The unitary dual problem

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Outline

Slides at `http://www-math.mit.edu/~dav/paper.html`

Gelfand's abstract harmonic analysis

Topological grp G acts on X, have questions about X.

Step 1. Attach to X Hilbert space $\mathcal{H}$ (e.g. $L^2(X)$).
Questions about $X \rightsquigarrow$ questions about $\mathcal{H}$.

Step 2. Find finest G-eqvt decomp $\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha$.
Questions about $\mathcal{H} \rightsquigarrow$ questions about each $\mathcal{H}_\alpha$.

Each $\mathcal{H}_\alpha$ is irreducible unitary representation of G: indecomposable action of G on a Hilbert space.

Step 3. Understand $\widehat{G} =$ all irreducible unitary representations of G: **unitary dual problem**.

Step 4. Answers about irr reps $\rightsquigarrow$ answers about X.

Today: $\widehat{G}$ for reductive Lie group G.

Why reductive G?

If $N \triangleleft G$, then $\widehat{G} \approx \widehat{N} \times \widehat{G/N}$ (Mackey...).

Example of Gelfand's program

$G = SL(2,\mathbb{R})$ acts on unit disc in $\mathbb{R}^2$; seek to understand/decompose V = functions on disc.

hyperbolic Laplacian Δ_h commutes with G.

Irr reps of G on functions = eigenspaces of Δ_h:

$$V_\lambda = \{v \in V \mid \Delta_h v = \lambda v\}.$$

Familiar case: $V_0 =$ harmonic functions on disc.

(harmonic fns) $\overset{\text{bdry values}}{\longleftrightarrow}$ fns on unit circle.

I can't do analysis: to make this true, replace fns by hyperfns.

General $\lambda \rightsquigarrow$ line bdle $\mathcal{L}_\lambda$ over unit circle.

$V_\lambda \overset{\text{bdry values}}{\longleftrightarrow}$ sections of $\mathcal{L}_\lambda$ on unit circle.

Conclusion: $L^2(\text{disc}) = \int_\lambda L^2(\text{circle}, \mathcal{L}_\lambda)$.

What's a unitary representation look like?

Gelfand's program says: to understand general action of G on X, write fns on X as "direct sum" of irreducible unitary representations.

In $SL(2, \mathbb{R})$ example, decomposed

functions on big space X (disc)

into pieces

secs of bdles on small space Y (*circle*).

This is approximately the general story.

First question: what are these small spaces Y?

Desideratum: fns on Y is nearly irr rep of G.

Introduction, concluded

Plan for today: focus on the question what are the nice small homogeneous spaces for G?

Reason to pick that small topic is that

1. the answer (spoiler alert: partial flag varieties) matters for lots of math, and
2. some aspects (spoiler alert: complex flag varieties) are not so familiar.

I'll include some long lists of unitary representations.

(Actually, just lists of people who made lists of unitary. . .)

Which nice small homog spaces?

This is a wonderful topic; I could talk for hours.

Here's the answer...

If $G(\mathbb{C}) =$ cplx alg grp, then $P(\mathbb{C}) \subset G(\mathbb{C})$ is parabolic if $G(\mathbb{C})/P(\mathbb{C})$ is compact.

If $G(\mathbb{R}) =$ real alg grp, a parabolic subgrp is real points $P(\mathbb{R})$ of complex parabolic defined $/\mathbb{R}$.

General structure theory: parabolic $P \subset G$ has unipotent radical $U \lhd P$; quotient $P/U =$ Levi quotient is a smaller reductive alg group.

General structure theory continued: algebraic G has finitely many conj classes of parabolic subgroups.

Examples of parabolic subgroups I

Example: V vec space; partial flag in V is subspaces

$$\mathcal{F} = \{0 = V_0 \subset V_1 \subset \cdots \subset V_m = V\}.$$

Any parabolic in $GL(V)$ has the form

$$P(\mathcal{F}) = \{g \in GL(V) \mid g \cdot V_i = V_i \quad (1 \le i \le m)\}$$

$$U(\mathcal{F}) = \text{unipotent radical}$$

$$= \{u \in P(\mathcal{F}) \mid u \cdot v \in v + V_{i-1} \quad (v \in V_i)\}$$

$$L(\mathcal{F}) = \text{Levi quotient} = P/U \simeq \prod_{i=1}^{m} GL(V_i/V_{i-1}).$$

Conjugacy classes of parabolic subgroups of $GL(V)$ are compositions of $n = \dim V$:

$$d_i = \dim(V_i/V_{i-1}) \quad (1 \le i \le m), \qquad \sum d_i = \dim V.$$

Levi quotient for $GL(V)$ is product of smaller GL.

Examples of parabolic subgroups II

Example: $(V, \langle,\rangle)$ orth space; isotropic flag in V is

$$\mathcal{I} = \{0 = V_0 \subset V_1 \subset \cdots \subset V_m \subset V_m^\perp \subset \cdots \subset V_1^\perp \subset V_0^\perp = V\}.$$

So subspaces V_i are isotropic, $V_i^\perp$ are coisotropic.
Any parabolic in $O(V)$ has the form

$$P(\mathcal{I}) = \{g \in O(V) \mid g \cdot V_i = V_i \quad (1 \le i \le m)\}$$

$$U(\mathcal{I}) = \text{unipotent radical}$$

$$= \{u \in P(\mathcal{I}) \mid u \cdot v \in v + V_{i-1} \quad (v \in V_i)\}$$

$$L(\mathcal{I}) = \text{Levi quot} = P/U \simeq O(V_m^\perp/V_m) \times \prod_{i=1}^{m} GL(V_i/V_{i-1}).$$

Conjugacy classes of parabolic subgroups of $O(V)$ are compositions of $r \le R = \dim(\text{max isotropic subspace})$:

$$r_i = \dim(V_i/V_{i-1}) \quad (1 \le i \le m), \qquad \sum r_i = \dim(V_m) \le R.$$

Levi quotient is prod of smaller GL, one smaller orth grp.

General nature of parabolic subgroups

Each real reductive G has finite number $\{P_1, \ldots, P_N\}$ of conj classes of real parabolic subgroups.

Each P_j has unip radical U_j, Levi quotient $L_j = P_j/U_j$.

Each L_j is nearly direct product of factors $GL(V_{j,\ell})$ and one complicated simple group $L_{j,0}$.

Each $\widehat{L_j}$ is nearly direct product of $\widehat{GL(V_{j,\ell})}$ and $\widehat{L_{j,0}}$.

What's a unitary representation look like?

Said that irr unitary reps are often sections of bundles on nice small homogeneous spaces.

Nice small homog spaces are G/P, P parabolic.

Equivariant Hilbert bdle on G/H is unitary rep (π_H, W_H) of H.

Conclusion: many irr unitary of reductive G are (π_G, W_G), $W_G =$ sections of Hilbert bundle

$$G \times_P W_L \to G/P, \qquad (\pi_L, W_L) \text{ irr of } L = P/U.$$

Mackey notation: $\pi_G = \mathrm{Ind}_P^G(\pi_L)$.

Big picture: for each maximal $P \subsetneq G$, $L = P/U$ Levi quo, get approximately an embedding

$$\widehat{L} \hookrightarrow \widehat{G}, \qquad \pi_L \mapsto \mathrm{Ind}_P^G(\pi_L)$$

Parametrizing this part of $\widehat{G}$ is easy: it's just $\widehat{L}$.

Understanding these reps of $G \leftrightsquigarrow$ understanding G/P.

Computing the unitary dual

Levi subgrp L in simple G approximately product of groups $GL(V_\ell)$ and at most one non-GL simple factor S.

Vector space V_ℓ can be real, complex, or quaternionic.

To parametrize $\widehat{L}$, must therefore

1. parametrize $\widehat{GL(m,\mathbb{F})}$ ($m \geq 1$, $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.
2. parametrize $\widehat{S}$ for all other simple S.

Here is early progress on the first question.

1. (1800s) Irr of $GL(1,\mathbb{R}) = \mathbb{R}^x$ is unitary char
$$t \mapsto |t|^{i\nu}\,\mathrm{sgn}(t)^\epsilon, \quad (\nu \in \mathbb{R}, \quad \epsilon \in \mathbb{Z}/2\mathbb{Z}).$$
2. (1800s) Irr of $GL(1,\mathbb{C}) = \mathbb{C}^x$ is unitary char
$$re^{i\theta} \mapsto r^{i\nu} e^{ie\theta}, \quad (\nu \in \mathbb{R}, \quad e \in \mathbb{Z}).$$
3. (1920s) Irr of $GL(1,\mathbb{H}) = \mathbb{H}^\times$ is
$$q \mapsto |q|^{i\nu} \cdot \xi_m(q), \quad (\nu \in \mathbb{R}, \quad m \in \mathbb{N}).$$
Here $\xi_m =$ irr m-diml rep of $\mathbb{H}^\times$.

Reps ξ_m in (3) (found by Hermann Weyl and Elie Cartan are a sign of trouble: more complicated than the unitary chars in (1) and (2), but not obtained by Mackey induction.

Computing unitary dual of $GL(m, \mathbb{F})$: begin

More progress on unitary dual of $GL(m, \mathbb{F})$:

1. (1947: Gelfand-Naimark) Unitary dual of $GL(2, \mathbb{C})$ is
   1.1 principal series induced from unitary chars of Levi quotient $\mathbb{C}^x \times \mathbb{C}^\times$ of minimal parabolic.
   1.2 complementary series indexed by $(0,1) \times \mathbb{Z}$
   1.3 one-diml unitary chars $g \mapsto |\det(g)|^{i\nu}(\det(g)/|\det(g)|)^e \quad (\nu \in \mathbb{R}, \quad e \in \mathbb{Z})$.
2. (1947: Bargmann) Unitary dual of $GL(2, \mathbb{R})$ is
   2.1 principal series induced from unitary chars of Levi quotient $\mathbb{R}^x \times \mathbb{R}^\times$ of minimal parabolic.
   2.2 relative discrete series indexed by unitary chars of $\mathbb{C}^\times$
   2.3 complementary series indexed by $(0,1) \times \mathbb{Z}/2\mathbb{Z}$
   2.4 one-diml unitary chars $g \mapsto |\det(g)|^{i\nu} \operatorname{sgn}(\det(g))^\epsilon \quad (\nu \in \mathbb{R}, \quad \epsilon \in \mathbb{Z}/2\mathbb{Z})$.
3. (Hirai (1962), Thieleker (1974)) Unitary dual of $GL(2, \mathbb{H})$: principal series, comp series, one-diml unitary chars.

Integer e in (2.2) ($re^{i\theta} \mapsto r^{i\nu}e^{ie\theta}$) is trouble: complicated reps not obtained by Mackey induction.

Complementary series in (1.2), (2.3), and (3) are trouble.

Computing unitary dual of $GL(m, \mathbb{F})$: continue

1948-1972: unitary dual of $GL(2, \mathbb{F})$ known. More...

1. (1950: Gelfand-Naimark): claimed to find $\widehat{G}$ for $G = GL(n, \mathbb{C}),\ Sp(2n, \mathbb{C}),\ SO(n, \mathbb{C})$.
2. (1967: Stein): Showed Gelfand-Naimark list for $GL(4, \mathbb{C})$ was shorter than for $SO(6, \mathbb{C})$, although groups are locally isomorphic. Beginning there, Stein found unitary reps of $GL(2n, \mathbb{C})$ missing from Gelfand-Naimark list for all $n \geq 2$.
3. (1967: Stein): Stein's list of unitary reps still far from complete.
4. (1986: Tadić, Vogan): proved Stein list $\widehat{GL(n, \mathbb{C})}$ was complete.
5. (1986: Tadić, Vogan): calculated $\widehat{GL(n, \mathbb{R})}$.
6. (1986: Vogan): calculated $\widehat{GL(n, \mathbb{H})}$.

All results have small trouble: complementary series.

All results except $\mathbb{F} = \mathbb{C}$ include big trouble: big families of reps not obtained by Mackey induction.

Unitary duals of other groups

(Thomas (1941), Dixmier (1961)): $SO(4,1)$.

(Takahashi (1963), Thieleker (1974)): $SO(n,1)$.

(Kraljević 1973): $SU(n,1)$

(Duflo 1979): $Sp(4,\mathbb{C}), G_2(\mathbb{C})$

(Baldoni Silva 1981): $Sp(n,1)$

(Baldoni Silva-Barbasch 1983): rank one F_4

(Barbasch 1989): all classical complex groups

(Vogan 1994): $G_2(\mathbb{R})$

This is slow progress, and there is a long distance to go.

All the answers exhibit small trouble as for $GL(n,\mathbb{F})$: complementary series.

Almost all the answers exhibit big trouble as for $GL(n,\mathbb{F})$: nice series of representations not obtained by Mackey induction.

Next topic: understanding some of the trouble.

Which nice small homog spaces? reprise

The list of real parabolic subgrps of reductive alg G is too short.

Induction gives principal series of $GL(2,\mathbb{R}) \leftrightsquigarrow \widehat{\mathbb{R}^\times \times \mathbb{R}^\times}$;
but omits relative discrete series $\leftrightsquigarrow \widehat{\mathbb{C}^\times}$

If $G(\mathbb{R}) = G(\mathbb{C})^\sigma$ = real alg grp, then θ-stable parabolic is by def cplx parabolic $Q(\mathbb{C}) \subset G(\mathbb{C})$ with $\sigma(Q)$ opposite to Q.

Follows that $Q \cap \sigma(Q) =_{\text{def}} L$ θ-Levi subgroup of Q is reductive subgp of G, defined$/\mathbb{R}$, isomorphic to Levi quotient Q/U.

General structure theory: θ-stable parabolic $Q \subset G \rightsquigarrow$ complex structure on $G(\mathbb{R})/L(\mathbb{R})$.

Reason: $Q(\mathbb{C}) \cap G(\mathbb{R}) = L(\mathbb{R})$, so $G(\mathbb{R})/L(\mathbb{R}) \hookrightarrow G(\mathbb{C})/Q(\mathbb{C})$ open.

Previous idea

most unitary irreducible reps are sections of bundles on nice small homogeneous spaces...

should be supplemented

...or holomorphic sections of holomorphic bundles on nice small holomorphic homogeneous spaces.

Examples of θ-stable parabolic subgroups

Example: V real; θ-stable flag in V is cplx subspaces

$$\mathcal{F}_\theta = \{0 = V_{0,\theta} \subset V_{1,\theta} \subset \cdots \subset V_{m,\theta} \subset W_{m,\theta} \subset \cdots \subset W_{0,\theta} = V(\mathbb{C})\}$$

subject to (being opposite to complex conjugate)

$$V_{i,\theta} \oplus \overline{W_{i,\theta}} = V(\mathbb{C}) \qquad (0 \le i \le m).$$

θ-stable flag $\leftrightsquigarrow$ direct sum decomp

$$V(\mathbb{C}) = E_{0,\theta}(\mathbb{C}) \oplus \sum_{i=1}^{m} E_{i,\theta} \oplus \overline{E_{i,\theta}}, \qquad E_0(\mathbb{C}) = W_{m,\theta} \cap \overline{W_{m,\theta}}.$$

by means of

$$E_{i,\theta} = V_{i,\theta} \cap \overline{W_{i-1,\theta}}, \qquad E_{0,\theta}, \qquad E_{0,\theta}(\mathbb{C}) = W_{m,\theta} \cap \overline{W_{m,\theta}}.$$

Any θ-stable parabolic in $GL(V)$ is $Q(\mathcal{F}_\theta)$

$$L(\mathcal{F}_\theta)(\mathbb{R}) = \theta\text{-Levi subgroup} \simeq GL(E_{0,\theta}(\mathbb{R})) \times \prod_{i=1}^{m} GL(E_{i,\theta}(\mathbb{C})).$$

$GL(V)$-conjugacy class of $Q(\mathcal{F}_\theta)$ given by

$$d_0 = \dim_{\mathbb{R}}(E_{0,\theta}(\mathbb{R})), \quad d_i = \dim_{\mathbb{C}}(E_{i,\theta}), \quad d_0 + 2\sum d_i = \dim_{\mathbb{R}}(V).$$

Examples of θ-stable parabolic subgroups II

Example: $(V, \langle,\rangle)$ real orth space; θ-isotropic flag in V is cplx subspaces

$$\mathcal{I}_\theta = \left\{0 = V_{0,\theta} \subset V_{1,\theta} \subset \cdots \subset V_{m,\theta} \subset V_{m,\theta}^\perp \subset \cdots \subset V_{1,\theta}^\perp \subset V_{0,\theta}^\perp = V(\mathbb{C})\right\}$$

subject to

$$V_{i.\theta} \oplus \overline{V_{i,\theta}^\perp} = V(\mathbb{C}).$$

θ-isotropic flag $\leftrightsquigarrow$ direct sum decomposition

$$V(\mathbb{C}) = E_{0,\theta}(\mathbb{C}) \oplus \sum_{i=1}^m E_{i,\theta} \oplus \overline{E_{i,\theta}}$$

by means of

$$E_{i,\theta} = V_{i,\theta} \cap \overline{V_{i-1,\theta}^\perp} \qquad E_{0,\theta}(\mathbb{C}) = V_{m,\theta}^\perp \cap \overline{V_{m,\theta}^\perp}.$$

$\langle,\rangle$ induces on $E_{0,\theta}(\mathbb{R})$ nondeg orth form $\langle,\rangle_0$, say sig (p_0, q_0).

$\langle,\rangle$ induces on $E_{i,\theta}$ nondeg herm form $\langle,\rangle_i$, say of sig (p_i, q_i).

If V has signature (p, q), then $(p, q) = (p_0 + 2\sum_i p_i, q_0 + 2\sum_i q_i)$.

θ-Levi subgp is $O(p_0, q_0) \times \prod_j U(p_j, q_j)$.

θ-stable parabolics in general

Each real reductive $G(\mathbb{R})$ has finite number $\{Q_1, \ldots, Q_M\}$ of conj classes of θ-stable parabolic subgroups.

Each Q_j has θ-Levi subgroup $L_j = Q_j \cap \overline{Q_j}$.

$L_j(\mathbb{R})$ is nearly direct product of factors $GL(V_{j,x})$, $U(W_{j,y})$, and (G simple) one complicated simple factor $L_{j,0}$.

Each $\widehat{L_j}$ is nearly direct product of $\widehat{GL(V_{j,x})}$, $\widehat{U(W_{j,y})}$, $\widehat{L_{j,0}}$.

What's a unitary rep look like? ($\mathbb{C}$ version)

Said that irr unitary reps can also be secs of holom bundles on small cplx homog spaces.

Nice cplx homog spaces are $G(\mathbb{R})/L(\mathbb{R})$, Q θ-stable.

Eqvt Hilbert bdle on $G(\mathbb{R})/L(\mathbb{R})$ is unitary rep (π_L, W_L) of $L(\mathbb{R})$.

Approximately: many irr unitary of reductive $G(\mathbb{R})$ are $(\pi_G(\mathbb{R}), W_G(\mathbb{R}))$, $W_G(\mathbb{R})$ = holom sections of Hilbert bundle

$$G(\mathbb{R}) \times_{L(\mathbb{R})} W_L \to G(\mathbb{R})/L(\mathbb{R}), \qquad (\pi_L, W_L) \text{ irr of } L(\mathbb{R}).$$

Details are painful: holomorphic bundles often have no sections, so need to replace sections by Dolbeault cohomology.

Idea of Kostant, Langlands, Schmid, put in final form by Zuckerman.

Getting unitary cohomology requires positivity hypotheses on π_L.

Idea of Kostant, Langlands, Schmid, final form by Zuckerman, DV.

Zuckerman notation: $\pi_G(\mathbb{R}) = \mathcal{R}_{\mathfrak{q}}^G(\mathbb{R})(\pi_L)$.

Big picture: for each θ-stable $Q \subsetneq G$, $L = \theta$-Levi, get approximately an embedding

$$\widehat{L}(\mathbb{R}) \hookrightarrow \widehat{G}(\mathbb{R}), \qquad \pi_L \mapsto \mathcal{R}_{\mathfrak{q}}^G(\mathbb{R})(\pi_L)$$

Understanding these reps of $G(\mathbb{R})$ $\leftsquigarrow$ understanding $G(\mathbb{R})/L(\mathbb{R})$.

Stopping in the middle

Cohomological induction from θ-stable parabolics explains more of the unitary dual calculations mentioned earlier.

$GL(2,\mathbb{R})$ example: relative discrete series all arise by cohomological induction from (unique) proper θ-stable parabolic, $L(\mathbb{R}) = GL(1,\mathbb{C})$.

For general real reductive G, after using induction from real and θ-stable proper parabolic subgroups, we are still missing three things to know $\widehat{G}$:

1. description and proof of unitarity of finitely many unipotent representations
2. description of all deformations of unipotent reps
3. proof that all other admissible reps are nonunitary.

Most of (1) is Arthur's special unipotent representations.

Fully defined by Adams-Barbasch-V (1992).

Unitarity proved for most G classical by Arthur(2013), and for all G exceptional by Adams-van Leeuwen-Miller-V using `atlas` software.

Parts (2) and (3) are hard, and not yet done in general.

Thank you…

…for the invitation

…for the perfect weather

did you notice the use of blue there?

…for the spectacular location

…for great mathematics (with parallel processing)

…for friends old and new (with parallel processing)