Lie groups and representations

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Outline

What's representation theory about?

Abstract symmetry and groups

Representation theory

Examples of representations

Spherical harmonics

Locally symmetric spaces

The talk in one slide

Two topics. . .

1. GROUPS: an abstract way to study symmetry.
2. REPRESENTATIONS: linear algebra to study groups.

REPRESENTATIONS connect groups (which are hard) to linear algebra (which is easy).

Talk will be about three examples of all these things:

1. EVEN AND ODD FUNCTIONS.
2. SPHERICAL HARMONICS.
3. SHIMURA VARIETIES.

In (1), the group is $\{\pm 1\}$.

In (2), the group is $SO(3)$, rotations of space.

In (3), the group is $Sp(2n, \mathbb{R})$, invertible $2n \times 2n$ matrices preserving a symplectic form.

Two cheers for linear algebra

My favorite mathematics is linear algebra.

It is hard enough to describe interesting things.

It is easy enough to calculate with.

If you have a linear map $S\colon V \to V$ you can calculate the eigenvalues and eigenvectors of S.

First example: $V =$ functions on $\mathbb{R}$,

$$S = \text{change of variables } x \mapsto -x.$$

This means $S(x^3 - 2x^2 - 7x + 1) = -x^3 - 2x^2 + 7x + 1$.

The eigenvalues of S are $+1$ and -1.

Eigenspace for $+1$ is even functions (like $\cos(x)$, x^2).

Eigenspace for -1 is odd functions (like $\sin(x)$, x^3).

Linear algebra says: to study sign changes in x, write any function as even function plus odd function.

Translation and Fourier transform

Second example: $V =$ functions on $\mathbb{R}$, $T_t =$ translate by t.

Simultaneous eigenvectors of (commuting) linear maps T_t are multiples of $e^{i\lambda x}$: $T_t(e^{i\lambda x}) = e^{i\lambda(x-t)} = e^{-i\lambda t}e^{i\lambda x}$.

The exponential function $e^{i\lambda x}$ is an eigenvector of T_t with eigenvalue $e^{-i\lambda t}$.

Linear algebra says: to study translation in x, write

any function as "sum" of exponentials.

Fourier transform and Laplace transform do that: the "sum" is an integral.

The third cheer for linear algebra

Best part about linear algebra is noncommutativity...

Try to study both translation T_t and sign change S.

Problem: functions $e^{i\lambda x}$ are neither even nor odd.

Representation theory idea: look at smallest subspaces preserved by both S and T_t.

$$W_{\pm\lambda} = \mathrm{Span}\Big(\underbrace{e^{i\lambda x}, e^{-i\lambda x}}_{\text{eigenfunctions of } T_t}\Big) = \mathrm{Span}\Big(\underbrace{\cos(\lambda x)}_{\text{even}}, \underbrace{\sin(\lambda x)}_{\text{odd}}\Big).$$

These two bases of $W_{\pm\lambda}$ are good for different things.

First is convenient for solving differential equations.

Second is convenient for describing a vibrating string.

No one basis is good for everything.

What is essential is the two-dimensional space $W_{\pm\lambda}$.

Plan of the talk

Remind you of the definition of symmetry group.

Talk about continuous groups, called Lie groups.

Outline Gelfand program for using representation theory in any problem about groups.

Define group representation carefully.

Describe all representations for the simplest groups discussed so far (sign changes and translation on $\mathbb{R}$).

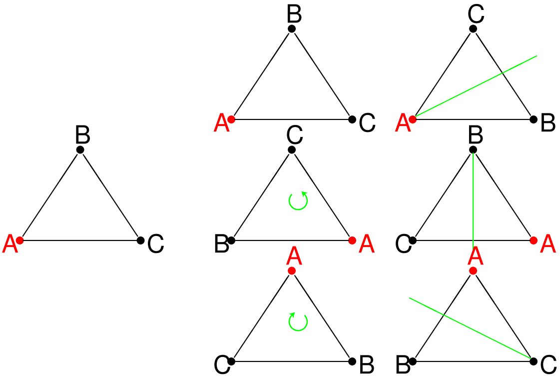
Talk about spherical harmonics: use representations to describe functions on the sphere $S^2 \subset \mathbb{R}^3$.

Talk about Shimura varieties: how representations can illuminate manifolds arising in number theory.

Symmetry group of a triangle

A basic idea in mathematics is symmetry.

A symmetry of X is a way of rearranging X so that nothing you care about changes.



The symmetry group of the triangle consists of these six rearrangements: nothing, two rotations, three reflections.

Symmetry group of $\mathbb{R}$

Suppose you care only about distance in $\mathbb{R}$.

What rearrangements of $\mathbb{R}$ preserve distance?

Once possibility is translation by t: $T_t(x) = x + t$.

Another is sign change: $S(x) = -x$.

Sign change is the same as reflection around 0.

This suggests reflection around s: $S_s(x) = 2s - x$.

Translations T_t and reflections S_s are all distance-preserving rearrangements of $\mathbb{R}$.

They make up the motion group of $\mathbb{R}$, $M(1) = \mathbb{R} \rtimes O(1)$.

The continuous families of symmetries T_t and S_s make $M(1)$ a Lie group.

Symmetry group of a vector space V.

Suppose V is a finite-dimensional real vector space.

This means we care about addition of vectors and scalar multiplication.

A symmetry of V is a rearrangement $T\colon V \to V$ respecting these two operations.

This means

1. $T\colon V \to V$ is invertible (rearrangement).
2. $T(v + w) = T(v) + T(w)$ $(v, w \in V)$ (respect addition).
3. $T(\lambda \cdot v) = \lambda \cdot T(v)$ $(v \in V, \lambda \in \mathbb{R})$ (respect scalar mult.).

That is, the group of symmetries of V is the group $GL(V)$ of invertible linear maps from V to V.

Since linear maps come in continuous families, $GL(V)$ is also a Lie group.

Approaching symmetry

Normal person's approach to symmetry:

1. look at something interesting;
2. find the symmetries.

Normal approach $\rightsquigarrow$ standard model in physics.

Explains everything that you can see without LIGO.

Mathematician's approach to symmetry:

1. find all multiplication tables for abstract groups;
2. pick an interesting abstract group;
3. find something it's the symmetry group of;
4. decide that something must be interesting.

Math approach $\rightsquigarrow$ Conway group (which has 8,315,553,613,086,720,000 elements) and Leech lattice (critical for packing 24-dimensional cannonballs).

Anyway, this is a math lecture. . .

How many Lie groups are there?: examples

Math approach to continuous symmetry:

how do you classify Lie groups?

Better to isolate part of the question:

how do you classify compact simple Lie groups?

Three infinite families of examples:

1. $O(n) = n \times n$ real orthogonal matrices
$= \mathbb{R}$-linear distance symmetries of $\mathbb{R}^n$
2. $U(n) = n \times n$ complex unitary matrices
$= \mathbb{C}$-linear distance symmetries of $\mathbb{C}^n$
3. $Sp(n) = n \times n$ quaternionic unitary matrices
$= \mathbb{H}$-linear distance symmetries of $\mathbb{H}^n$

These are compact (nearly) simple Lie groups:

$\dim O(n) = n(n-1)/2, \ \dim U(n) = n^2, \ \dim Sp(n) = 2n^2 + n.$

For $p + q = n$, $U(n)$ acts on Grassmannian manifold $M_{p,q}$.

$$\dim M_{p,q} = 2pq, \quad \chi(M_{p,q}) = \binom{n}{p}, \quad \sum_{p=0}^{n} \chi(M_{p,q}) = 2^n.$$

How many Lie groups are there?: classification

Found a compact almost simple Lie group $O(n, \mathbb{D})$ for each $n \geq 1$ and finite-dimensional division algebra $\mathbb{D}/\mathbb{R}$.

$$O(n, \mathbb{R}) = O(n) \qquad \text{dimension } n(n-1)/2$$

$$O(n, \mathbb{C}) = U(n) \qquad \text{dimension } n^2$$

$$O(n, \mathbb{H}) = Sp(n) \qquad \text{dimension } 2n^2 + n$$

Theorem (Cartan-Killing) With five exceptions, every compact simple Lie group appears above. Exceptions:

G_2	dim 14	F_4	dim 52
E_6	dim 78	E_7	dim 133
E_8	dim 248		

$G_2 \subset SO(7)$ acts on S^6; maybe related to unsolved problem
is S^6 is a complex manifold?

Look for interesting manifolds where these groups act.

E_8 acts on compact manifolds M_0, M_{112}, M_{128} of dims $0, 112, 128$.

$\chi(M_0) = 1$, $\chi(M_{112}) = 120$, $\chi(M_{128}) = 135$, $1 + 120 + 135 = 2^8$

G_2 acts on M_0, M_8, $\chi(M_0) = 1, \chi(M_8) = 3$, $1 + 3 = 2^2$.

Gelfand program. . .

. . . for using groups to do other math.

Say G is a group of symmetries of X.

Step 1: LINEARIZE. $X \rightsquigarrow V(X)$ vec space of fns on X. Now G acts on $V(X)$ by linear maps.

Step 2: DIAGONALIZE. Decompose $V(X)$ into minimal G-invariant subspaces.

Step 3: REPRESENTATION THEORY. Understand all ways that G can act by linear maps.

Step 4: GELFAND'S GREAT IDEA. Use understanding of $V(X)$ to answer questions about X.

One hard step is 3: how can G act by linear maps?

Definition of representation

G group; representation of G is

1. (complex) vector space V, and
2. collection of linear maps $\{\pi(g) \colon V \to V \mid g \in G\}$

subject to $\pi(g)\pi(h) = \pi(gh), \quad \pi(e) = \text{identity}$.

Reformulate: group homomorphism $\pi \colon G \to GL(V)$.

Subrepresentation is subspace $W \subset V$ such that

$$\pi(g)W = W \qquad (\text{all } g \in G).$$

Rep is irreducible if only subreps are $\{0\} \neq V$.

Irreducible subrepresentations are minimal nonzero subspaces of V preserved by all $\pi(g)$.

Infinite-dimensional complications

Linear algebra on infinite-dimensional spaces is harder.

For example, eigenvalues make sense, but there is no theorem saying every linear map has eigenvalues.

Functional analysis addresses these difficulties.

A Hilbert space is a complex vector space V with an inner product $\langle\ ,\ \rangle\colon V \times V \to \mathbb{C}$ so that

1. $\langle v, w\rangle = \overline{\langle w, v\rangle}, \quad \langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$
2. $\langle v, v\rangle \geq 0$, with equality only if $v = 0$.
3. The metric $d(v, w) = \langle v - w, v - w\rangle^{1/2}$ makes V a complete metric space.

A unitary representation of a topological group G is a representation (π, V) of G on a Hilbert space V, so that

1. The map $G \times V \to V, \quad (g, v) \mapsto \pi(g)v$ is continuous; and
2. π preserves the inner product: $\langle\pi(g)v, \pi(g)w\rangle = \langle v, w\rangle$.

A unitary representation (π, V) is irreducible if V has exactly two closed invariant subspaces.

Diagonalizing groups

Irreducible representations are a group-theory version of eigenspaces.

There's a theorem like eigenspace decomposition:

Theorem. Suppose G is a finite group.

1. $\exists$ finitely many irreducible reps $\tau_1, \ldots, \tau_\ell$ of G.
2. Any rep π of G is sum of copies of irreducibles:
$$\pi = n_\pi(\tau_1)\tau_1 + n_\pi(\tau_2)\tau_2 + \cdots + n_\pi(\tau_\ell)\tau_\ell.$$
3. Nonnegative integers $n_\pi(\tau)$ uniquely determined by π.
4. $|G| = (\dim \tau_1)^2 + \cdots + (\dim \tau_\ell)^2$.
5. G is abelian $\iff \dim \tau_j = 1$, all j.

Dimensions of irreducible representations
$\leftrightsquigarrow$ how non-abelian G is.

Extending this theorem to infinite groups encounters problems of infinite-dimensional linear algebra.

Diagonalizing infinite groups

Hilbert spaces and unitary representations address problems of infinite-dimensional linear algebra.

Theorem. If G is a separable locally compact group

1. $\exists$ nice measure space $\widehat{G}$ of irreducible unitary reps of G.
2. Unitary rep π of G is direct integral of copies of irrs:
$$\pi = \int_{\widehat{G}} n_\pi(\tau) d\mu_\pi(\tau).$$
3. Multiplicities $n_\pi(\tau)$, measure $d\mu_\pi(\tau)$ determined by π.
4. G is abelian $\iff$ $\dim \tau = 1$, all $\tau \in \widehat{G}$.

Making this theorem precise and true requires more functional analysis work.

Good reference is Dixmier's book *Les C^*-algèbres et leurs représentations*, translated to English as *C^*-algebras*.

Representations of $O(1) = \{\pm 1\}$

Now look at some irr unitary reps, see what they say about Gelfand's idea for understanding symmetry.

Start with the two-element group $O(1) = \{e, S\}$ of symmetries of $\mathbb{R}$; $S(x) = -x$.

Representation (π, V) of $O(1)$ is $\pi \colon O(1) \to GL(V)$.

Same thing: linear map $\pi(S) \colon V \to V$, $\quad \pi(S)^2 = I_V$.

Same thing: direct sum decomposition $V = V_+ \oplus V_-$ (± 1 eigenspaces of $\pi(S)$).

Two irr reps of $O(1)$: $(\tau_\pm, \mathbb{C})$, $\tau_\pm(S) = \pm 1$

Decomposition of any rep (π, V) as sum of irrs is $V = V_+ \oplus V_-$: $V_\pm =$ sum of copies of $\tau_\pm$.

Example: $V =$ fns on $\mathbb{R} \supset V_+ =$ even fns, $V_- =$ odd fns.

Gelfand idea: think about even and odd functions separately.

In computing $\int_{-1}^{1} f(x)dx$, case of odd f is easier.

Representations of the motion group of $\mathbb{R}$

Two kinds of distance-preserving symmetries of $\mathbb{R}$.

First kind is translation by t, $T_t(x) = x + t$.

Second kind is reflection around s: $S_s(x) = 2s - x$.

Union of two kinds is motion group $M(1)$.

By thinking about functions on $\mathbb{R}$, easily found two-dimensional reps $\tau_{\pm\lambda}$ of $M(1)$ on

$$W_{\pm\lambda} = \mathrm{Span}(e^{i\lambda x}, e^{-i\lambda x}) = \mathrm{Span}(\cos(\lambda x), \sin(\lambda x)) \quad (\lambda > 0)$$

Use $W_{\pm\lambda} \simeq \mathbb{C}^2$ from either basis: $\tau_{\pm\lambda}$ is irr unitary rep.

1-diml irr unitary reps τ_{+0}, τ_{-0}, $\tau_{\pm 0}(T_t) = 1$, $\tau_{\pm 0}(S_s) = \pm 1$.

$$\widehat{M(1)} = \{\tau_{\pm\lambda} \mid \lambda > 0\} \cup \{\tau_{+0}, \tau_{-0}\}$$

Topology/measure space structure: $\mathbb{R}_{>0} \cup$ double point.

Reps of $M(1)$ and functions on $\mathbb{R}$

We now know irr reps of motion group $M(1)$ for $\mathbb{R}$.

Gelfand idea: understand motions using fns on $\mathbb{R}$.

Reasonable choice: $L^2(\mathbb{R})$, Hilbert space of functions.

Get unitary rep of $M(1)$ on $L^2(\mathbb{R})$.

Decompose $L^2(\mathbb{R})$ into irr reps of $M(1)$:

Theorem (Plancherel) $L^2(\mathbb{R}) = \int_{\mathbb{R}_{>0}} V_{\pm\lambda}\, d\lambda$.

This is direct integral of (almost) all irr unitary reps of $M(1)$.

Explicitly: any function $f \in L^2(\mathbb{R})$ is

$$f(x) = \int_{\mathbb{R}_{>0}} [a_+(\lambda)e^{i\lambda x} + a_-(\lambda)e^{-i\lambda x}]\, d\lambda.$$

Here the Fourier transform of f is

$$\hat{f}(\xi) = \begin{cases} a_+(\xi) & (\xi > 0) \\ a_-(-\xi) & (\xi < 0). \end{cases}$$

Are 1-diml reps $\tau_{\pm 0}$ in Plancherel thm? is not well-posed: msre = 0.

Representations of $SO(3)$, part 1

$$\begin{aligned} SO(3) &= \text{rotations of } \mathbb{R}^3 \\ &= \text{orthogonal matrices of size 3, determinant one} \\ &= \{3 \times 3 \text{ real } g \mid \det(g) = 1, \quad g \cdot g^t = I_3\} \end{aligned}$$

Symmetries preserving origin and distance and orientation.

Have rep τ_1 of $SO(3)$ on three-dimensional space

$$V_1 = \mathbb{C}\text{-valued linear functions on } \mathbb{R}^3 = \mathrm{Span}(x, y, z).$$

The representation τ_1 is unitary and irreducible.

Similarly, get a natural representation σ_m on space

$$\begin{aligned} S^m &= \text{polynomials on } \mathbb{R}^3 \text{ homogeneous of degree } m \\ &= \mathrm{Span}(x^m, x^{m-1}y, \ldots, z^m). \end{aligned}$$

This representation has dimension $(m+1)(m+2)/2$.

Representations of $SO(3)$, part 2

$m \geq 2$: rep (σ_m, S^m) (polys of degree m) is not irreducible.

Has $SO(3)$-invariant subspace $(x^2 + y^2 + z^2)S^{m-2}$ of polynomials divisible by $(x^2 + y^2 + z^2)$.

Theorem. Let (σ_m, S^m) be the rep of $SO(3)$ on polynomials homogeneous of degree m. Write

$$r^2 = x^2 + y^2 + z^2, \qquad \Delta = (\partial/\partial x)^2 + (\partial/\partial y)^2 + (\partial/\partial z)^2.$$

1. $r^2 S^{m-2}$ is an $SO(3)$-invariant subspace of S^m.
2. The quotient representation τ_m of $SO(3)$ on $V_m = S^m/(r^2 S^{m-2})$ is irreducible, of dimension $2m + 1$.
3. The orthogonal complement of $r^2 S^{m-2}$ in S^m is
$$H^m = \{p \in S^m \mid \Delta p = 0\},$$
harmonic polys of degree m, also of dim $2m + 1$.
4. $\widehat{SO(3)}$ = irr unitary reps of $SO(3) = \{\tau_m \mid m \geq 0\}$

One irr rep τ_m for each odd dimension $2m + 1$.

Reps of $SO(3)$ and functions on S^2

$SO(3)$ is symmetries of the two-diml sphere S^2.

Gelfand idea: understand rotations with functions on S^2.

Get unitary rep of $SO(3)$ on $L^2(S^2)$.

Decompose $L^2(S^2)$ into irr reps of $SO(3)$:

Theorem. $L^2(\mathbb{S}^2) = \sum_{m \geq 0} V_m$.

$V_m \simeq H_m \simeq$ restrictions to S^2 of harmonic polys of degree m.

$$\begin{aligned} V_0 &= \mathbb{C} = \text{constant functions} \\ V_1 &= \text{Span}_{\mathbb{C}}(x, y, z) \\ V_2 &= \text{Span}_{\mathbb{C}}(xy, yz, xz, x^2 - y^2, y^2 - z^2) \\ &\vdots \end{aligned}$$

Theorem. V_m is the $m(m+1)$-eigenspace of Laplacian Δ_{S^2}.

Reason: Δ_{S^2} commutes with $SO(3)$, so eigenspaces $SO(3)$ reps.

Theorem helps solve Schrödinger equation for hydrogen.

What's an automorphic form?

Number theory is about integer solutions of polynomials.

Explicitly: $P_1, \ldots, P_m$ polys in $\xi_1, \ldots, \xi_n$ with $\mathbb{Z}$ coeffs.

Seek $\xi \in \mathbb{Z}^n$ satisfying $P_i(\xi) = 0$, all i.

Intersect alg surface $S(P) = \{x \in \mathbb{R}^n \mid P_i(x) = 0\}$ with $\mathbb{Z}^n$.

Coord-free way: (fixed $S(P) \subset \mathbb{R}^n$) $\cap$ (varying lattice $L \subset \mathbb{R}^n$).

Approximate definition: automorphic form on $GL(n)$ is a function on the space $Z(n) =$ lattices in $\mathbb{R}^n$.

$$GL(n, \mathbb{R})/GL(n, \mathbb{Z}) \simeq Z(n),$$

$$g \in GL(n, \mathbb{R}) \mapsto \text{lattice spanned by columns of } g.$$

$GL(n, \mathbb{R})$ acts by translation on functions on $Z(n)$.

$$\text{number theory} \rightsquigarrow \underbrace{\text{functions on } Z(n)}_{\text{automorphic forms}} \rightsquigarrow \underbrace{\text{reps of } GL(n, \mathbb{R})}_{\text{automorphic reps}}.$$

Gelfand, Langlands: reps of $GL(n, \mathbb{R})$ control aut forms.

Locally symmetric spaces

Interesting automorphic forms are (nearly) rotationally fixed.

So (nearly) functions on $X(n) = O(n)\backslash GL(n,\mathbb{R})/GL(n,\mathbb{Z})$.

HOW TO THINK ABOUT $X(n)$.

$$\begin{aligned} O(n)\backslash GL(n,\mathbb{R}) &\simeq \text{ positive definite quad forms in } n \text{ variables} \\ &\simeq \text{ positive definite symmetric } n\times n \text{ matrices} \\ &\simeq_{\log} \text{ all symmetric } n\times n \text{ matrices} \simeq \mathbb{R}^{n(n-1)/2}. \end{aligned}$$

Universal cover[1] of $X(n) = O(n)\backslash GL(n,\mathbb{R}) \simeq \mathbb{R}^{n(n-1)/2}$.

$\pi_1(X(n)) = GL(n,\mathbb{Z})$; $X(n)$ is an Eilenberg-MacLane space.

Consequence: topology of $X(n)$ is controlled by the (number-theoretic!) structure of the discrete group $GL(n,\mathbb{Z})$.

Basic number theory problem: understand de Rham cohomology $H^*(X(n),\mathbb{R})$.

Want to use representation theory to approach this problem.

[1] Torsion in $GL(n,\mathbb{Z})$ makes this statement slightly incorrect

Cohomology of locally symmetric spaces

$X(n) = O(n)\backslash GL(n,\mathbb{R})/GL(n,\mathbb{Z})$ locally symmetric space.

Cohom(X) $\leftrightsquigarrow$ derived functors(loc const functions on X).

For reps: functor $(\pi, V) \mapsto V^G$ has derived functors $H^i(G,\pi)$.

Theorem (Matsushima) If $X = K\backslash G/\Gamma$ is a compact locally symmetric manifold, then

$$H^i(X,\mathbb{R}) \simeq \sum_{\pi\in\widehat{G}} m_Z(\pi) \cdot H^i(G,\pi).$$

Here $m_Z(\pi)$ = mult of π in aut forms on $Z = G/\Gamma$.

$X(n)$ not smooth compact, so thm doesn't apply. But almost.

GELFAND: find irr reps π with $H^i(G,\pi) \neq 0$, then find $m_Z(\pi)$.

Irr unitary rep $\pi \in \widehat{G}$ with $H^i(G,\pi) \neq 0$ is cohomological.

Matsushima: cohom(loc symm X) $\leftrightsquigarrow$ cohom reps.

Come to question: what does a cohomological rep look like?

Cohomological representations. . .

. . . were described by V-Zuckerman (1984). Here are results for $G = GL(n, \mathbb{R})$ (Speh 1983), $n = 2m + 1$ (to simplify).

First example: trivial rep $(\tau, \mathbb{C})$.

By definition $H^0(G, \tau) = \mathbb{C}^G = \mathbb{C}$.

$H^i(G, \tau) \simeq H^i_{\text{deRham}}(O(n) \backslash U(n))$.

Need also $H^i(GL(p, \mathbb{C}), \tau) \simeq H^i_{\text{deRham}}(U(p))$.

Theorem (Speh 1983). Subgroups of $GL(2m + 1, \mathbb{R})$ which are centralizers of compact tori are

$$L = L(m_0, m_1, \dots, m_r) \simeq GL(2m_0 + 1, \mathbb{R}) \times \cdots \times GL(m_r, \mathbb{C}),$$

$m = m_0 + \cdots + m_r$. From each such subgroup one can construct an irreducible unitary rep $\pi(m_0, \dots, m_r)$ so that

$$H^{i+N}(G, \pi(m_0, \dots, m_r)) \simeq H^i(L, \tau(L))$$

$$\simeq \sum_{i_0 + \cdots + i_r = i} H^{i_0}_{\text{deRh}}(O(2m_0 + 1) \backslash U(2m_0 + 1)) \otimes \cdots \otimes H^{i_r}_{\text{deRh}}(U(m_r)).$$

Here $N = (1/2)(\dim(G/L) - \dim(K/L \cap K))$, a shift making the right side satisfy Poincaré duality.

Summary of automorphic ideas

Number theory for $G \rightsquigarrow$ locally symm $K\backslash G/\Gamma$.

Topology of $K\backslash G/\Gamma \leftrightsquigarrow$ cohomological reps of G.

cohom rep of $G \leftrightsquigarrow L = \mathrm{Cent}_G(T)$ (T compact torus).

$H^*(G, \pi(L)) = H^*_{\text{deRham}}(U_L/L \cap K)$.

$H^*_{\text{deRham}}(U_L/L \cap K)$ understood by Lie group theory.

Examples of $U_L/L \cap K$: E_8 manifolds M_0, M_{112}, M_{128} of Euler characteristics 1, 120, 135 summing to 2^8.

These worlds are full of interesting facts that we understand only a little.