Coherent sheaves on nilpotent cones

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Outline

Introduction

What are the questions?

Equivariant K-theory

K-theory and representations

Complex groups: ∞-diml reps and algebraic geometry

Lusztig's conjecture and generalizations

Slides at `http://www-math.mit.edu/~dav/paper.html`

Why coherent sheaves?

Compact groups K are relatively easy...

Noncompact groups G are relatively hard.

Harish-Chandra *et al.* idea:

$$\text{understand } \pi \in \widehat{G} \rightsquigarrow \text{understand } \pi|_K$$

(nice compact subgroup $K \subset G$).

Get an invariant of a repn $\pi \in \widehat{G}$:

$$m_\pi \colon \widehat{K} \to \mathbb{N}, \qquad m_\pi(\mu) = \text{mult of } \mu \text{ in } \pi|_K.$$

1. What's the support of m_π? (subset of $\widehat{K}$)
2. What's the rate of growth of m_π?
3. What functions on $\widehat{K}$ can be m_π?

Answers $\leftrightsquigarrow$ sheaves on nilpotent cones.

Examples

1. $G = GL(n,\mathbb{C})$, $K = U(n)$. Typical restriction to K is
$$\pi|_K = \mathrm{Ind}_{U(1)^n}^{U(n)}(\gamma) = \sum_{\mu \in \widehat{U(n)}} m_\mu(\gamma)\gamma \quad (\gamma \in \widehat{U(1)^n}):$$
$m_\pi(\mu) =$ mult of μ is $m_\mu(\gamma) =$ dim of γ wt space.

2. $G = GL(n,\mathbb{R})$, $K = O(n)$. Typical restriction to K is
$$\pi|_K = \mathrm{Ind}_{O(1)^n}^{O(n)}(\gamma) = \sum_{\mu \in \widehat{O(n)}} m_\mu(\gamma):$$
$m_\pi(\mu) =$ mult of μ in π is $m_\mu(\gamma) =$ mult of γ in μ.

3. G split of type E_8, $K = Spin(16)$. Typical res to K is
$$\pi|_{Spin(16)} = \mathrm{Ind}_M^{Spin(16)}(\gamma) = \sum_{\mu \in \widehat{Spin(16)}} m_\mu(\gamma)\gamma;$$
here $M \subset Spin(16)$ subgp of order 512, cent ext of $(\mathbb{Z}/2\mathbb{Z})^8$.

Moral: may compute m_π using compact groups.

Plan for today

Work with real reductive Lie group $G(\mathbb{R})$.

Describe (old) assoc cycle $\mathcal{AC}(\pi)$ for $\pi \in \widehat{G(\mathbb{R})}$:

$\approx$ geom shorthand for approximating $\pi|_{K(\mathbb{R})}$.

Describe (new) algorithm for computing $\mathcal{AC}(\pi)$.

A *real* algorithm is one that's been implemented on a computer. This one has not, but should be possible soon.

Assumptions

$G(\mathbb{C}) = G =$ cplx conn reductive alg gp.

$G(\mathbb{R}) =$ group of real points for a real form.

Could allow fin cover of open subgp of $G(\mathbb{R})$, so allow nonlinear.

$K(\mathbb{R}) \subset G(\mathbb{R})$ max cpt subgp; $K(\mathbb{R}) = G(\mathbb{R})^\theta$.

$\theta =$ alg inv of G; $K = G^\theta$ possibly disconn reductive.

Harish-Chandra idea:

∞-diml reps of $G(\mathbb{R}) \leftrightsquigarrow$ alg gp $K \curvearrowright$ cplx Lie alg $\mathfrak{g}$

$(\mathfrak{g}, K)$-module is vector space V with

1. repn π_K of algebraic group K: $V = \sum_{\mu \in \widehat{K}} m_V(\mu)\mu$
2. repn $\pi_{\mathfrak{g}}$ of cplx Lie algebra $\mathfrak{g}$
3. $d\pi_K = \pi_{\mathfrak{g}}|_{\mathfrak{k}}, \qquad \pi_K(k)\pi_{\mathfrak{g}}(X)\pi_K(k^{-1}) = \pi_{\mathfrak{g}}(\mathrm{Ad}(k)X)$.

In module notation, cond (3) reads $k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot (k \cdot v)$.

Geometrizing representations

$G(\mathbb{R})$ real reductive, $K(\mathbb{R})$ max cpt, $\mathfrak{g}(\mathbb{R})$ Lie alg

$\rightsquigarrow$ K cplx reductive alg gp $\curvearrowright$ $\mathfrak{g}$ cplx reduc Lie alg.

$\mathcal{N}^* =$ cone of nilpotent elements in $\mathfrak{g}^*$.

$\mathcal{N}^*_\theta = \mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^*$, finite # nilpotent K orbits.

Goal 1: Attach nilp orbits to repns in theory.

Goal 2: Compute them in practice.

"In theory there is no difference between theory and practice. In practice there is." Jan L. A. van de Snepscheut (or not).

$$\begin{array}{c} V \text{ irr } (\mathfrak{g}, K)\text{-module} \\ \downarrow \text{ assoc cycle of gr} \\ \mathcal{AC}(V) \text{ closed union of } K \text{ orbits on } \mathcal{N}^*_\theta \end{array}$$

So Goal 1 is completed. Turn to Goal 2...

Associated varieties

$\mathcal{F}(\mathfrak{g}, K)$ = finite length $(\mathfrak{g}, K)$-modules...

noncommutative world we care about.

$\mathcal{C}(\mathfrak{g}, K) =$ f.g. $(S(\mathfrak{g}/\mathfrak{k}), K)$-modules, support $\subset \mathcal{N}_\theta^*$...

commutative world where geometry can help.

$$\mathcal{F}(\mathfrak{g}, K) \overset{\text{gr}}{\rightsquigarrow} \mathcal{C}(\mathfrak{g}, K)$$

Prop. gr induces surjection of Grothendieck groups

$$K\mathcal{F}(\mathfrak{g}, K) \xrightarrow{\text{gr}} K\mathcal{C}(\mathfrak{g}, K);$$

image records restriction to K of HC module.

So restrictions to K of HC modules sit in equivariant coherent sheaves on nilpotent cone in $(\mathfrak{g}/\mathfrak{k})^*$

$$K\mathcal{C}(\mathfrak{g}, K) =_{\text{def}} K^K(\mathcal{N}_\theta^*),$$

equivariant K-theory of the K-nilpotent cone.

Goal 2: compute $K^K(\mathcal{N}_\theta^*)$ and the map **Prop.**

Equivariant K-theory

Setting: (complex) algebraic group K acts on (complex) algebraic variety X.

$\mathrm{Coh}^K(X) =$ abelian categ of coh sheaves on X with K action.

$K^K(X) =_{\mathrm{def}}$ Grothendieck group of $\mathrm{Coh}^K(X)$.

Example: $\mathrm{Coh}^K(\mathrm{pt}) = \mathrm{Rep}(K)$ (fin-diml reps of K).

$K^K(\mathrm{pt}) = R(K) =$ rep ring of K; free $\mathbb{Z}$-module, basis $\widehat{K}$.

Example: $X = K/H$; $\mathrm{Coh}^K(K/H) \simeq \mathrm{Rep}(H)$

$E \in \mathrm{Rep}(H) \rightsquigarrow \mathcal{E} =_{\mathrm{def}} K \times_H E$ eqvt vector bdle on K/H

$K^K(K/H) = R(H)$.

Example: $X = V$ vector space (repn of K).

$E \in \mathrm{Rep}(K) \rightsquigarrow$ proj module $\mathcal{O}_V(E) =_{\mathrm{def}} \mathcal{O}_V \otimes E \in \mathrm{Coh}^K(X)$

proj resolutions $\implies$ $K^K(V) \simeq R(K)$, basis $\{\mathcal{O}_V(\tau)\}$.

Doing nothing carefully

Suppose $K \curvearrowright X$ with finitely many orbits:

$X = Y_1 \cup \cdots \cup Y_r, \qquad Y_i = K \cdot y_i \simeq K/K^{y_i}.$

Orbits partially ordered by $Y_i \geq Y_j$ if $Y_j \subset \overline{Y_i}$.

$(\tau, E) \in \widehat{K^{y_i}} \rightsquigarrow \mathcal{E}(\tau) \in \mathrm{Coh}^K(Y_i)$.

Choose (always possible) K-eqvt coherent extension

$$\widetilde{\mathcal{E}}(\tau) \in \mathrm{Coh}^K(\overline{Y_i}) \rightsquigarrow [\widetilde{\mathcal{E}}] \in K^K(\overline{Y_i}).$$

Class $[\widetilde{\mathcal{E}}]$ on $\overline{Y}_i$ unique modulo $K^K(\partial Y_i)$.

Set of all $[\widetilde{\mathcal{E}}(\tau)]$ (as Y_i and τ vary) is basis of $K^K(X)$.

Suppose $M \in \mathrm{Coh}^K(X)$; write class of M in this basis

$$[M] = \sum_{i=1}^{r} \sum_{\tau \in \widehat{K^{y_i}}} n_\tau(M)[\widetilde{\mathcal{E}}(\tau)].$$

Maxl orbits in Supp(M) = maxl Y_i with some $n_\tau(M) \neq 0$.

Coeffs $n_\tau(M)$ on maxl Y_i ind of choices of exts $\widetilde{\mathcal{E}}(\tau)$.

Our story so far

We have found

1. homomorphism

$$\text{virt } G(\mathbb{R}) \text{ reps } \ K\mathcal{F}(\mathfrak{g}, K) \xrightarrow{\text{gr}} K^K(\mathcal{N}_\theta^*) \ \text{ eqvt } K\text{-theory}$$

2. geometric basis $\left\{[\widetilde{\mathcal{E}(\tau)}]\right\}$ for $K^K(\mathcal{N}_\theta^*)$, indexed by irr reps of isotropy gps
3. expression of $[\text{gr}(\pi)]$ in geom basis $\rightsquigarrow \mathcal{AC}(\pi)$.

Problem is computing such expressions...

Teaser for the next section: Kazhdan and Lusztig taught us how to express π using std reps $I(\gamma)$:

$$[\pi] = \sum_\gamma m_\gamma(\pi)[I(\gamma)], \qquad m_\gamma(\pi) \in \mathbb{Z}.$$

$\{[\text{gr } I(\gamma)]\}$ is another basis of $K^K(\mathcal{N}_\theta^*)$.

Last goal is compute chg of basis matrix: to write

$$[\widetilde{\mathcal{E}}(\tau)] = \sum_\gamma n_\gamma(\tau)[\text{gr } I(\gamma)].$$

The last goal

Studying cone $\mathcal{N}_\theta^* =$ nilp lin functionals on $\mathfrak{g}/\mathfrak{k}$.

Found (for free) basis $\left\{[\widetilde{\mathcal{E}(\tau)}]\right\}$ for $K^K(\mathcal{N}_\theta^*)$, indexed by orbit K/K^i and irr rep τ of K^i.

Found (by rep theory) second basis $\{[\mathrm{gr}\, I(\gamma)]\}$, indexed by (parameters for) std reps of $G(\mathbb{R})$.

To compute associated cycles, enough to write

$$[\mathrm{gr}\, I(\gamma)] = \sum_{\text{orbits}} \sum_{\substack{\tau \text{ irr for} \\ \text{isotropy}}} N_\tau(\gamma)[\widetilde{\mathcal{E}}(\tau)].$$

Equivalent to compute inverse matrix

$$[\widetilde{\mathcal{E}}(\tau)] = \sum_\gamma n_\gamma(\tau)[\mathrm{gr}\, I(\gamma)].$$

Need to relate

geom of nilp cone $\leftrightsquigarrow$ geom of std reps.

Use parabolic subgps and Springer resolution.

Introducing Springer

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ Cartan decomp, $\mathcal{N}_\theta^* \simeq \mathcal{N}_\theta =_{\text{def}} \mathcal{N} \cap \mathfrak{s}$ nilp cone in $\mathfrak{s}$.

Kostant-Rallis, Jacobson-Morozov: nilp $X \in \mathfrak{s} \rightsquigarrow Y \in \mathfrak{s}$, $H \in \mathfrak{k}$

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H,$$

$$\mathfrak{g}[k] = \mathfrak{k}[k] \oplus \mathfrak{s}[k] \qquad (\text{ad}(H) \text{ eigenspace}).$$

$\rightsquigarrow \mathfrak{g}[\geq 0] =_{\text{def}} \mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ θ-stable parabolic.

Theorem (Kostant-Rallis) Write $\mathcal{O} = K \cdot X \subset \mathcal{N}_\theta$.

1. $\mu \colon \mathcal{O}_Q =_{\text{def}} K \times_{Q \cap K} \mathfrak{s}[\geq 2] \to \overline{\mathcal{O}}$, $\quad (k, Z) \mapsto \text{Ad}(k)Z$ is proper birational map onto $\overline{\mathcal{O}}$.
2. $K^X = (Q \cap K)^X = (L \cap K)^X (U \cap K)^X$ is a Levi decomp; so $\widehat{K^X} = [(L \cap K)^X]\widehat{\ }$.

So have resolution of singularities of $\overline{\mathcal{O}}$:

$$\begin{array}{ccc} & K \times_{Q \cap K} \mathfrak{s}[\geq 2] & \\ \text{vec bdle} \swarrow & & \searrow \mu \\ K/Q \cap K & & \overline{\mathcal{O}} \end{array}$$

Use it (*i.e.*, copy McGovern, Achar) to calculate equivariant K-theory...

Using Springer to calculate K-theory

$X \in \mathcal{N}_\theta$ represents $\mathcal{O} = K \cdot X$.

$\mu\colon \mathcal{O}_Q =_{\text{def}} K \times_{Q\cap K} \mathfrak{s}[\geq 2] \to \overline{\mathcal{O}}$ Springer resolution.

Theorem Recall $\widehat{K^X} = [(L \cap K)^X]^\frown$.

1. $K^K(\mathcal{O}_Q)$ has basis of eqvt vec bdles:
$$(\sigma, F) \in \text{Rep}(L\cap K) \rightsquigarrow \mathcal{F}(\sigma).$$
2. Get extension of $\mathcal{E}(\sigma|_{(L\cap K)^X})$ from $\mathcal{O}$ to $\overline{\mathcal{O}}$
$$[\overline{\mathcal{F}}(\sigma)] =_{\text{def}} \sum_i (-1)^i [R^i \mu_*(\mathcal{F}(\sigma))] \in K^K(\overline{\mathcal{O}}).$$
3. Compute (very easily) $[\overline{\mathcal{F}}(\sigma)] = \sum_\gamma n_\gamma(\sigma)[\text{gr}\, I(\gamma)]$.
4. Each irr $\tau \in [(L\cap K)^X]^\frown$ extends to (virtual) rep $\sigma(\tau)$ of $L \cap K$; can choose $\overline{\mathcal{F}(\sigma(\tau))}$ as extension of $\mathcal{E}(\tau)$.

Now we can compute associated cycles

Recall $X \in \mathcal{N}_\theta \rightsquigarrow \mathcal{O} = K \cdot X$; $\tau \in [(L \cap K)^X]\widehat{\ }$.

We now have explicitly computable formulas

$$[\widetilde{\mathcal{E}}(\tau)] = [\overline{\mathcal{F}(\sigma(\tau))}] = \sum_\gamma n_\gamma(\tau)[\mathrm{gr}\, I(\gamma)].$$

Here's why this does what we want:

1. inverting matrix $n_\gamma(\tau) \rightsquigarrow$ matrix $N_\tau(\gamma)$ writing $[\mathrm{gr}\, I(\gamma)]$ in terms of $[\widetilde{\mathcal{E}}(\tau)]$.
2. multiplying $N_\tau(\gamma)$ by Kazhdan-Lusztig matrix $m_\gamma(\pi)$ $\rightsquigarrow$ matrix $n_\tau(\pi)$ writing $[\mathrm{gr}\, \pi]$ in terms of $[\widetilde{\mathcal{E}}(\tau)]$.
3. Nonzero entries $n_\tau(\pi) \rightsquigarrow \mathcal{AC}(\pi)$.

Side benefit: algorithm for $G(\mathbb{R})$ cplx also computes a bijection (conj Lusztig, proof Bezrukavnikov)

$$(\text{dom wts}) \leftrightarrow (\text{pairs } (\mathcal{O}, \tau))\ldots$$

Complex groups regarded as real

G_1 = cplx conn reductive alg gp $\leftrightsquigarrow$ old $G(\mathbb{R})$).

σ_1 = cplx conj for compact real form of G_1.

$G = G_1 \times G_1$ complexification of G_1...

1. $\sigma(x, y) = (\sigma_1(y), \sigma_1(x))$ cplx conj for real form G_1:
$$G(\mathbb{R}) = G^\sigma = \{(x, \sigma_1(x) \mid x \in G_1\} \simeq G_1.$$
2. $\theta(x, y) = (y, x)$ Cartan inv: $K = G^\theta = (G_1)_\Delta$.

K-nilp cone $\mathcal{N}_\theta^* \subset \mathfrak{g}^* \simeq G_1$-nilp cone $\mathcal{N}_1^* \subset \mathfrak{g}_1^*$.

$H_1 \subset G_1$, $H = H_1 \times H_1 \subset G$, $T = (H_1)_\Delta \subset K$ max tori.

$\mathfrak{a} = \mathfrak{h}^{-\theta} = \{(Z, -Z) \mid Z \in \mathfrak{h}_1\}$ Cartan subspace.

Param for princ series rep is $\gamma = (\lambda, \nu) \in X^*(T) \times \mathfrak{a}^*$:

1. $I(\lambda, \nu)|_K \simeq \mathrm{Ind}_T^K(\lambda)$;
2. virt rep $[I(w_1 \cdot \lambda, w_1 \cdot \nu)]$ indep of $w_1 \in W_1$;
3. $[\mathrm{gr}\ I(\lambda, \nu)] \in K^K(\mathcal{N}_\theta^*) \simeq K^{G_1}(\mathcal{N}_1^*)$ indep of ν.

Conclusion: the set of all $[\mathrm{gr}\ I(\lambda)] \simeq \mathrm{Ind}_T^K(\lambda)$ ($\lambda \in X^*(T)$ dom) is basis for (virt HC-mods of G_1)$|_K$.

Connection with Weyl char formula

$K \simeq G_1$ cplx conn reductive alg, $T \simeq H_1$ max torus.

Asserted "$\{\mathrm{Ind}_T^K(\lambda)\}$ basis for (virt HC-mods of G_1) $|_K$."

What's that mean or tell you?

Fix $(F, \mu) \in \widehat{K}$ of highest weight $\mu \in X^{\mathrm{dom}}(T)$.

(F, μ) also irr (fin diml) HC-mod for G_1; $(F, \mu)|_K = (F, \mu)$.

Assertion means $F = \sum_{\gamma \in X^{\mathrm{dom}}(T)} m_\gamma(F) \mathrm{Ind}_T^K(\gamma)$.

Such a formula is a version of Weyl char formula:

$$(F, \mu) = \sum_{w \in W(K,T)} (-1)^{\ell(w)} \mathrm{Ind}_T^K(\mu + \rho - w\rho)$$

$$= \sum_{B \subset \Delta^+(\mathfrak{k},\mathfrak{t})} (-1)^{|\Delta^+|-|B|} \mathrm{Ind}_T^K(\mu + 2\rho - 2\rho(B)).$$

One meaning: if $(E, \gamma) \in \widehat{K}$, then

$$\sum_{w \in W} (-1)^{\ell(w)} m_{E,\gamma}(\mu + \rho - w \cdot \rho) = \begin{cases} 1 & (\gamma = \mu) \\ 0 & (\gamma \neq \mu). \end{cases}$$

Lusztig's conjecture

$G \supset B \supset H$ complex reductive algebraic.

$X^*(H) \supset X^{\mathrm{dom}}(H)$ dominant weights.

$\mathcal{N}^* =$ cone of nilpotent elements in $\mathfrak{g}^*$.

Lusztig conjecture: there's a bijection

$$X^{\mathrm{dom}} \leftrightsquigarrow \text{pairs } (\xi,\tau)/G \text{ conjugation;}$$

$\xi \in \mathcal{N}^*$, $\tau \in \widehat{G^\xi} \leftrightsquigarrow$ eqvt vec bdle $\mathcal{E}(\tau) = G \times_{G^\xi} \tau$

Thm (Bezrukavnikov). There is a preferred virt extension $\widetilde{\mathcal{E}}(\tau)$ to $\overline{G \cdot \xi}$ so

$$[\widetilde{\mathcal{E}}(\tau)] = \pm[\mathrm{gr}\, I(\lambda(\xi,\tau))] + \sum_{\gamma < \lambda(\xi,\tau)} n_\gamma(\xi,\tau)[\mathrm{gr}\, I(\gamma)].$$

Upper triangularity characterizes Lusztig bijection.

Calculating Lusztig's bijection

Proceed by upward induction on nilpotent orbit.

Start with (ξ, τ), $\xi \in \mathcal{N}^*$, $\tau \in \widehat{G^\xi}$.

JM parabolic $Q = LU$, $\xi \in (\mathfrak{g}/\mathfrak{q})^*$; $G^\xi = Q^\xi = L^\xi U^\xi$.

Choose virt rep $[\sigma(\tau)] \in R(L)$ extension of τ.

Write formula for corr ext of $\mathcal{E}(\tau)$ to $\overline{G \cdot \xi}$:

$$[\overline{\mathcal{F}(\sigma(\tau))}] = \sum_\lambda m_{\sigma(\tau)}(\lambda) \sum_{B \subset \Delta^+(\mathfrak{l},\mathfrak{h})} (-1)^{|\Delta^+(\mathfrak{l},\mathfrak{h})| - |B|} \sum_{A \subset \Delta(\mathfrak{g}[1],\mathfrak{h})} (-1)^{|A|}$$

$$[\text{gr } I(\lambda + 2\rho_L - 2\rho(A) - 2\rho(B))].$$

Rewrite with $[\text{gr } I(\lambda')]$, λ' dominant.

Loop: find largest λ'.

If $\lambda' \leftrightsquigarrow (\xi', \tau')$ for smaller $G \cdot \xi'$, subtract

$$m_{\sigma(\tau)}(\lambda') \times \text{formula for } (\xi', \tau');$$

$\rightsquigarrow$ new formula for (ξ, τ) with smaller leading term.

When loop ends, $\lambda' = \lambda(\xi, \tau)$.

What to do next

Sketched effective algorithms for computing
assoc cycles for HC modules, Lusztig bijection.

What should we (this means you) do now?

Software implementations of these?

Pramod Achar $\rightsquigarrow$ `gap` script for Lusztig bij in type A.
Marc van Leeuwen $\rightsquigarrow$ `atlas` software for (std rep)$|_K$.

Real group version of Lusztig bijection?

Algorithm still works, but bijection not quite true.
Failure partitions $\widehat{K}$ into small finite sets.

Closed form information about algorithms?

formula for smallest $\lambda \leftrightsquigarrow$ (one orbit, any τ);
Would bound below infl char of HC-mod $\leftrightsquigarrow$ orbit.