Regular polyhedra in *n* dimensions

David Vogan

Department of Mathematics Massachusetts Institute of Technology

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Regular polyhedra in *n* dimensions

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Outline

Introduction

Ideas from linear algebra

Flags in polyhedra

Reflections and relations

Relations satisfied by reflection symmetries

Presentation and classification

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Introduction Linear algebra Flags Reflections Relations Classification

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Want to understand the possibilities for a regular polyhedron P_n of dimension n.

Schläfli symbol is string $\{m_1, \ldots, m_{n-1}\}$.

Meaning of m_1 : two-dimensional faces are regular m_1 -gons.

Equivalent: *m*₁ edges ("1-faces") in a fixed 2-face.

Meaning of m_2 : fixed vertex $\subset m_2$ 2-faces \subset fixed 3-face.

fixed k - 1-face $\subset m_{k+1} k + 1$ -faces \subset fixed k + 2-face.

What are the possible Schläfli symbols, and why do they characterize P_n ?

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One regular *m*-gon for every $m \ge 3$

m = 3: equilateral triangle

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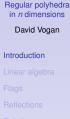
Classification



Linear algebra Flags Reflections Relations



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Classification



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One regular *m*-gon for every $m \ge 3$ m = 3: equilateral triangle

Schläfli symbol {3}

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Schläfli symbol {3}

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m = 7: regular heptagon



One regular *m*-gon for every $m \ge 3$ m = 3: equilateral triangle



Schläfli symbol {3}

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Schläfli symbol {7}

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Five regular polyhedra.

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Five regular polyhedra.

tetrahedron



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Five regular polyhedra.





Schläfli symbol $\{3,3\}$

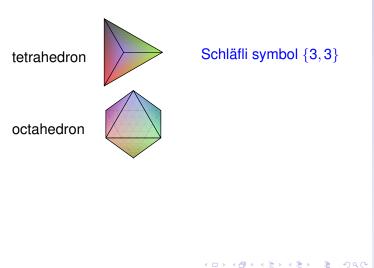
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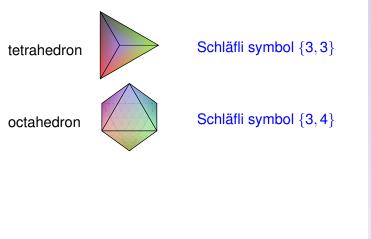
Five regular polyhedra.



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Five regular polyhedra.

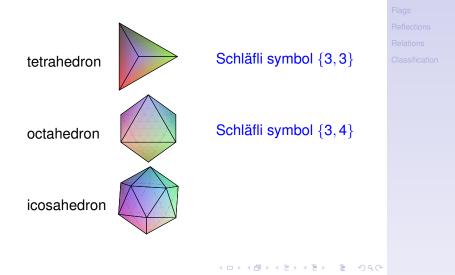


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Five regular polyhedra.

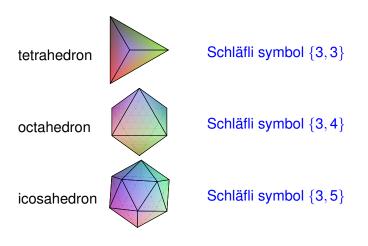


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Introduction

Five regular polyhedra.



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One regular 1-gon.

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One regular 1-gon.

interval ———

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Symmetry group: two elements $\{1, s\}$

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Symmetry group: two elements $\{1, s\}$

There is also just one regular 0-gon:



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Symmetry group: two elements $\{1, s\}$

There is also just one regular 0-gon: point

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Symmetry group: two elements $\{1, s\}$

There is also just one regular 0-gon: point •

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Symmetry group: two elements $\{1, s\}$

There is also just one regular 0-gon:

point

 Schläfli symbol undefined

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Symmetry group: two elements $\{1, s\}$

There is also just one regular 0-gon:

• Schläfli symbol undefined

Symmetry group trivial (zero gens of order 2).

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Something really symmetrical...like a square

FIX one vertex inside one edge inside square.

Two building block symmetries.

 s_0 takes red vertex to adj vertex along red edge; s_1 takes red edge to adj edge at red vertex.

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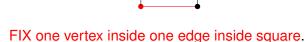
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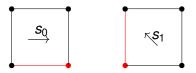
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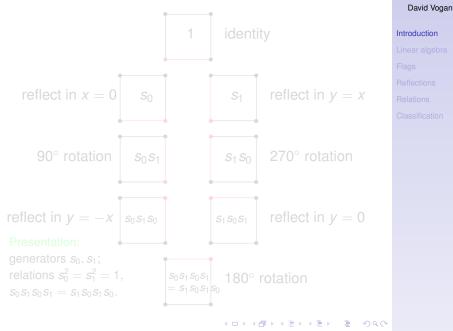


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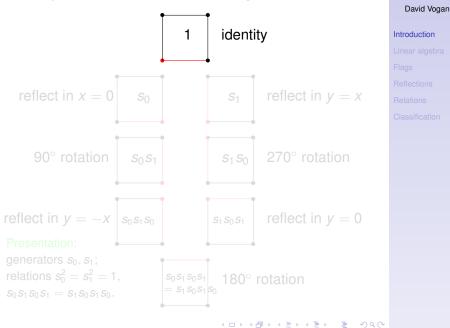
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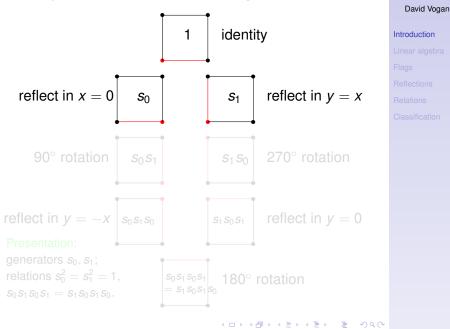
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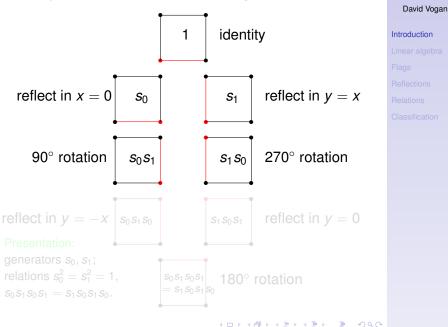
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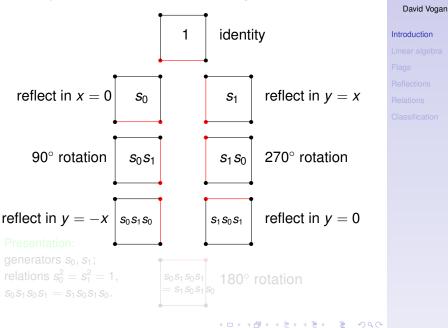
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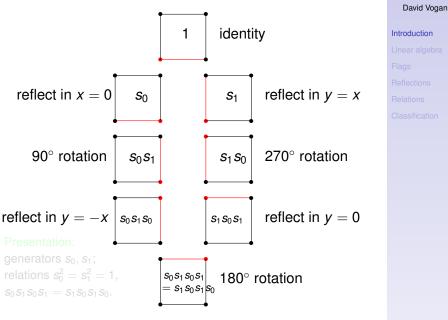
Regular polyhedra



Regular polyhedra

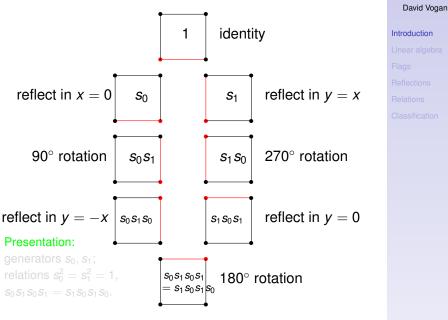


Regular polyhedra



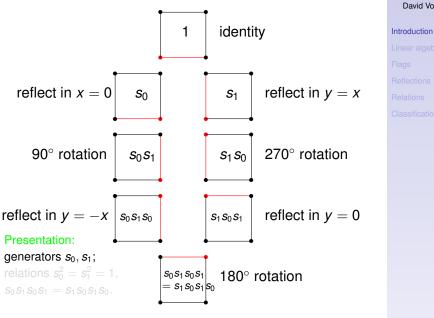
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Regular polyhedra



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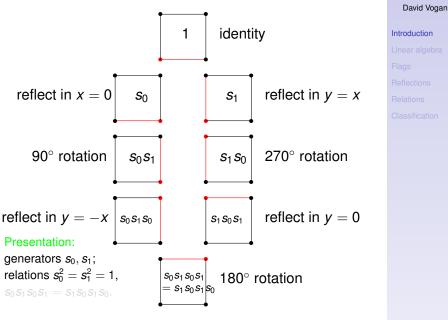
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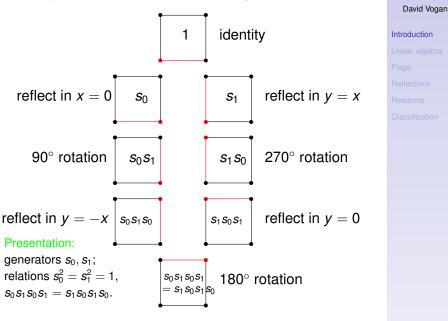
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Regular polyhedra



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Regular polyhedra

Define a flag as a chain of faces like vertex \subset edge.

Introduce basic symmetries like s_0 , s_1 which change a flag as little as possible.

Find a presentation of the symmetry group.

Reconstruct the polyhedron from this presentation.

Decide which presentations are possible.

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Introduction Linear algebra Flags Reflections Relations Classification

V n-diml vec space \rightsquigarrow *GL*(*V*) invertible linear maps.

complete flag in V is chain of subspaces \mathcal{F}

 $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V, \quad \dim V_i = i.$

Stabilizer $B(\mathcal{F})$ called Borel subgroup of GL(V).

Example

 $V = k^n$, $V_i = \{(x_1, \dots, x_i, 0, \dots, 0) \mid x_i \in k\} \simeq k^i$. Stabilizer of this flag is upper triangular matrices. **Theorem**

- 1. GL(V) acts transitively on flags.
- Stabilizer of one flag is isomorphic to group o invertible upper triangular matrices.

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Regular polyhedra in *n* dimensions

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Linear algebra

Flags

Reflections

Relations

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- 2. Stabilizer of one flag is isomorphic to group of invertible upper triangular matrices.

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Relations

Fix integers $\mathbf{d} = (0 = d_0 < d_1 < \cdots < d_r = n)$

Partial flag of type **d** is chain of subspaces \mathcal{G} $W_0 \subset W_1 \subset \cdots \subset V_{r-1} \subset W_r$, dim $W_i = d_i$

Stabilizer $P(\mathcal{G})$ is a parabolic subgroup of GL(V).

Theorem Fix a complete flag (0 = $V_0 \subset \cdots \subset V_n = V$), and consider the n - 1 partial flags

 $\mathcal{G}_p = (V_0 \subset \cdots \subset V_p \subset \cdots \subset V_n) \quad 1 \le p \le n-1$ obtained by omitting one proper subspace.

GL(V) is generated by the n — 1 subgroups P(G₀).
 P(G₀) is isomorphic to block upper-triangular matrices with a single 2 × 2 block.

So build all linear transformations from two by two matrices and upper triangular matrices.

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Fix integers $\mathbf{d} = (0 = d_0 < d_1 < \cdots < d_r = n)$ Partial flag of type \mathbf{d} is chain of subspaces \mathcal{G} $W_0 \subset W_1 \subset \cdots \subset V_{r-1} \subset W_r$, dim $W_j = d_j$. Stabilizer $P(\mathcal{G})$ is a parabolic subgroup of GL(V).

Theorem Fix a complete flag (0 = $V_0 \subset \cdots \subset V_n = V$), and consider the n - 1 partial flags

 $\mathcal{G}_p = (V_0 \subset \cdots \subset V_p \subset \cdots \subset V_n) \quad 1 \le p \le n-1$ obtained by omitting one proper subspace.

GL(V) is generated by the n — 1 subgroups P(g_n)
 P(g₀) is isomorphic to block upper-triangular matrices with a single 2 × 2 block.

So build all linear transformations from two by two matrices and upper triangular matrices.

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Suppose P_n compact *n*-diml convex polyhedron. A (complete) flag \mathcal{F} in P is a chain $P_0 \subset P_1 \subset \cdots \subset P_n$, dim $P_i = i$ with P_{i-1} a face of P_i .

Example

Two flags in two-diml P. Symmetry group (generated by reflections in x and y axes) is transitive on edges, not transitive on flags.

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Definition *P* regular if symmetry group acts transitively on flags. Regular polyhedra in *n* dimensions

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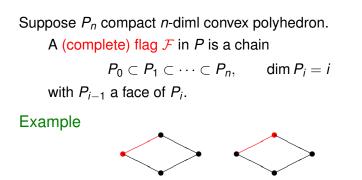
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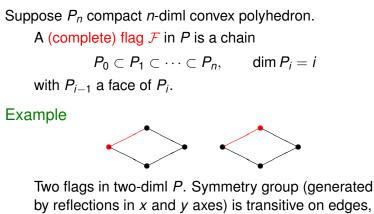


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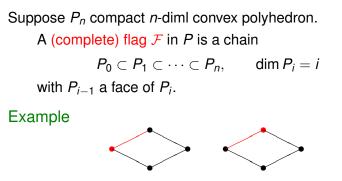
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Flags

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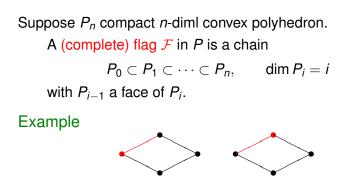
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$$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \quad \text{dim } P_i = i$$

complete flag in *n*-diml compact convex polyhedron.

A flag $\mathcal{F}' = (P'_0 \subset P'_1 \subset \cdots \subset P'_n)$ is *i*-adjacent to \mathcal{F} if $P_j = P'_i$ for all $j \neq i$, and $P_i \neq P'_i$.



Three flags adjacent to \mathcal{F} , i = 0, 1, 2.

Symmetry doesn't matter for this!

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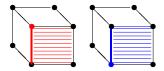
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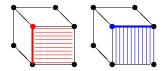
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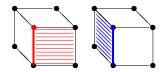
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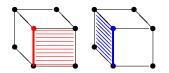
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Three flags adjacent to \mathcal{F} , i = 0, 1, 2. \mathcal{F}'_2 : move face P_2 only.

There is exactly one \mathcal{F}' *i*-adjacent to \mathcal{F} (each i = 0, 1, ..., n - 1).

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Lemma

Suppose $\mathcal{F} = (P_0 \subset P_1 \subset \cdots)$ is a complete flag in *n*-dimensional compact convex polyhedron P_n . Any affine map T preserving \mathcal{F} acts trivially on P_n .

Proof. Induction on *n*. If n = -1, $P_n = \emptyset$ and result is true.

Suppose $n \ge 0$ and the the result is known for n - 1.

Write p_n = center of mass of P_n . Since center of mass is preserved by affine transformations, $Tp_n = p_n$.

By inductive hypothesis, *T* acts trivially on n - 1-diml affine span(P_{n-1}) spanned by P_{n-1} .

Easy to see that $p_n \notin \text{span}(P_{n-1})$, so p_n and (n-1)-diml span (P_{n-1}) must generate *n*-diml span (P_n) .

Since T trivial on gens, trivial on span(P_n). Q.E.D.

Compactness matters; result fails for $P_1 = [0, \infty)$.

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Write p_i = center of mass of P_i .

Theorem

There is exactly one symmetry w of P_n for each complete flag \mathcal{G} , characterized by w $\mathcal{F} = \mathcal{G}$.

Corollary

Define $\mathcal{F}'_i =$ unique flag (i)-adj to \mathcal{F} ($0 \le i < n$). There is a unique symmetry s_i of P_n characterized by $s_i(\mathcal{F}) = \mathcal{F}'_i$. It satisfies

 $S_1(\mathcal{F}_1) = \mathcal{F}_1 S_1^2 = 1$

2. s, thes the (n - 1)-dimb hyperplane through the npoints $\{\rho_1, ..., \rho_{i-1}, \overline{\rho_i}, ..., \rho_{i-1}\}$ Regular polyhedra in *n* dimensions

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Define $\mathcal{F}'_i =$ unique flag (i)-adj to \mathcal{F} ($0 \le i < n$). There is a unique symmetry s_i of P_n characterized by $s_i(\mathcal{F}) = \mathcal{F}'_i$. It satisfies

1. $s_i(\mathcal{F}'_i) = \mathcal{F}, s_i^2 = 1.$

s_i fixes the (n − 1)-diml hyperplane through the n points {p₀,..., p_{i−1}, p_i, p_{i+1},..., p_n}.

Regular polyhedra in *n* dimensions

David Vogan

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From now on P_n is a compact convex regular polyhedron with fixed flag

 $\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \quad \text{dim } P_i = i$

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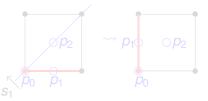
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Examples of basic symmetries s_i



This is s_0 , which changes \mathcal{F} only in P_0 , so acts trivially on the line through p_1 and p_2 .

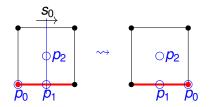


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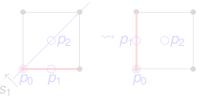
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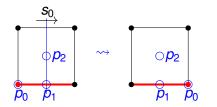


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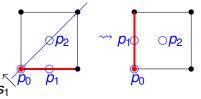
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On vector space V (characteristic not 2), a linear map s with $s^2 = 1$, dim(-1 eigenspace) = 1. -1 eigenspace $L_s =$ span of nonzero vector $\alpha^{\vee} \in V$ $L_s = \{v \in V \mid sv = -v\} =$ span (α^{\vee}) . +1 eigenspace $H_s =$ kernel of nonzero $\alpha \in V^*$

 $H_{s} = \{ v \in V \mid sv = v \} = \ker(\alpha).$

$$sv = s_{(lpha, lpha^{ee})}(v) = v - 2 rac{\langle lpha, v
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Definition of reflection does not mention "orthogonal." If *V* has quadratic form \langle, \rangle identifying $V \simeq V^*$, then

s is orthogonal $\iff \alpha$ is proportional to α^{\vee} .

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$$sv = v - 2 \frac{\langle lpha_s, v \rangle}{\langle lpha_s, lpha_s^{ee} \rangle} lpha_s^{ee}, \qquad tv = v - 2 \frac{\langle lpha_t, v \rangle}{\langle lpha_t, lpha_t^{ee}
angle} lpha_t^{ee}.$$

Assume $V = L_s \oplus L_t \oplus (H_s \cap H_t)$. Subspace $L_s \oplus L_t$ has basis { $\alpha_s^{\lor}, \alpha_t^{\lor}$ }, $c_{st} = 2\langle \alpha_s, \alpha_t^{\lor} \rangle / \langle \alpha_s, \alpha_s^{\lor} \rangle$

$$s = \begin{pmatrix} -1 & c_{st} \\ 0 & 1 \end{pmatrix}, t = \begin{pmatrix} 1 & 0 \\ c_{ts} & -1 \end{pmatrix}, st = \begin{pmatrix} -1 + c_{st}c_{ts} & c_{st} \\ c_{ts} & -1 \end{pmatrix}.$$

st has eigenvalues $z, z^{-1}, z + z^{-1} = c_{st}c_{ts} - 2$

$$z, z^{-1} = e^{\pm i\theta}, \qquad \theta = \cos^{-1}(-1 + c_{st}c_{ts}/2)).$$

Suppose -1 + $c_{st}c_{ts}/2 = \zeta + \zeta^{-1}$ for a primitive m^{th} root ζ . Then st is a rotation of order m in the plane $L_s \oplus L_t$. Otherwise st has infinite order. So

- 1. m = 2 if and only if $C_{sl} = C_b = 0$;
- 2 m = 3 if and only if $C_{at}C_{b} = 1$;
- 3. m = 4 if and only if $c_{a}c_{b} = 2$
- 4. m = 6 if and only if $c_{q}c_{p} = -6$

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 $\det(st) = 1, \qquad \operatorname{tr}(st) = -2 + c_{st}c_{ts},$

st has eigenvalues $z, z^{-1}, \quad z + z^{-1} = c_{st}c_{ts} - 2$.

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- m = 2 if and only if $c_{st} = c_{ts} = 0$;
- 2 m = 3 if and only if $c_{\rm sf}c_{\rm ff} = 1$,
- 3. m = 4 if and only if $c_{\rm ef}c_{\rm ff} = 2$
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Regular polyhedra in *n* dimensions

David Vogan

$$sv = v - 2 \frac{\langle lpha_s, v \rangle}{\langle lpha_s, lpha_s^{ee} \rangle} lpha_s^{ee}, \qquad tv = v - 2 \frac{\langle lpha_t, v \rangle}{\langle lpha_t, lpha_t^{ee} \rangle} lpha_t^{ee}.$$

Assume $V = L_s \oplus L_t \oplus (H_s \cap H_t)$.

Subspace $L_s \oplus L_t$ has basis $\{\alpha_s^{\lor}, \alpha_t^{\lor}\}$, $c_{st} = 2\langle \alpha_s, \alpha_t^{\lor} \rangle / \langle \alpha_s, \alpha_s^{\lor} \rangle$;

$$s = \begin{pmatrix} -1 & c_{st} \\ 0 & 1 \end{pmatrix}, t = \begin{pmatrix} 1 & 0 \\ c_{ts} & -1 \end{pmatrix}, st = \begin{pmatrix} -1 + c_{st}c_{ts} & c_{st} \\ c_{ts} & -1 \end{pmatrix}.$$

 $\det(st) = 1, \qquad \operatorname{tr}(st) = -2 + c_{st}c_{ts},$

st has eigenvalues $z, z^{-1}, \quad z + z^{-1} = c_{st}c_{ts} - 2.$

$$z, z^{-1} = e^{\pm i\theta}, \qquad \theta = \cos^{-1}(-1 + c_{st}c_{ts}/2)).$$

Proposition

Suppose $-1 + c_{st}c_{ts}/2 = \zeta + \zeta^{-1}$ for a primitive m^{th} root ζ . Then st is a rotation of order m in the plane $L_s \oplus L_t$. Otherwise st has infinite order. So

1.
$$m = 2$$
 if and only if $c_{st} = c_{ts} = 0$;
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 $\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \quad \dim P_k = k, \quad p_k = \operatorname{ctr} of mass(P_k).$

 $s_k =$ symmetry preserving all P_j except P_k $(0 \le k < n)$.

 s_k must be orthogonal reflection in hyperplane

 $H_k = \operatorname{span}(p_0, p_1, \ldots, p_{k-1}, \widehat{p_k}, p_{k+1}, \ldots, p_n)$

(unique affine hyperplane through these *n* points). Write equation of H_k

 $H_k = \{ v \in \mathbb{R}^n \mid \langle \alpha_k, v \rangle = c_k \}.$

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For $0 \le k \le n-2$, have seen that $s_k s_{k+1}$ must be rotation of some order m_{k+1} in a plane inside span(P_{k+2}), fixing P_{k-1} .

Proposition

Suppose P_n is an n-dimensional regular polyhedron. Then the rotation $s_k s_{k+1}$ acts transitively on the k-dimensional faces of P_n that are contained between P_{k-1} and P_{k+2} . Therefore the Schläfli symbol of P_n is $\{m_1, m_2, \ldots, m_{n-1}\}$.

We turn next to computing m_{k+1} .

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$$\begin{split} & \mathcal{P}_n \text{ cpt cvx reg polyhedron in } \mathbb{R}^n, \text{ flag} \\ & \mathcal{F} = (\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_n), \quad \dim \mathcal{P}_k = k, \quad p_k = \text{ctr of mass}(\mathcal{P}_k). \\ & s_k = \text{orthogonal reflection in hyperplane} \\ & H_k = \text{span}(\mathcal{p}_0, \mathcal{p}_1, \dots, \mathcal{p}_{k-1}, \widehat{\mathcal{p}_k}, \mathcal{p}_{k+1}, \dots, \mathcal{p}_n) \\ & \text{For } 0 \leq k \leq n-2, \text{ have seen that } s_k s_{k+1} \text{ must be} \\ & \text{rotation of some order } m_{k+1} \text{ in a plane inside} \\ & \text{span}(\mathcal{P}_{k+2}), \text{ fixing } \mathcal{P}_{k-1}. \end{split}$$

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Seek to relate coordinates for P_n to geometry... Translate so center of mass is at the origin: $p_n = 0$. Rotate so center of mass of n - 1-face is on x-axis: $p_{n-1} = (a_n, 0, ...), \quad a_n > 0$. Now P_{n-1} is perp. to x-axis: span $(P_{n-1}) = \{x_1 = a_n\}$. Botate around the x axis so center of mass of (n - 2)-

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Now span(P_{n-2}) = { $x_1 = a_n, x_2 = a_{n-1}$ }.

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Relations

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 $\begin{array}{l} P_n \mbox{ cpt cvx reg polyhedron in } \mathbb{R}^n, \mbox{ flag} \\ \mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \quad \dim P_i = i, \quad p_i = \mbox{ ctr of mass}(P_i). \\ \mbox{ Seek to relate coordinates for } P_n \mbox{ to geometry.} \\ \mbox{ Translate so center of mass is at the origin: } p_n = 0. \\ \mbox{ Rotate so center of mass of } n - 1 \mbox{ face is on } x \mbox{ -axis: } \\ p_{n-1} = (a_n, 0, \ldots), \quad a_n > 0. \\ \mbox{ Now } P_{n-1} \mbox{ is perp. to } x \mbox{ -axis: } \mbox{ span}(P_{n-1}) = \{x_1 = a_n\}. \end{array}$

Rotate around the *x* axis so center of mass of (n - 2)-face is in the x - y plane: $p_{n-2} = (a_n, a_{n-1}, 0...), a_{n-1} > 0.$

Now span(P_{n-2}) = { $x_1 = a_n, x_2 = a_{n-1}$ }.

 $p_{n-k} = (a_n, \dots, a_{n-k+1}, 0 \dots), \quad a_{n-k+1} > 0.$ $span(P_{n-k}) = \{x_1 = a_n, x_2 = a_{n-1} \dots x_k = a_{n-k+1}\}.$

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Regular polyhedra

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Relations

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Regular polyhedra

in *n* dimensions David Vogan

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 $p_{n-1}=(a_n,0,\ldots), \quad a_n>0.$

Now P_{n-1} is perp. to *x*-axis: span $(P_{n-1}) = \{x_1 = a_n\}$.

Rotate around the *x* axis so center of mass of (n - 2)-face is in the x - y plane: $p_{n-2} = (a_n, a_{n-1}, 0...), a_{n-1} > 0.$

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David Vogan

Introduction Linear algebra Flags Reflections Relations

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Good coordinates

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Regular polyhedra in *n* dimensions

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Regular polyhedra in *n* dimensions

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Classification

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Classification

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 $\boldsymbol{P}_n = \{\boldsymbol{x} \in \mathbb{R}^n \mid -1 \leq x_i \leq 1 \quad (1 \leq i \leq n)\}.$

Choose flag $P_k = \{x \in P_n \mid x_1 = \dots = x_{n-k} = 1\}$, ctr of mass $p_k = (1, \dots, 1, 0, \dots, 0)$ $(n - k \ 1s)$. $s_k = \text{refl in } \alpha_k = (0, \dots, 1, -1, \dots, 0) = e_{n-k} - e_{n-k+1}$ = transpos of coords n - k, n - k + 1 $(1 \le k < n)$. $s_0 = \text{refl in } \alpha_0 = (0, \dots, 0, 1) = e_n$ = sign change of coord n. $s_k s_{k+1} = \text{rot by } \cos^{-1} \left(\frac{-1^4 - 1^4 - 1^4 + 1^4}{1^4 + 1^4 + 1^4} \right) = 2\pi/3$ $(1 \le k)$ $s_0 s_1 = \text{rotation by } \cos^{-1} \left(\frac{-1^4 + 1^4}{1^4 + 1^4} \right) = 2\pi/4$

Symmetry grp = permutations, sign changes of coords

$$= \langle s_0, \dots s_{n-1} \rangle / \langle s_k^2 = 1, (s_k s_{k+1})^3 = 1, (s_0 s_1)^4 = 1 \rangle$$

(0 \le j < n, 1 \le k < n - 1)

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 $P_n = \{x \in \mathbb{R}^n \mid -1 \le x_i \le 1 \mid (1 \le i \le n)\}.$ Choose flag $P_k = \{x \in P_n \mid x_1 = \cdots = x_{n-k} = 1\}$, ctr of mass $p_k = (1, ..., 1, 0, ..., 0)$ $(n - k \ 1s)$.

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Relations

David Vogan $P_n = \{x \in \mathbb{R}^n \mid -1 \le x_i \le 1 \mid (1 \le i \le n)\}.$ Choose flag $P_k = \{x \in P_n \mid x_1 = \cdots = x_{n-k} = 1\}$, ctr of mass $p_k = (1, ..., 1, 0, ..., 0)$ $(n - k \ 1s)$. $s_k = \text{refl in } \alpha_k = (0, \dots, 1, -1, \dots, 0) = e_{n-k} - e_{n-k+1}$ Relations = transpos of coords n - k, n - k + 1 ($1 \le k \le n$).

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Regular polyhedra

David Vogan $P_n = \{x \in \mathbb{R}^n \mid -1 \le x_i \le 1 \quad (1 \le i \le n)\}.$ Choose flag $P_k = \{x \in P_n \mid x_1 = \dots = x_{n-k} = 1\}$, ctr of mass $p_k = (1, ..., 1, 0, ..., 0)$ $(n - k \ 1s)$. $s_k = \text{refl in } \alpha_k = (0, \dots, 1, -1, \dots, 0) = e_{n-k} - e_{n-k+1}$ Relations = transpos of coords n - k, n - k + 1 ($1 \le k \le n$). $s_0 = \text{refl in } \alpha_0 = (0, \dots, 0, 1) = e_n$ = sign change of coord *n*.

Symmetry grp = permutations, sign changes of coords

$$= \langle s_0, \dots s_{n-1} \rangle / \langle s_k^2 = 1, (s_k s_{k+1})^3 = 1, (s_0 s_1)^4 = 1 \rangle$$

(0 \le j < n, 1 \le k < n - 1)

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$$\cos(\theta_{k+1}) = \left(\frac{-1 + r_k - r_{k+1} - r_k r_{k+1}}{1 + r_k + r_{k+1} + r_k r_{k+1}}\right).$$

When k = 0, some terms disappear:

 $\cos(\theta_1) = \frac{-1+r_1}{1+r_1}, \qquad r_1 = \frac{1+\cos(\theta_1)}{1-\cos(\theta_1)}.$

These recursion formulas give all r_k in terms of all θ_k . Next formula is

$$r_2 = -\frac{\cos(\theta_1) + \cos(\theta_2)}{1 + \cos(\theta_2)}.$$

Formula makes sense (defines strictly positive r_2) iff $\cos(\theta_1) + \cos(\theta_2) < 0$.

Regular polyhedra in *n* dimensions

David Vogan

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$$s_k^2 = 1$$
, $s_k s_{k'} = s_{k'} s_k (|k - k'| > 1)$, $(s_k s_{k+1})^{m_{k+1}} = 1$.
Here $m_{k+1} > 3$. Rotation angle for $s_k s_{k+1}$ must be

 $\theta_{k+1} = 2\pi/m_{k+1} \in \{120^\circ, 90^\circ, 72^\circ, 60^\circ \ldots\},\$

$$\cos(\theta_k) \in \left\{-\frac{1}{2}, \ 0, \ \frac{\sqrt{5}-1}{4}, \ \frac{1}{2}, \ldots\right\},$$

Group-theoretic information recorded in Coxeter graph

 $\underbrace{m_{n-1}}_{m_{n-2}} \underbrace{m_2}_{m_1} \underbrace{m_1}_{m_2}$

Recursion formulas give r_k from $\cos(\theta_k) = \cos(2\pi/m_k)$. Condition $\cos(\theta_2) + \cos(\theta_1) < 0$ says

one of m_{k+1} , m_k must be 3; other at most 5.

Regular polyhedra in *n* dimensions

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Regular polyhedra in *n* dimensions

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Classification

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Group-theoretic information recorded in Coxeter graph

 $\underbrace{m_{n-1}}_{\bullet} \underbrace{m_{n-2}}_{\bullet} \underbrace{m_2}_{\bullet} \underbrace{m_1}_{\bullet}$

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Finite Coxeter groups with one line

Same ideas lead (Coxeter) to classification of all graphs for which recursion gives positive r_k .

type	diagram	G	G	reg poly	
An	••-•	symm gp S_{n+1}	<i>n</i> !	<i>n</i> -simplex	
BCn	•_•··••	cube group	2 ⁿ ∙ n!	hypercube hyperoctahedron	Classificatio
<i>l</i> ₂ (<i>m</i>)	<i>m</i> ● <u></u>	dihedral gp	2 <i>m</i>	<i>m</i> -gon	
H_3	•••	H ₃	120	icosahedron dodecahedron	
H_4	••••	H_4	14400	600-cell 120-cell	
F_4	4 ●—● <u>—</u> ●—●	F ₄	1152	24-cell	
For much more, see Bill Casselman's amazing website					
http://www.math.ubc.ca/~cass/coxeter/crm.html					

Regular polyhedra

in *n* dimensions David Vogan