma_0(\pi) \rightsquigarrow$ Springer rep $\sigma(\pi) \in \widehat{W}$, and so a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^*$.

So $\Sigma(\pi)$ determines GK dimension of π.