Geometry and representations of reductive groups

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Outline

Abstract harmonic analysis

Say Lie group G acts on manifold M. Can ask about

- topology of M
- solutions of G-invariant differential equations
- special functions on M (automorphic forms, etc.)

Method step 1: LINEARIZE. Replace M by Hilbert space $L^2(M)$. Now G acts by unitary operators.

Method step 2: DIAGONALIZE. Decompose $L^2(M)$ into minimal G-invariant subspaces.

Method step 3: REPRESENTATION THEORY. Study minimal pieces: irreducible unitary repns of G.

Difficult questions: how does DIAGONALIZE work, and what kind of minimal pieces do you get?

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Plan of talks

- Outline strategy for decomposing $L^2(M)$, by analogy with "double centralizers" in finite-dimensional algebra.
- Strategy $\rightsquigarrow$ **Kirillov-Kostant philosophy:**

  irreducible unitary representations of Lie group G

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  (nearly) symplectic manifolds with (nearly) transitive Hamiltonian action of G
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Decomposing a representation

Given ops on Hilbert space $\mathcal{H}$, want to decompose $\mathcal{H}$ in operator-invt way. Fin-diml theory:

$V/\mathbb{C}$ fin-diml, $\mathcal{A} \subset \mathrm{End}(V)$ cplx semisimple alg of ops.

Classical structure theorem:

$W_1, \ldots, W_r$ all simple $\mathcal{A}$-modules; then

$$\mathcal{A} \simeq \mathrm{End}(W_1) \times \cdots \times \mathrm{End}(W_r).$$

$$V \simeq m_1 W_1 + \cdots + m_r W_r.$$

Positive integer m_i is *multiplicity* of W_i in V.

Slicker version: define *multiplicity space* $M_i = \mathrm{Hom}_{\mathcal{A}}(W_i, V)$; then $m_i = \dim M_i$, and

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Commuting algebras and all that

$V/\mathbb{C}$ fin-diml, $\mathcal{A} \subset \mathrm{End}(V)$ cplx semisimple alg of ops.

Define

$$\mathcal{Z} = \mathrm{Cent}_{\mathrm{End}(V)}(\mathcal{A}),$$

a new semisimple algebra of operators on V.

Theorem

Say $\mathcal{A}$ and $\mathcal{Z}$ are complex semisimple algebras of operators on V as above.

1. $\mathcal{A} = \mathrm{Cent}_{\mathrm{End}(V)}(\mathcal{Z})$.
2. *There is a natural bijection between irr modules W_j for $\mathcal{A}$ and irr modules M_j for $\mathcal{Z}$, given by*

$$M_j = \mathrm{Hom}_{\mathcal{A}}(W_j, V), \qquad W_j = \mathrm{Hom}_{\mathcal{Z}}(M_j, V).$$

3. $V \simeq \sum_j M_j \otimes W_j$ *as a module for $\mathcal{A} \times \mathcal{Z}$.*

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Example of commuting algebras

G finite group, $V = L^2(G)$.

$\mathcal{A}$ = alg gen by left translations in $G \subset \mathrm{End}(V)$.

$\mathcal{A}$ is the group algebra of G.

$\mathcal{Z}$ = alg gen by right translations in $G \subset \mathrm{End}(V)$.

$\mathcal{Z}$ is *also* the group algebra of G.

Set of simple $\mathcal{A}$-modules is

$$\{W_i\} = \text{all irr reps of } G.$$

Set of simple $\mathcal{Z}$-modules is

$$\{M_i\} = \text{all irr reps of } G, \qquad M_i = W_i^*.$$

Decomposition of $L^2(G)$ is Peter-Weyl theorem:

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$GL(V)$ acts on nth tensor power $T^n(V)$: define

$$\mathcal{A} = \text{ends of } T^n(V) \text{ gen by } GL(V).$$

Quotient of group alg of $GL(V)$;

simple $\mathcal{A}$-mods $\{W_i\}$ = irr reps of $GL(V)$ on $T^n(V)$.

Symmetric group S_n also acts on $T^n(V)$: define

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Theorem (Schur-Weyl duality)

Algebras $\mathcal{A}$ and $\mathcal{Z}$ acting on $T^n(V)$ as mutual centralizers:

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Summands $\leftrightsquigarrow$ partitions of n into at most $\dim V$ *parts.*

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$$\mathcal{A} = \text{ends of } T^n(V) \text{ gen by } GL(V).$$

Quotient of group alg of $GL(V)$;

$$\text{simple } \mathcal{A}\text{-mods } \{W_i\} = \text{irr reps of } GL(V) \text{ on } T^n(V).$$

Symmetric group S_n also acts on $T^n(V)$: define

$$\mathcal{Z} = \text{ends of } T^n(V) \text{ gen by symm group } S_n.$$

Quotient of group alg of S_n;

$$\text{simple } \mathcal{Z}\text{-mods } \{M_i\} = \text{irr reps of } S_n \text{ on } T^n(V).$$

Theorem (Schur-Weyl duality)

Algebras $\mathcal{A}$ and $\mathcal{Z}$ acting on $T^n(V)$ as mutual centralizers:

$$T^n(V) = \sum M_i \otimes W_i.$$

Summands $\leftrightsquigarrow$ partitions of n into at most $\dim V$ *parts.*

Infinite-dimensional representations

Need framework to study ops on inf-diml V.

Finite-diml $\leftrightarrow$ infinite-diml dictionary

finite-diml V	$\leftrightarrow$	$C^\infty(M)$
repn of G on V	$\leftrightarrow$	action of G on M
$\mathrm{End}(V)$	$\leftrightarrow$	$\mathrm{Diff}(M)$
$\mathcal{A} = \mathrm{im}(\mathbb{C}[G]) \subset \mathrm{End}(V)$	$\leftrightarrow$	$\mathcal{A} = \mathrm{im}(U(\mathfrak{g})) \subset \mathrm{Diff}(M)$
$\mathcal{Z} = \mathrm{Cent}_{\mathrm{End}(V)}(\mathcal{A})$	$\leftrightarrow$	$\mathcal{Z}$ = diff ops comm with G

Which differential operators commute with G?

Answer $\rightsquigarrow$ generalizations of dictionary…

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Differential operators and symbols

$$\mathrm{Diff}_n(M) = \text{diff operators of order} \leq n.$$

Increasing filtration, $(\mathrm{Diff}_p)(\mathrm{Diff}_q) \subset \mathrm{Diff}_{p+q}$.

Theorem (Symbol calculus)

1. *There is an isomorphism of graded algebras*

$$\sigma\colon \mathrm{gr}\,\mathrm{Diff}(M) \to \mathrm{Poly}(T^*(M))$$

   to fns on $T^*(M)$ that are polynomial in fibers.

2. $$\sigma_n\colon \mathrm{Diff}_n(M)/\mathrm{Diff}_{n-1}(M) \to \mathrm{Poly}^n(T^*(M)).$$

3. *Commutator of diff ops $\rightsquigarrow$ Poisson bracket $\{,\}$ on $T^*(M)$: for $D \in \mathrm{Diff}_p(M)$, $D' \in \mathrm{Diff}_q(M)$,*

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Poisson structure and Lie group actions

X mfld w. Poisson $\{,\}$ on fns (e.g. $T^*(M)$).

Bracket with $f \rightsquigarrow \xi_f \in \text{Vect}(X)$: $\xi_f(g) = \{f, g\}$.

Vector fields ξ_f called *Hamiltonian*; flows preserve $\{,\}$. Map $f \mapsto \xi_f$ is Lie alg homomomorphism.

Lie group action on $X \rightsquigarrow$ Lie alg homom $Y \mapsto \xi_Y$ from Lie(G) to Vect(X).

Call X *Hamiltonian G-space* if given Lie alg homom $Y \mapsto f_Y$ from Lie(G) to $C^\infty(X)$ with $\xi_Y = \xi_{f_Y}$.

G acts on $M \rightsquigarrow T^*(M)$ is Hamiltonian G-space: Lie alg elt $Y \rightsquigarrow$ vec fld ξ_Y^M on $M \rightsquigarrow$ function f_Y on $T^*(M)$:

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Our story so far. . .

G acts on $M \leftrightsquigarrow T^*(M)$ Hamiltonian G-space.

G-decomp of $C^\infty(M) \leftrightsquigarrow (\text{Diff } M)^G$-modules.

$(\text{Diff } M)^G \overset{\sigma}{\leftrightsquigarrow} C^\infty(T^*(M))^G \leftrightsquigarrow C^\infty((T^*(M))//G)$.

Hope $C^\infty(M)$ irr $\Leftrightarrow$ G has dense orbit on $T^*(M)$.

Suggests generalization. . .

Hamiltonian G-cone $X \rightsquigarrow$ graded alg Poly(X).

Seek filtered alg $\mathcal{D}$, symbol calc gr $\mathcal{D} \overset{\sigma}{\to}$ Poly(X) carrying [,] on $\mathcal{D}$ to $\{,\}$ on Poly(X).

Seek to lift G action on Poly(X) to G action on $\mathcal{D}$ via Lie alg hom $\mathfrak{g} \to \mathcal{D}_1$.

Seek simple $\mathcal{D}$-module $\mathcal{W}$ (analogue of $C^\infty(M)$).

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$(\mathrm{Diff}\, M)^G \overset{\sigma}{\leftrightsquigarrow} C^\infty(T^*(M))^G \leftrightsquigarrow C^\infty((T^*(M))//G)$.

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Suggests generalization...

Hamiltonian G-cone $X \rightsquigarrow$ graded alg Poly(X).

Seek filtered alg $\mathcal{D}$, symbol calc gr $\mathcal{D} \xrightarrow{\sigma}$ Poly(X) carrying $[,]$ on $\mathcal{D}$ to $\{,\}$ on Poly(X).

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Vals of Hamiltonian vec flds $\rightsquigarrow$ integrable distn $\mathcal{S} \rightsquigarrow$ foliation of X by embedded submflds.

Symp form on $\mathcal{S}_x$: $\omega_x(\xi_f(x), \xi_g(x)) = \{f, g\}(x)$.

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Includes classification of symp homog spaces for G. (Riem homog spaces hopelessly complicated.)

Kirillov-Kostant philosophy of coadjt orbits suggests

$$\{\text{irr unitary reps of } G\} = \widehat{G} \leftrightsquigarrow \mathfrak{g}^*/G. \qquad (\star)$$

Bij $(\star)$ true for G simply conn nilp (Kirillov).

Other G: restr rt side to "admissible" orbits (integrality cond). Expect "almost all" of $\widehat{G}$: enough for interesting harmonic analysis.

Duflo: $(\star)$ for algebraic G reduces to reductive G.

Two ways to do repn theory:

1. start with coadjt orbit, look for repn. Hard.
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