Signatures of Hermitian forms and unitary representations

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Deforming to $\nu = 0$

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Unitarity algorithm

Outline

Lecture 1: $SL(2,\mathbb{R})$

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Lecture 2: Character formulas and Hermitian forms

Character formulas
Hermitian forms
Jantzen filtrations and invariant forms

Lecture 3: Hermitian KL polynomials

Defining Hermitian KL polynomials
Illuminating calculations in $SL(2,\mathbb{R})$
Computing easy Hermitian KL polynomials
Deforming to $\nu = 0$

Lecture 4: Unitarity algorithm

Unitarity algorithm

What's representation theory for?

Example. $\int_{-\pi}^{\pi} \sin^5(t)dt = ?$ Zero!

Generalize: $f = f_{\text{even}} + f_{\text{odd}}, \quad \int_{-a}^{a} f_{\text{odd}}(t)dt = 0$. Reps of $\{\pm 1\}$.

Example. Evolution of initial temp distn of hot ring

$T(0,\theta) = A + B\cos(m\theta)$? $T(t,\theta) = A + Be^{-c\cdot m^2 t}\cos(m\theta)$.

Generalize: Fourier series of initial temp. Reps of circle group.

Example. X compact (arithmetic) locally symmetric manifold of dim 128; $\dim\left(H^{28}(X,\mathbb{C})\right) = ?$ Eight!

Same as H^{28} for compact globally symmetric space.

Generalize: $X = \Gamma\backslash G/K$, $H^p(X,\mathbb{C}) = H^p_{\text{cont}}(G, L^2(\Gamma\backslash G))$. Decomp L^2:

$$L^2(\Gamma\backslash G) = \sum_{\pi \text{ irr rep of } G} m_\pi(\Gamma)\mathcal{H}_\pi \qquad (m_\pi = \text{dim of some aut forms})$$

Deduce $H^p(X,\mathbb{C}) = \sum_\pi m_\pi(\Gamma) \cdot H^p_{\text{cont}}(G, \mathcal{H}_\pi)$.

General principal: group G acts on vector space V; decompose V; study pieces separately.

Gelfand's abstract harmonic analysis

Topological grp G acts on X, have questions about X.

Step 1. Attach to X Hilbert space $\mathcal{H}$ (e.g. $L^2(X)$).
Questions about $X \rightsquigarrow$ questions about $\mathcal{H}$.

Step 2. Find finest G-eqvt decomp $\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha$.
Questions about $\mathcal{H} \rightsquigarrow$ questions about each $\mathcal{H}_\alpha$.

Each $\mathcal{H}_\alpha$ is irreducible unitary representation of G: indecomposable action of G on a Hilbert space.

Step 3. Understand $\widehat{G}_u =$ all irreducible unitary representations of G: unitary dual problem.

Step 4. Answers about irr reps $\rightsquigarrow$ answers about X.

Topic for these lectures: **Step 3** for Lie group G.

Mackey theory (normal subgps) $\rightsquigarrow$ case G reductive.

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What's a unitary dual look like?

$G(\mathbb{R})$ = real points of complex connected reductive alg G

Problem: find $\widehat{G(\mathbb{R})}_u$ = irr unitary reps of $G(\mathbb{R})$.

Harish-Chandra: $\widehat{G(\mathbb{R})}_u \subset \widehat{G(\mathbb{R})}$ = "all" irr reps.

Unitary reps = "all" reps with pos def invt form.

Example: $G(\mathbb{R})$ compact $\Rightarrow \widehat{G(\mathbb{R})}_u = \widehat{G(\mathbb{R})}$ = discrete set.

Example: $G(\mathbb{R}) = \mathbb{R}$;

$$\widehat{G(\mathbb{R})} = \{\chi_z(t) = e^{zt} \ \ (z \in \mathbb{C})\} \qquad \simeq \mathbb{C}$$

$$\widehat{G(\mathbb{R})}_u = \{\chi_{i\xi} \ \ (\xi \in \mathbb{R})\} \qquad \simeq i\mathbb{R}$$

Suggests: $\widehat{G(\mathbb{R})}_u$ = real pts of cplx var $\widehat{G(\mathbb{R})}$. Almost...

$\widehat{G(\mathbb{R})}_h$ = reps with invt form: $\widehat{G(\mathbb{R})}_u \subset \widehat{G(\mathbb{R})}_h \subset \widehat{G(\mathbb{R})}$.

Approximately (Knapp): $\widehat{G(\mathbb{R})}$ = cplx alg var, real pts $\widehat{G(\mathbb{R})}_h$; subset $\widehat{G(\mathbb{R})}_u$ cut out by real algebraic ineqs.

These lectures: algorithm computing inequalities.

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Example: $SL(2,\mathbb{R})$ spherical reps

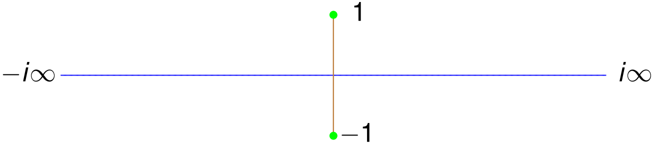
$G(\mathbb{R}) = SL(2,\mathbb{R})$ acts on upper half plane $\mathbb{H}$.

$\rightsquigarrow$ repn $E(\nu) = \{f \in C^\infty(\mathbb{H}) \mid \Delta_{\mathbb{H}} f = (\nu^2 - 1)f\}$.

$\nu \in \mathbb{C}$ parametrizes line bdle on circle where bdry values live.

Most $E(\nu)$ irr; always unique irr subrep $J(\nu) \subset E(\nu)$.

Spherical reps for $SL(2,\mathbb{R}) \leftrightsquigarrow \mathbb{C}/\pm 1$



Spectrum of $\Delta_{\mathbb{H}}$ on $L^2(\mathbb{H})$ is $(-\infty, -1]$. Gives unitary reps
unitary principal series $\leftrightsquigarrow \{E(\nu) \mid \nu \in i\mathbb{R}\}$.

Trivial representations $\leftrightsquigarrow$ [const fns on $\mathbb{H}$] $= J(\pm 1)$.

$J(\nu)$ is Herm. $\Leftrightarrow J(\nu) \simeq J(-\overline{\nu}) \Leftrightarrow \nu \in i\mathbb{R} \cup \mathbb{R}$.
By continuity, signature stays positive from 0 to ± 1.

complementary series reps $\leftrightsquigarrow \{E(t) \mid t \in (-1, 1)\}$.

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Digression about technical difficulties

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The space $E(\nu) = \{f \in C^\infty(\mathbb{H}) \mid \Delta_{\mathbb{H}} f = (\nu^2 - 1)f\}$ is never a unitary representation, even for ν purely imaginary.

Reason: if $B =$ upper triangular matrices, bdry of $\mathbb{H}$ is

$$\mathbb{R} \cup \{\infty\} = \mathbb{RP}^1 \simeq SL(2,\mathbb{R})/B,$$

Complex number ν defines character

$$\xi_\nu \colon B \to \mathbb{C}^\times, \qquad \xi_\nu \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = |a|^{\nu+1}$$

$\rightsquigarrow$ eqvt line bdle $\mathcal{L}(\nu) \to \mathbb{RP}^1$.

$\rightsquigarrow$ rep of $SL(2,\mathbb{R})$ on secs $I(\nu)$ of $\mathcal{L}(\nu)$. Which sections?

$$\underset{\text{analytic}}{I^\omega(\nu)} \subset \underset{\text{smooth}}{I^\infty(\nu)} \subset \underset{\text{square-integrable}}{I^{(2)}(\nu)} \subset \underset{\text{distribution}}{I^{-\infty}(\nu)} \subset \underset{\text{hyperfunction}}{I^{-\omega}(\nu)}.$$

Helgason theorem: if $\mathrm{Re}(\nu) \le 0$, then $E(\nu) \overset{\text{bdry val}}{\simeq} I^{-\omega}(\nu)$.

Hilbert space structure only on subspace $I^{(2)}(\nu)$.

Harish-Chandra soln: study vecs finite under cpt subgrp.

Digression about new branch of math

Often study solutions $Df = 0$ of diff op D.

Ex. $D = \Delta_{\mathbb{H}} - (\nu^2 - 1)$,

$E(\nu) =_{\text{def}} \{f \text{ generalized function on } \mathbb{H} \mid Df = 0\}$.

Could instead study cosolutions

$$E^*(\nu) =_{\text{def}} \{\text{test densities on } \mathbb{H}\}/\{D\delta\}.$$

Spaces $E(\nu)$ and $E^*(\nu)$ are topologically dual by $\int_{\mathbb{H}}$.

$E(\nu)$ big and fat $\rightsquigarrow$ diff eqns have lots of solns; but

$E^*(\nu)$ small and thin $\rightsquigarrow$ elts cptly supp, integrals converge.

Boundary value map $E^*(\nu) \rightsquigarrow$ analytic secs $I^\omega(-\nu)$.

Contrast solutions $\leftrightsquigarrow$ cosolutions very dramatic for Cauchy-Riemann eqns.

Solns are holomorphic fns, widely known; cosolns don't seem to appear much.

For Hermitian forms, prefer spaces like $E^*(\nu)$: Schmid's *minimal globalization*.

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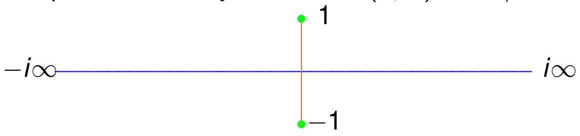
The moral[s] of the picture. . .

. . . and a preview of more general groups.

Spherical unitary dual for $SL(2,\mathbb{R}) \leftrightsquigarrow \mathbb{C}/\pm 1$



$SL(2,\mathbb{R})$	$G(\mathbb{R})$
$E(\nu), \nu \in \mathbb{C}$	$I(\nu), \nu \in \mathfrak{a}^*_{\mathbb{C}}$
$E(\nu),\ \nu \in i\mathbb{R}$	$I(\nu),\ \nu \in i\mathfrak{a}^*_{\mathbb{R}}$
$J(\nu) \hookrightarrow E(\nu)$	$I(\nu) \twoheadrightarrow J(\nu)$
$[-1,1]$	polytope in $\mathfrak{a}^*_{\mathbb{R}}$

Will deform Herm forms
unitary axis $i\mathfrak{a}^*_{\mathbb{R}} \rightsquigarrow$
real axis $\mathfrak{a}^*_{\mathbb{R}}$.

Deformed form pos $\rightsquigarrow$
unitary rep.

Reps appear in families, param by ν in cplx vec space $\mathfrak{a}^*$.

Pure imag params $\leftrightsquigarrow$ L^2 harm analysis $\leftrightsquigarrow$ unitary.

Each rep in family has distinguished irr piece $J(\nu)$.

Difficult unitary reps $\leftrightarrow$ deformation in real param

Reducibility of $E(\nu)$

Earlier used reps $E(\nu) = (\nu^2 - 1)$-eigenspace of $\Delta_{\mathbb{H}}$, Laplacian eigenspace on upper half plane

$$\mathbb{H} \simeq SL(2,\mathbb{R})/SO(2).$$

These are all reps (π, V) of $SL(2,\mathbb{R})$ having $SO(2)$-fixed $\lambda \in V^*$:

$$V \to C^\infty(\mathbb{H}), \qquad v \mapsto f_v(gK) = \lambda(\pi(g^{-1}v)).$$

For special ν, $E(\nu)$ is reducible.
$\{\text{const fns}\} = \mathbb{C} \subset \{\text{harm fns on } \mathbb{H}\} = E(\pm 1)$.
$\nu = \pm(2m+1)$ odd integer; $J(2m+1) = 2m+1$-diml irr rep of $SL(2,\mathbb{R})$ has $SO(2)$ wts

$$2m,\ 2m-2,\ \cdots,\ -2m$$

including zero.
Get $SO(2)$-fixed $\lambda \in J(2m+1)^*$, so inclusion

$$J(2m+1) \hookrightarrow E(2m+1).$$

Turns out all other $E(\nu)$ are irreducible.

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Signatures for $SL(2,\mathbb{R})$

Recall $E(\nu) = (\nu^2 - 1)$-eigenspace of $\Delta_{\mathbb{H}}$.

Need "signature" of Herm form on this inf-diml space.

Harish-Chandra (or Fourier) idea:
use $K = SO(2)$ break into fin-diml subspaces

$$E(\nu)_{2m} = \{f \in E(\nu) \mid \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot f = e^{2im\theta} f\}.$$

$$E(\nu) \supset \sum_m E(\nu)_{2m}, \qquad \text{(dense subspace)}$$

Decomp is orthogonal for any invariant Herm form.

Signature is $+$ or $-$ for each m. For $3 < |\nu| < 5$

$\cdots$	-6	-4	-2	0	$+2$	$+4$	$+6$	$\cdots$
$\cdots$	$+$	$+$	$-$	$+$	$-$	$+$	$+$	$\cdots$

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Deforming signatures for $SL(2,\mathbb{R})$

Here's how signatures of the reps $E(\nu)$ change with ν.

$\nu = 0$, $E(0)$ "$\subset$" $L^2(\mathbb{H})$: unitary, signature positive.

$0 < \nu < 1$, $E(\nu)$ irr: signature remains positive.

$\nu = 1$, form finite pos on $J(1) \subset E(1) \leftrightsquigarrow SO(2)$ rep 0.

$\nu = 1$, form has pole, pos residue on $E(1)/J(1)$.

$1 < \nu < 3$, across pole at $\nu = 1$, signature changes.

$\nu = 3$, Herm form finite $- + -$ on $J(3)$.

$\nu = 3$, Herm form has pole, neg residue on $E(3)/J(3)$.

$3 < \nu < 5$, across pole at $\nu = 3$, signature changes. ETC.

Conclude: $J(\nu)$ unitary, $\nu \in [0,1]$; nonunitary, $\nu \in [1,\infty)$.

$\cdots$	-6	-4	-2	0	$+2$	$+4$	$+6$	$\cdots$	$SO(2)$ reps
$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$\nu = 0$
$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$0 < \nu < 1$
$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$\nu = 1$
$\cdots$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$\cdots$	$1 < \nu < 3$
$\cdots$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$\cdots$	$\nu = 3$
$\cdots$	$+$	$+$	$-$	$+$	$-$	$+$	$+$	$\cdots$	$3 < \nu < 5$

From $SL(2,\mathbb{R})$ to reductive G

Calculated signatures of invt Herm forms on spherical reps of $SL(2,\mathbb{R})$.

Seek to do "same" for real reductive group. Need...

List of irr reps $=$ ctble union (cplx vec space)/(fin grp).

reps for purely imag points "$\subset$" $L^2(G)$: unitary!

Natural (orth) decomp of any irr (Herm) rep into fin-diml subspaces $\rightsquigarrow$ define signature subspace-by-subspace.

Compute signature at $\nu + i\tau$ by analytic continuation in t: $t\nu + i\tau$, $0 \leq t \leq 1$.

Precisely: start with pos def signature at $t = 0$; add contributions of sign changes from zeros/poles of odd order in $0 \leq t \leq 1 \rightsquigarrow$ signature at $t = 1$.

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Our story so far…

Yesterday: what's the unitary dual of a Lie group?

Gave part of answer for $SL(2,\mathbb{R})$: union of rational polyhedra in $\mathbb{C}$-vector spaces defined over $\mathbb{Q}$.

Looked at how to find this $SL(2,\mathbb{R})$ answer:

> start with Harish-Chandra's "tempered" unitary reps
> deform parameter, keep track of sign changes where Herm form becomes singular.

Answer for general reductive G has same shape, but with more complicated polyhedra.

Today: introduce technology (Langlands classification, Kazhdan-Lusztig theory of irreducible characters) needed to calculate in general reductive groups.

Categories of representations

G cplx reductive alg $\supset G(\mathbb{R})$ real form $\supset K(\mathbb{R})$ max cpt.

Rep theory of $G(\mathbb{R})$ modeled on Verma modules...

$H \subset B \subset G$ maximal torus in Borel subgp,

$\mathfrak{h}^* \leftrightarrow$ highest weight reps

$V(\lambda)$ Verma of hwt $\lambda \in \mathfrak{h}^*$, $L(\lambda)$ irr quot

Put cplxification of $K(\mathbb{R}) = K \subset G$, reductive algebraic.

$(\mathfrak{g}, K)$-mod: cplx rep V of $\mathfrak{g}$, compatible alg rep of K.

Harish-Chandra: irr $(\mathfrak{g}, K)$-mod $\leftrightsquigarrow$ "arb rep of $G(\mathbb{R})$."

X parameter set for irr $(\mathfrak{g}, K)$-mods

$I(x)$ std $(\mathfrak{g}, K)$-mod $\leftrightarrow x \in X$ $J(x)$ irr quot

Set X described by Langlands, Knapp-Zuckerman: countable union (subspace of $\mathfrak{h}^*$)/(subgroup of W).

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Character formulas

Can decompose Verma module into irreducibles

$$V(\lambda) = \sum_{\mu \leq \lambda} m_{\mu,\lambda} L(\mu) \qquad m_{\mu,\lambda} \in \mathbb{N})$$

or write a formal character for an irreducible

$$L(\lambda) = \sum_{\mu \leq \lambda} M_{\mu,\lambda} V(\mu) \qquad (M_{\mu,\lambda} \in \mathbb{Z})$$

Can decompose standard HC module into irreducibles

$$I(x) = \sum_{y \leq x} m_{y,x} J(y) \qquad (m_{y,x} \in \mathbb{N})$$

or write a formal character for an irreducible

$$J(x) = \sum_{y \leq x} M_{y,x} I(y) \qquad (M_{y,x} \in \mathbb{Z})$$

Matrices m and M upper triang, ones on diag, mutual inverses. Entries are KL polynomials eval at 1:

$$m_{y,x} = Q_{y,x}(1), \quad M_{y,x} = P_{y,x}(1) \quad (Q_{y,x}, P_{y,x} \in \mathbb{N}[q]).$$

What are we computing?

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Def of $(\mathfrak{g}, K)$-module $V \rightsquigarrow$

$$V|_K = \sum_{\mu\in\widehat{K}} m_V(\mu)\mu \qquad (m_V(\mu) \in \mathbb{N}\cup\{\infty\})$$

Harish-Chandra thm: V irr or std $\Rightarrow m_V(\mu) < \infty$.

$m_V\colon \widehat{K} \to \mathbb{N}$ *multiplicity function of V.*

$\exists$ algorithm (Hecht-Schmid pf of Blattner conj, etc.) computing function m_V, any V irr. or std.

Take functions m_I, I std, as known.

Non-deg K-invt Hermitian form $\langle,\rangle_V \rightsquigarrow$

$(p_V, q_V)\colon \widehat{K} \to \mathbb{N}\times\mathbb{N}$ *signature function of $\langle,\rangle_V$.*

Will compute sig fns $p_V, q_V \leftrightsquigarrow$ each irr Herm V.

"Compute" $\leftrightsquigarrow$ "write as fin int comb of mult fns m_I"

Character formulas for $SL(2,\mathbb{R})$

Recall (eigenspace of $\Delta_{\mathbb{H}}$) $= E(\nu) \leftrightarrow J(\nu)$. Prefer

$$\text{dual of } E(\nu) = I_{\text{ev}}(\nu) \twoheadrightarrow J_{\text{ev}}(\nu).$$

Need discrete series $I_{\text{hol/antihol}}(n)$ $(n = 1, 2, \ldots)$ char by

$$I_+(n)|_{SO(2)} = n+1,\ n+3,\ n+5\cdots$$
$$I_-(n)|_{SO(2)} = -n-1,\ -n-3,\ -n-5\cdots$$

Discrete series reps are irr: $I_{\text{hol/antihol}}(n) = J_{\text{hol/antihol}}(n)$

Decompose principal series

$$I_{\text{ev}}(2m+1) = J_{\text{ev}}(2m+1) + J_{\text{hol}}(2m+1) + J_{\text{antihol}}(2m+1).$$

Character formula

$$J_{\text{ev}}(2m+1) = I_{\text{ev}}(2m+1) - I_{\text{hol}}(2m+1) - I_{\text{antihol}}(2m+1).$$

Kazhdan-Lusztig matrix

$P_{x,y}$	$I_{\text{ev}}(2m+1)$	$I_{\text{hol}}(2m+1)$	$I_{\text{antihol}}(2m+1)$
$I_{\text{ev}}(2m+1)$	1	-1	-1
$I_{\text{hol}}(2m+1)$	0	1	0
$I_{\text{antihol}}(2m+1)$	0	0	1

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Forms and dual spaces

V cplx vec space (or alg rep of K, or $(\mathfrak{g}, K)$-module...)

Hermitian dual of V

$$V^h = \{\xi : V \to \mathbb{C} \text{ additive} \mid \xi(zv) = \overline{z}\xi(v)\}$$

(If V is K-rep, also require ξ is K-finite.)

Sesquilinear pairings between V and W

$$\mathrm{Sesq}(V, W) = \{\langle, \rangle \colon V \times W \to \mathbb{C}, \text{linear in } V, \text{conj-lin in } W\}$$

$$\mathrm{Sesq}(V, W) \simeq \mathrm{Hom}(V, W^h), \quad \langle v, w\rangle_T = (Tv)(w).$$

Cplx conj of forms defines (conjugate linear) isomorphism

$$\mathrm{Sesq}(V, W) \simeq \mathrm{Sesq}(W, V).$$

Corresponding (conj linear) isom is Hermitian transpose

$$\mathrm{Hom}(V, W^h) \simeq \mathrm{Hom}(W, V^h), \quad (T^h w)(v) = (Tv)(w).$$

Sesq form $\langle, \rangle_T$ on one space V is Hermitian if

$$\langle v, v'\rangle_T = \overline{\langle v', v\rangle}_T \Leftrightarrow T^h = T.$$

Defining Herm dual repn(s)

Suppose V is a $(\mathfrak{g}, K)$-module. Write π for repn map.

Recall Hermitian dual of V

$$V^h = \{\xi : V \to \mathbb{C} \text{ additive} \mid \xi(zv) = \overline{z}\xi(v)\}$$

Want to construct functor

$$\text{cplx linear rep } (\pi, V) \rightsquigarrow \text{cplx linear rep } (\pi^h, V^h)$$

using Hermitian transpose map of operators.

REQUIRES twist by conjugate linear automorphism of $\mathfrak{g}$.

Assume $\sigma\colon G \to G$ antiholom aut, $\quad \sigma(K) = K$.

Define $(\mathfrak{g}, K)$-module $\pi^{h,\sigma}$ on V^h,

$$\pi^{h,\sigma}(X) \cdot \xi = [\pi(-\sigma(X))]^h \cdot \xi \qquad (X \in \mathfrak{g}, \xi \in V^h).$$

$$\pi^{h,\sigma}(k) \cdot \xi = [\pi(\sigma(k)^{-1})]^h \cdot \xi \qquad (k \in K, \xi \in V^h).$$

Classically $\sigma_0 \leftrightsquigarrow G(\mathbb{R})$. We use also $\sigma_c \leftrightsquigarrow$ compact form of G

Different $\sigma \rightsquigarrow$ different Hermitian dual rep $\pi^{h.\sigma}$.

Big idea: choose σ to make calculations easy.

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Invariant Hermitian forms

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$V = (\mathfrak{g}, K)$-module, σ antihol aut of G preserving K.

A σ-invt sesq form on V is sesq pairing $\langle , \rangle$ such that

$$\langle X \cdot v, w\rangle = \langle v, -\sigma(X)\cdot w\rangle, \quad \langle k \cdot v, w\rangle = \langle v, \sigma(k^{-1})\cdot w\rangle$$
$$(X \in \mathfrak{g}; k \in K; v, w \in V).$$

Proposition

σ-invt sesq form on V $\leftrightsquigarrow$ $(\mathfrak{g}, K)$-map $T\colon V \to V^{h,\sigma}$:

$$\langle v, w\rangle_T = (Tv)(w).$$

Form is Hermitian iff $T^h = T$.
Assume V is irreducible.
$V \simeq V^{h,\sigma} \Leftrightarrow \exists$ invt sesq form $\Leftrightarrow \exists$ invt Herm form
A σ-invt Herm form on V is unique up to real scalar.

$T \to T^h \leftrightsquigarrow$ real form of cplx line $\mathrm{Hom}_{\mathfrak{g},K}(V, V^{h,\sigma})$.

Invariant forms on standard reps

Recall multiplicity formula

$$I(x) = \sum_{y \le x} m_{y,x} J(y) \qquad (m_{y,x} \in \mathbb{N})$$

for standard $(\mathfrak{g}, K)$-mod $I(x)$.

Want parallel formulas for σ-invt Hermitian forms.

Need forms on standard modules.

Form on irr $J(x) \xrightarrow{\text{deformation}}$ Jantzen filt $I^k(x)$ on std, nondeg forms $\langle , \rangle^k$ on I^k/I^{k+1}.

Details (proved by Beilinson-Bernstein):

$$I(x) = I^0 \supset I^1 \supset I^2 \supset \cdots, \qquad I^0/I^1 = J(x)$$

I^k/I^{k+1} completely reducible

$$[J(y) \colon I^k/I^{k+1}] = \text{coeff of } q^{(\ell(x)-\ell(y)-k)/2} \text{ in KL poly } Q_{y,x}$$

Hence $\langle , \rangle_{I(x)} \overset{\text{def}}{=} \sum_k \langle , \rangle^k$, nondeg form on gr $I(x)$.

Restricts to original form on irr $J(x)$.

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What's a Jantzen filtration?

V cplx, $\langle,\rangle_t$ $\mathbb{R}$-analytic fam of Herm forms, generically nondeg.

$$V = V^0(t) \supset V^1(t) = \text{Rad}(\langle,\rangle_t), \quad J(t) = V^0(t)/V^1(t)$$

$$(p^0(t), q^0(t)) = \text{signature of } \langle,\rangle_t \text{ on } J(t).$$

Question: how does $(p^0(t), q^0(t))$ change with t?

First answer: locally constant on open set $V^1(t) = 0$.

Refine answer. . . define form on $V^1(t)$

$$\langle v, w\rangle^1(t) = \lim_{s\to t} \frac{1}{s-t} < v, w >_s, \qquad V_2(t) = \text{Rad}(\langle,\rangle^1(t))$$

$$(p^1(t), q^1(t)) = \text{signature of } \langle,\rangle^1(t).$$

Continuing gives Jantzen filtration

$$V = V^0(t) \supset V^1(t) \supset V^2(t) \cdots \supset V^{m+1}(t) = 0$$

From $t-\epsilon$ to $t+\epsilon$, signature changes on odd levels:

$$p(t+\epsilon) = p(t-\epsilon) + \sum [p^{2k+1}(t) + q^{2k+1}(t)].$$

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Example of Jantzen filtrations

Example: $V = \mathbb{C}$; non-triv family of Herm forms $\leftrightsquigarrow$ non-zero real-analytic $f(t) = \langle 1, 1 \rangle_t$.

$$V^1(t) = \begin{cases} \{0\}, & f(t) \neq 0 \\ \mathbb{C}, & f(t) = 0. \end{cases}$$

Form $\langle , \rangle^1(t) = 0$ (on zero vec space) if $f(t) \neq 0$.

$$\langle 1, 1 \rangle^1(t) = f'(t) \qquad \text{if } f(t) = 0.$$

General formula is

$$V^k(t) = \begin{cases} \{0\}, & f^{(m)}(t) \neq 0, \text{ some } m < k \\ \mathbb{C} & 0 = f(t) = f'(t) = \cdots = f^{(k-1)}(t). \end{cases}$$

$V^k(t)/V^{k+1}(t) \neq 0 \Leftrightarrow f^{(k}(t)$ first nonzero deriv of f.

Then signature of $\langle , \rangle^k(t) \leftrightsquigarrow \operatorname{sgn} f^{(k)}(t)$.

Formula $p(t+\epsilon) = p(t-\epsilon) + \sum [p^{2k+1}(t) - q^{2k+1}(t)]$ says

analytic functions change sign at zeros of odd order.

Where we are

Have classification of irr reps.

Parameter $x \in X \rightsquigarrow$ std rep $I(x) \rightsquigarrow$ irr quotient $J(x)$

Character formula $J(x) = \sum_{y \leq x} M_{y,x} I(y)$

Integers $M_{y,x}$ are computable (Kazhdan-Lusztig).

Choice of complex conjugation $\sigma \rightsquigarrow$ Hermitian dual operation $J \mapsto J^{h,\sigma}$ on irr reps and (therefore) $x \mapsto \sigma(x)$ on parameter $x \in X$.

Action of σ on X is "real structure" whose fixed pts are the Herm reps.

If $J(x)$ has invt Herm form, Jantzen filtration of $I(x)$ $\rightsquigarrow$ invt Herm form on gr $I(x)$

Tomorrow: introduce Herm KL polys relating signatures on irrs and stds.

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Where we were

Classification of irr reps (of $G(\mathbb{R})$ real reductive).

Param $x \in X \rightsquigarrow$ std rep $I(x) \rightsquigarrow$ irr quotient $J(x)$

Character formula $J(x) = \sum_{y \leq x} M_{y,x} I(y)$.

Integers $M_{y,x}$ are computable (Kazhdan-Lusztig).

Choice of complex conjugation $\sigma \rightsquigarrow$ Hermitian dual operation $J \mapsto J^{h,\sigma}$ on irr reps and (therefore) $x \mapsto \sigma(x)$ on parameter $x \in X$.

If $J(x)$ has invt Herm form, Jantzen filtration of $I(x)$ $\rightsquigarrow$ invt Herm form on gr $I(x)$

Today: introduce Herm KL polys relating signatures on irrs and stds.

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Virtual Hermitian forms

$$\mathbb{Z} = \text{Groth group of vec spaces.}$$

Integers are mults of irr reps in virtual reps. Hence

Groth grp of fin lgth reps $\simeq \mathbb{Z}[X]$,

$$V \mapsto \sum_{x \in X} m_V(J(x)) \cdot x;$$

coeffs are mults of irrs as composition factors.

For invariant forms...

$$\mathbb{W} = \mathbb{Z} \oplus s\mathbb{Z} = \text{Groth grp of fin diml forms.}$$

$s \leftrightsquigarrow$ one-diml space with negative Herm form.

Ring structure (tensoring forms) is $\mathbb{Z}[s]/(s^2 - 1)$:

$$(p, q)(p', q') = (pp' + qq', pq' + q'p).$$

Mult of irr-with-forms in virtual-with-forms is in $\mathbb{W}$:

$$\mathbb{W}[X] \approx \text{Groth grp of fin lgth reps with invt forms.}$$

Two problems: invt form $\langle , \rangle_J$ may not exist for irr J; and $\langle , \rangle_J$ may not be preferable to $-\langle , \rangle_J$.

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Hermitian KL polynomials: multiplicities

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Fix σ-invt Hermitian form $\langle,\rangle_{J(x)}$ on each irr having one; recall Jantzen form $\langle,\rangle^n$ on $I(x)^n/I(x)^{n+1}$.

MODULO problem of irrs with no invt form, write

$$(I(x)^n/I(x)^{n+1},\langle,\rangle^n)=\sum_{y\le x} w_{y,x}(n)(J(y),\langle,\rangle_{J(y)}),$$

coeffs $w(n)=(p(n),q(n))\in\mathbb{W}$; summand means

$$p(n)(J(y),\langle,\rangle_{J(y)})\oplus q(n)(J(y),-\langle,\rangle_{J(y)})$$

Define Hermitian KL polynomials

$$Q^\sigma_{y,x}=\sum_n w_{y,x}(n)q^{(l(x)-l(y)-n)/2}\in\mathbb{W}[q]$$

Eval in $\mathbb{W}$ at $q=1 \leftrightsquigarrow$ form $\langle,\rangle_{I(x)}$ on gr(std).

Reduction to $\mathbb{Z}[q]$ by $\mathbb{W}\to\mathbb{Z} \leftrightsquigarrow$ KL poly $Q_{y,x}$.

Hermitian KL polynomials: characters

Matrix $Q^\sigma_{y,x}$ is upper tri, 1s on diag: INVERTIBLE.

$$P^\sigma_{x,y} \stackrel{\text{def}}{=} (-1)^{l(x)-l(y)}((x,y) \text{ entry of inverse}) \in \mathbb{W}[q].$$

Definition of $Q^\sigma_{x,y}$ says

$$(\text{gr } I(x), \langle,\rangle_{I(x)}) = \sum_{y \leq x} Q^\sigma_{x,y}(1)(J(y), \langle,\rangle_{J(y)});$$

inverting this gives

$$(J(x), \langle,\rangle_{J(x)}) = \sum_{y \leq x} (-1)^{l(x)-l(y)} P^\sigma_{x,y}(1)(\text{gr } I(y), \langle,\rangle_{I(y)})$$

Next question: how do you compute $P^\sigma_{x,y}$? Stay tuned…

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Encouraging example: $SL(2,\mathbb{R})$

$G(\mathbb{R}) = SL(2,\mathbb{R})$, $K = SO(2)$

Easy case: sph princ series $I_{\text{ev}}(1) \twoheadrightarrow J_{\text{ev}}(1) =$ triv rep, $I_{\text{hol/antihol}}(1)$ first discrete series reps.

Put pos def σ_0-invt form on each irr $J_{\text{ev}}(1)$, $J_{\text{hol/antihol}}(1)$.

Jantzen filtration of $I_{\text{ev}}(1)$ is

$$\underbrace{I_{\text{ev}}(1)}_{I^0_{\text{ev}}(1)} \supset \underbrace{J_{\text{hol}}(1) \oplus J_{\text{antihol}}(1)}_{I^1_{\text{ev}}(1)} \supset \underbrace{0}_{I^2_{\text{ev}}(1)}, \qquad I^0/I^1 = J_{\text{ev}}(1).$$

Previous calculation of signature

$\cdots$	-6	-4	-2	0	$+2$	$+4$	$+6$	$\cdots$	$SO(2)$ reps
$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$\nu = 1$

shows Jantzen form on $I^1_{\text{ev}}(1)$ (lim from *above*) negative.

So KL polys $Q^{\sigma_0}_{\text{hol}(1)/\text{antihol}(1),\text{ev}(1)} = s$. Not too bad...

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Less encouraging $SL(2,\mathbb{R})$ example

$G(\mathbb{R}) = SL(2,\mathbb{R})$, $K = SO(2)$

Sph princ series $I_{\text{ev}}(3) \twoheadrightarrow J_{\text{ev}}(3) =$ 3-diml, $I_{\text{hol/antihol}}(3)$ discrete series reps.

Put pos def form on $J_{\text{hol/antihol}}(1)$; form on $J_{\text{ev}}(3)$ pos on $SO(2)$-invt.

Jantzen filtration of $I_{\text{ev}}(3)$ is

$$\underbrace{I_{\text{ev}}(3)}_{I^0_{\text{ev}}(3)} \supset \underbrace{J_{\text{hol}}(3) \oplus J_{\text{antihol}}(3)}_{I^1_{\text{ev}}(3)} \supset \underbrace{0}_{I^2_{\text{ev}}(3)}, \qquad I^0/I^1 = J_{\text{ev}}(3).$$

Previous calculation of signature

$\cdots$	-6	-4	-2	0	$+2$	$+4$	$+6$	$\cdots$	$SO(2)$ reps
$\cdots$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$\cdots$	$\nu = 3$

shows that Jantzen form on $I^1_{\text{ev}}(3)$ is positive

So KL polys $Q^{\sigma_0}_{\text{hol(3)/antihol(3)},\text{ev(3)}} = 1$. Starts to sounds complicated. . .

Discouraging $SL(2,\mathbb{R})$ example

$G(\mathbb{R}) = SL(2,\mathbb{R})$, $K = SO(2)$

Nonsph princ series $I_{\mathrm{odd}}(2) \twoheadrightarrow J_{\mathrm{odd}}(2) =$ 2-diml, $I_{\mathrm{hol/antihol}}(2)$ discrete series reps.

Put pos def form on $J_{\mathrm{hol/antihol}}(2)$; form on $J_{\mathrm{ev}}(3)$ pos on $+1$ $SO(2)$-type, neg on -1 $SO(2)$-type.

Jantzen filtration of $I_{\mathrm{odd}}(2)$ is

$$\underbrace{I_{\mathrm{ev}}(2)}_{I^0_{\mathrm{ev}}(2)} \supset \underbrace{J_{\mathrm{hol}}(2) \oplus J_{\mathrm{antihol}}(2)}_{I^1_{\mathrm{ev}}(2)} \supset \underbrace{0}_{I^2_{\mathrm{ev}}(2)}, \qquad I^0/I^1 = J_{\mathrm{ev}}(2).$$

Calculation of signature gives

$\cdots$	-5	-3	-1	$+1$	$+3$	$+5$	$\cdots$	$SO(2)$ reps
$\cdots$	$-$	$-$	$-$	$+$	$+$	$+$	$\cdots$	$\nu = 2$

Jantzen form on $I^1_{\mathrm{ev}}(2)$ is neg on antihol, pos on hol

So KL polys $Q^{\sigma_0}_{\mathrm{antihol}(2),\mathrm{odd}(2)} = 1$, $Q^{\sigma_0}_{\mathrm{hol}(2),\mathrm{odd}(2)} = s$.

Sounds impossible...

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σ_c-invariant forms for $SL(2,\mathbb{R})$

σ_c = antihol aut of $G \leftrightsquigarrow$ compact real form.

For $SL(2)$, $\sigma_c(g) = {}^t\overline{g}^{-1}$; fixed points $SU(2)$.

Finite-diml reps have pos def σ_c-invt forms.

σ_c-invt forms on disc ser $I_{\text{hol/antihol}}(m)$ alternate in sign

choose pos on $\pm(m+1)$, then neg on $\pm(m+3)$, etc.

σ_c-invt forms on sph princ series $I_{ev}(\nu)$:

$\cdots$	-6	-4	-2	0	$+2$	$+4$	$+6$	$\cdots$	$SO(2)$ reps
$\cdots$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$\cdots$	$\nu = 0$
$\cdots$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$\cdots$	$0 < \nu < 1$
$\cdots$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$\cdots$	$\nu = 1$
$\cdots$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$\cdots$	$1 < \nu < 3$
$\cdots$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$\cdots$	$\nu = 3$
$\cdots$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$\cdots$	$3 < \nu < 5$
$\cdots$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$\cdots$	$\nu = 5$

Jantzen form always positive on LKT of I^1.

Interesting Herm KL polys $Q^{\sigma_c}_{xy}$ always $= 1$ (for this $SL(2,\mathbb{R})$ example).

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Herm KL polys for σ_c

$\sigma_c =$ cplx conj for cpt form of G, $\sigma_c(K) = K$.

Plan: study σ_c-invt forms, relate to σ_0-invt forms.

Proposition

Suppose $J(x)$ irr $(\mathfrak{g}, K)$-module, real infl char. Then $J(x)$ has σ_c-invt Herm form $\langle , \rangle^c_{J(x)}$, characterized by

$\langle , \rangle^c_{J(x)}$ *is pos def on the lowest K-types of $J(x)$.*

Proposition $\Longrightarrow$ Herm KL polys $Q^{\sigma_c}_{x,y}$, $P^{\sigma_c}_{x,y}$ well-def.

Coeffs in $\mathbb{W} = \mathbb{Z} \oplus s\mathbb{Z}$; $s = (0, 1) \leftrightsquigarrow$ one-diml neg def form.

Fact: $Q^{\sigma_c}_{x,y}(q) = s^{\frac{\ell_o(x)-\ell_o(y)}{2}} Q_{x,y}(qs), \quad P^{\sigma_c}_{x,y}(q) = s^{\frac{\ell_o(x)-\ell_o(y)}{2}} P_{x,y}(qs).$

Equiv: if $J(y)$ occurs at level k of Jantzen filt of $I(x)$, then Jantzen form is $(-1)^{(l(x)-l(y)-k)/2}$ times $\langle , \rangle_{J(y)}$.

. . . except for a small complicating sign from ℓ_o. . .

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Orientation number

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Fact without ℓ_o $\leftrightsquigarrow$ KL polys $\leftrightsquigarrow$ *integral* roots.

Fact without $\ell_o \Rightarrow$ Jantzen-Zuckerman translation across non-integral root walls preserves signatures of (σ_c-invariant) Hermitian forms.

It ain't necessarily so.

$SL(2,\mathbb{R})$: translating spherical principal series from (real non-integral positive) ν to (negative) $\nu - 2m$ changes sign of form iff $\nu \in (0,1) + 2\mathbb{Z}$.

Orientation number $\ell_o(x)$ is

1. # pairs $(\alpha, -\theta(\alpha))$ cplx nonint, pos on x; PLUS
2. # real β s.t. $\langle x, \beta^\vee\rangle \in (0,1) + \epsilon(\beta, x) + 2\mathbb{N}$.

$\epsilon(\beta, x) = 0$ spherical, 1 non-spherical.

Deforming to $\nu = 0$

Have computable formula (proof not yet written down)

$$(J(x), \langle,\rangle^c_{J(x)}) = \sum_{y \le x} (-1)^{l(x)-l(y)} s^{\frac{\ell_o(x)-\ell_o(y)}{2}} P_{x,y}(s)(\mathrm{gr}\, I(y), \langle,\rangle^c_{I(y)})$$

for σ^c-invt forms in terms of forms on stds at same inf char.

Polys $P_{x,y}$ are KL polys, computed by `atlas` software.

Difficulty: forms on gr $I(\nu)$ change with continuous parameter ν.

$$Z =_{\mathrm{def}} \{z \in X \mid \text{ continuous parameter } \nu(z) = 0\}$$
$$= \{z \in X \mid I(z) \text{ tempered, real infl char}\}$$

Z is countable, discrete; prefer to write $\langle,\rangle^c_{J(x)}$ using $\langle,\rangle^c_{I(z)}$.

Method: consider $I(t) = I(t\nu),\ t \ge 0$. Deform $t = 1 \rightsquigarrow t = 0$.

Plan: keep track of signature changes, so rewrite each signature

$$\langle,\rangle^c_{I(y)} = \sum_{z \in Z} b_{y,z} \langle,\rangle^c_{I(z)} \qquad (b_{y,z} \in \mathbb{W}).$$

Combining these two formulas will give

$$\langle,\rangle^c_{J(x)} = \sum_{z \in Z} v_{x,z} \langle,\rangle^c_{I(z)} \qquad (v_{x,z} \in \mathbb{W}).$$

Coeffs all computable, and signatures on right also computable.

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Deforming to $\nu = 0$ for $SL(2, \mathbb{R})$

σ_c = antihol aut of G ⇝ compact real form.

For $SL(2)$, $\sigma_c(g) = {}^t\overline{g}^{-1}$; fixed points $SU(2)$.

σ_c-invt forms on gr $I_{ev}(\nu)$ (level one in red):

$\cdots$	-6	-4	-2	0	$+2$	$+4$	$+6$	$\cdots$	$SO(2)$ reps
$\cdots$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$\cdots$	$0 \le \nu < 1$
$\cdots$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$\cdots$	$\nu = 1$
$\cdots$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$\cdots$	$1 < \nu < 3$
$\cdots$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$\cdots$	$\nu = 3$
$\cdots$	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$\cdots$	$3 < \nu < 5$

Deduce formulas σ_c-invariant signatures

$$I(\nu) = \begin{cases} I(0) & 0 \le \nu < 1 \\ I(0) + (1-s)\left[I_{\mathrm{hol}}(1) + I_{\mathrm{anti}}(1)\right] & 1 \le \nu < 3 \\ I(0) + (1-s)\left[I_{\mathrm{hol}}(1) + I_{\mathrm{anti}}(1) + I_{\mathrm{hol}}(3) + I_{\mathrm{anti}}(3)\right] & 3 \le \nu < 5 \end{cases}$$

Same for general G: for std rep $I(\nu)$ with cont param ν, and $t > 0$, formula for signature of σ_c-invt form

$$\text{gr } I(t\nu) = \text{gr } I((t-\epsilon)\nu) + (1-s)(\text{odd levels of Jantzen filt of } I(t)).$$

What happened in the last three episodes

X = parameters for irr reps $\supset Z$ = parameters with continuous part zero.

Suppose irr $J(x)$ admits invt Hermitian form $\langle,\rangle^{\sigma}_{J(x)}$.

Std $I(x) \twoheadrightarrow J(x) \rightsquigarrow$ Jantzen filt $I(x) = I^0 \supset I^1 \supset I^2 \supset \cdots$;

$\rightsquigarrow$ nondeg form $\langle,\rangle^{k,\sigma}_{I}$ on I^k/I^{k+1}, $I^0/I^1 = J$, $\langle,\rangle^{0,\sigma}_{I} = \langle,\rangle^{\sigma}_{J}$

$\rightsquigarrow$nondeg form $\langle,\rangle^{\sigma}_{I}$ on gr I.

$\rightsquigarrow$ Herm KL polys $Q^{\sigma}_{x,y}$, $P^{\sigma}_{x,y}$ (coeffs in $\mathbb{W} = \mathbb{Z} \oplus s\mathbb{Z}$)

In terms of these polys, can

1. write (signature of) $\langle,\rangle^{\sigma}_{I(x)}$ using $\langle,\rangle^{\sigma}_{J(y)}$;
2. invert formula to write $\langle,\rangle^{\sigma}_{J(x)}$ using $\langle,\rangle_{I(y)}$;
3. $\nu \rightsquigarrow 0 \rightsquigarrow$ write $\langle,\rangle^{\sigma}_{J(x)}$ using $\langle,\rangle_{I(z)}$ $(z \in Z)$;

For σ_c-invariant forms, computed everything explicitly.

For this choice, Herm KL polys $\approx$ ordinary KL polys.

Last step to unitarity: relating $\langle,\rangle^{\sigma_c} \leftrightsquigarrow \langle,\rangle^{\sigma_0}$

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From σ_c to σ_0

Cplx conjs σ_c (compact form) and σ_0 (our real form) differ by Cartan involution θ: $\sigma_0 = \theta \circ \sigma_c$.

Irr $(\mathfrak{g}, K)$-mod $J \rightsquigarrow J^\theta$ (same space, rep twisted by θ).

Proposition

J admits σ_0-invt Herm form if and only if $J^\theta \simeq J$. If $T_0 \colon J \xrightarrow{\sim} J^\theta$, and $T_0^2 = \mathrm{Id}$, then

$$\langle v, w \rangle_J^0 = \langle v, T_0 w \rangle_J^c.$$

$T \colon J \xrightarrow{\sim} J^\theta \Rightarrow T^2 = z \in \mathbb{C} \Rightarrow T_0 = z^{-1/2} T \rightsquigarrow \sigma$-invt Herm form.

To convert formulas for σ_c invt forms $\rightsquigarrow$ formulas for σ_0-invt forms need intertwining ops $T_J \colon J \xrightarrow{\sim} J^\theta$, consistent with decomp of std reps.

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Equal rank case

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$\operatorname{rk} K = \operatorname{rk} G \Rightarrow$ Cartan inv inner: $\exists \tau \in K$, $\operatorname{Ad}(\tau) = \theta$.

$\theta^2 = 1 \Rightarrow \tau^2 = \zeta \in Z(G) \cap K$.

Study reps π with $\pi(\zeta) = c$. Fix square root $c^{1/2}$.

If ζ acts by c on V, and $\langle,\rangle_V^c$ is σ_c-invt form, then
$\langle v, w\rangle_V^0 \overset{\text{def}}{=} \langle v, c^{-1/2}\tau \cdot w\rangle_V^c$ is σ_0-invt form.

$$\langle,\rangle_{J(x)}^c = \sum_{z\in Z} v_{x,z}\langle,\rangle_{I(z)}^c \qquad (v_{x,z} \in \mathbb{W}).$$

translates to

$$\langle,\rangle_{J(x)}^0 = \sum_{z\in Z} v_{x,z}\langle,\rangle_{I(z)}^0 \qquad (v_{x,z} \in \mathbb{W}).$$

$I(z)$ has LKT $\mu(z) \Rightarrow \langle,\rangle_{I(z)}^0$ definite, sign $c^{-1/2}\mu(\tau)$.

$J(x)$ unitary $\Leftrightarrow$ summands are definite of same sign

General case

Fix "distinguished involution" δ_0 of G inner to θ

Define extended group $G^\Gamma = G \rtimes \{1, \delta_0\}$.

Can arrange $\theta = \mathrm{Ad}(\tau\delta_0)$, some $\tau \in K$.

Define $K^\Gamma = \mathrm{Cent}_{G^\Gamma}(\tau\delta_0) = K \rtimes \{1, \delta_0\}$.

Study $(\mathfrak{g}, K^\Gamma)$-mods $\leftrightsquigarrow$ $(\mathfrak{g}, K)$-mods V with $D_0 \colon V \xrightarrow{\sim} V^{\delta_0}$, $D_0^2 = \mathrm{Id}$.

Beilinson-Bernstein localization: $(\mathfrak{g}, K^\Gamma)$-mods $\leftrightsquigarrow$ action of δ_0 on K-eqvt perverse sheaves on G/B.

Should be computable by mild extension of Kazhdan-Lusztig ideas. Not done yet!

Now translate σ_c-invt forms to σ_0 invt forms

$$\langle v, w\rangle_V^0 \overset{\text{def}}{=} \langle v, c^{-1/2}\tau\delta_0 \cdot w\rangle_V^c$$

on $(\mathfrak{g}, K^\Gamma)$-mods as in equal rank case.

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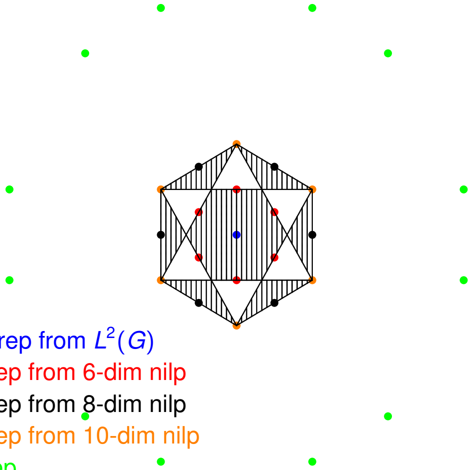
Example of $G_2(\mathbb{R})$

Real parameters for spherical unitary reps of $G_2(\mathbb{R})$



- Unitary rep from $L^2(G)$
- Arthur rep from 6-dim nilp
- Arthur rep from 8-dim nilp
- Arthur rep from 10-dim nilp
- Trivial rep

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Possible unitarity algorithm

Hope to get from these ideas a computer program; enter

- real reductive Lie group $G(\mathbb{R})$
- general representation π

and ask whether π is unitary.

Program would say either

- π has no invariant Hermitian form, or
- π has invt Herm form, indef on reps μ_1, μ_2 of K, or
- π is unitary, or
- I'm sorry Dave, I'm afraid I can't do that.

Answers to finitely many such questions $\rightsquigarrow$ complete description of unitary dual of $G(\mathbb{R})$.

This would be a good thing.

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