Introduction

Compact groups

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The translation principle

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Outline

Introduction

Structure of compact groups

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Root data and representations of compact groups

The translation principle

Weyl group action on virtual reps

What the translation principle can do for you

The big idea

(finite-diml) irr reps of K

$$\uparrow$$

compact conn Lie gp K
max torus T, $X^* = \widehat{T}$

$\downarrow$ cplxify

cplx reductive alg gp G
maximal cplx torus H

$\downarrow$ real form

real red alg gp $G(\mathbb{R})$
real max torus $H(\mathbb{R})$

$$\downarrow$$

(inf-diml) irr reps of $G(\mathbb{R})$

Theorem (Cartan-Weyl)

$$\widehat{K} \leftrightsquigarrow X^*/W.$$

$$\dim(E(\lambda)) = \text{poly in } \lambda.$$

compact Lie group
$\updownarrow$
root datum
$\downarrow$
lattice X^*, Weyl gp W

Theorem (Zuckerman)
Irred repns of $G(\mathbb{R})$ appear in "translation families" $\pi(\lambda)$ ($\lambda \in X^*$). Some invts of $\pi(\lambda)$ are poly fns of λ.

Root data: abelian case

Torus (compact conn ABELIAN Lie gp) $\rightsquigarrow$

$$X^*(T) = \text{Hom}(T, \text{circle}) = \text{lattice of characters},$$

$$X_*(T) = \text{Hom}(\text{circle}, T) = \text{lattice of 1-param subgps}.$$

$$\text{Hom}(\text{circle}, \text{circle}) \simeq \mathbb{Z}, \qquad (z \mapsto z^n) \leftrightsquigarrow n.$$

$\lambda \in X^*(T),\ \xi \in X_*(T) \rightsquigarrow \langle \lambda, \xi \rangle = \lambda \circ \xi \in \text{Hom}(\text{circle}, \text{circle}) \simeq \mathbb{Z};$

Pairing identifies two lattices as dual:

$$X_*(T) = \text{Hom}_{\mathbb{Z}}(X^*(T), \mathbb{Z}), \quad X^*(T) = \text{Hom}_{\mathbb{Z}}(X_*(T), \mathbb{Z}).$$

Can recover T from (either) lattice:

$$T \simeq \text{Hom}_{\mathbb{Z}}(X^*(T), \text{circle}), \qquad t \mapsto [\lambda \mapsto \lambda(t)],$$

$$X_*(T) \otimes_{\mathbb{Z}} \text{circle} \simeq T, \qquad \xi \otimes z \mapsto \xi(z).$$

Functor $T \to X^*(T)$ is contravariant equiv of categories from compact conn abelian Lie groups to lattices.

Functor $T \to X_*(T)$ is covariant equiv of categories from compact conn abelian Lie groups to lattices.

Root data: torus representations

T torus, $X^*(T) = \mathrm{Hom}(T, \text{circle})$ char lattice.

$\mathbb{C}_\lambda$ = 1-diml rep of T, $t \rightsquigarrow$ mult by $\lambda(t)$.

V any rep of T; define

$$V^\lambda = \mathrm{Hom}_T(\mathbb{C}_\lambda, V), \quad V_\lambda = \{v \in V \mid t \cdot v = \lambda(t)v\}.$$

$$V^\lambda \otimes \mathbb{C}_\lambda \simeq V_\lambda, \qquad \phi \otimes z \mapsto \phi(z).$$

$$V_T = \sum_{\lambda \in X^*(T)} V_\lambda = \sum_\lambda V^\lambda \otimes \mathbb{C}_\lambda.$$

Multiplicity of λ in V $=_{\text{def}} \dim V^\lambda = m_V(\lambda)$.

Weights of V $=_{\text{def}} \Delta(V, T) = \{\lambda \in X^*(T) \mid V_\lambda \neq 0\}$.

Rep ring of T $=_{\text{def}}$ Groth gp of fin-diml reps of T

$=$ gp ring of $X^*(T) = \mathbb{Z}[X^*(T)]$,

$$[V] \rightsquigarrow \sum_\lambda m_V(\lambda)\lambda$$

Root data: compact groups

Compact conn Lie gp K; fix max torus T.

$$X^*(T) = \mathrm{Hom}(T, \text{circle}) = \text{lattice of characters},$$

$$X_*(T) = \mathrm{Hom}(\text{circle}, T) = \text{lattice of 1-param subgps}.$$

$\mathfrak{k} = \mathrm{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C}$ cplxified Lie algebra; K acts by Ad.

Roots of T in $\mathfrak{k}$ are the nonzero weights of Ad:

$$R(\mathfrak{k}, T) = \{\alpha \in X^*(T) - 0 \mid \mathfrak{k}_\alpha \neq 0\} \subset X^*(T).$$

Coroots of T in $\mathfrak{k}$ $\subset X_*(T)$ defined later, in bij with roots:

$$X^*(T) \supset R(\mathfrak{k}, T) \leftrightarrow R^\vee(\mathfrak{k}, T) \subset X_*(T), \quad \alpha \leftrightarrow \alpha^\vee.$$

Root α defines *root reflection*

$$s_\alpha \in \mathrm{Aut}(X^*(T)), \qquad s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha.$$

Weyl group $W(K, T)$ $\subset \mathrm{Aut}(X^*(T))$ is gp gen by all s_α.

Root datum of K is $(X^*, R, X_*, R^\vee)$: lattice, finite subset, dual lattice, finite subset in bijection.

Example of root data: $SU(2)$

$$K = SU(2) \supset \left\{ t_\theta =_{\text{def}} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in \mathbb{R} \right\} = T,$$

$$X^*(T) \simeq \mathbb{Z},\ \lambda_n(t_\theta) = e^{in\theta}; \quad X_*(T) \simeq \mathbb{Z},\ \xi_m(e^{i\theta}) = t_{m\theta}.$$

$$\mathfrak{k} = \mathfrak{sl}(2) = 2 \times 2 \text{ cplx matrices of trace } 0.$$

$$\mathfrak{k}_2 = \left\{ \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \right\},\ \mathfrak{k}_0 = \left\{ \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} \right\},\ \mathfrak{k}_{-2} = \left\{ \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \right\}.$$

$$R(\mathfrak{k}, T) = \{\pm 2\}, \qquad R^\vee(\mathfrak{k}, T) = \{\pm 1\}.$$

Root reflection is

$$s_2(n) = n - \langle n, 1 \rangle 2 = n - 2n = -n,$$

so $W(K, T) = \{\pm \mathrm{Id}\}$.

Example of root data: $SO(3)$

$$K = SO(3) \supset \left\{ t_\phi =_{\text{def}} \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \phi \in \mathbb{R} \right\} = T,$$

$$X^*(T) \simeq \mathbb{Z},\ \lambda_n(t_\phi) = e^{in\phi};\ \ X_*(T) \simeq \mathbb{Z},\ \xi_m(e^{i\phi}) = t_{m\phi}.$$

$$\mathfrak{k} = \mathfrak{so}(3) = 3 \times 3 \text{ cplx skew-symm matrices.}$$

$$\mathfrak{k}_{\pm 1} = \left\{ \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & \pm ri \\ -r & \mp ri & 0 \end{pmatrix} \right\},\ \ \mathfrak{k}_0 = \left\{ \begin{pmatrix} 0 & s & 0 \\ -s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

$$R(\mathfrak{k}, T) = \{\pm 1\}, \qquad R^\vee(\mathfrak{k}, T) = \{\pm 2\}.$$

Root reflection is

$$s_2(n) = n - \langle n, 2 \rangle 1 = n - 2n = -n,$$

so $W(K, T) = \{\pm \mathrm{Id}\}$.

Example of root data: $SU(n)$

$$K = SU(n) = n \times n \text{ unitary matrices of det } 1$$

$$T = \left\{ t_\theta =_{\text{def}} \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \mid \theta \in \mathbb{R}^n, \sum_j \theta_j = 0 \right\}.$$

$$X^*(T) \simeq \mathbb{Z}^n/\mathbb{Z}(1, \dots, 1), \quad \lambda_p(t_\theta) = e^{i \sum_j p_j \theta_j};$$

$$X_*(T) \simeq \left\{ m \in \mathbb{Z}^n \mid \sum_j m_j = 0 \right\}, \quad \xi_m(e^{i\theta}) = t_{\theta m}.$$

Write $\{e_j\}$ for standard basis of $\mathbb{Z}^n$, $\{e_{jk}\}$ for basis of $n \times n$ matrices. Roots are $e_j - e_k$ $(j \neq k)$; root spaces

$$\mathfrak{k}_{e_j - e_k} = \mathbb{C} e_{jk}.$$

$$R(\mathfrak{k}, T) = \{e_j - e_k \mid 1 \leq j \neq k \leq n\} \subset \mathbb{Z}^n/\mathbb{Z}(e_1 + \cdots + e_n),$$

$$R^\vee(\mathfrak{k}, T) = \{e_j - e_k\} \subset \left\{ m \in \mathbb{Z}^n \mid \sum_j m_j = 0 \right\}.$$

Root reflection is

$$s_{jk}(p) = p - \langle p, e_j - e_k \rangle (e_j - e_k) = p - (p_j - p_k)(e_j - e_k),$$

interchanging jth coord p_j with kth coord p_k.

$W(K, T) = S_n$, symmetric group on n letters.

Example of root data: E_8

$$K = \text{compact form of } E_8$$
$$= \text{unique 248-diml cpt simple Lie gp}$$

$$X^*(T) = \{\lambda \in \mathbb{Z}^8 + \mathbb{Z}(1/2, \cdots, 1/2) \mid \sum(\lambda_j \in 2\mathbb{Z}\} \subset \mathbb{R}^8.$$

Standard inner product on $\mathbb{R}^8$ identifies $X^* \simeq X_*$.

$$R(\mathfrak{k}, T) = \{\pm e_j \pm e_k \mid 1 \le j \ne k \le 8\}$$
$$\cup \{(1/2)(\epsilon_1, \ldots, \epsilon_8) \mid \epsilon_j = \pm 1, \prod \epsilon_j = 1\}.$$

This is root system D_8 for $Spin(16)$ (first 112 roots) and wts of pos half spin rep of $Spin(16)$ (last 128 roots).

What's special about 8 is wts of spin have length 2.

$$R^\vee(\mathfrak{k}, T) = R(\mathfrak{k}, T).$$

$\#W(K, T) = 696729600$; $\#(\text{subgp } W(D_8)) = 5160960$; index 135.

Orbit of $W(E_8)$ on gp of $2^8 = 256$ elts of order 2 in T.

Another realization of E_8

Previous construction of root datum of E_8 based on subgp $Spin(16)$. Another construction:

$$X^*(T) = \{\lambda \in \mathbb{Z}^9 + (\mathbb{Z}/3)(2,2,2,-1,\cdots,-1) \mid \sum \lambda_j = 0\}.$$

Standard inner product on $\mathbb{R}^9$ identifies $X^* \simeq X_*$.

$$R(\mathfrak{k}, T) = \{\pm(e_j - e_k) \mid 1 \le j \ne k \le 9\}$$
$$\cup \{e_p + e_q + e_r - (1/3)(1,\ldots,1) \mid 1 \le p < q < r \le 9\}$$
$$\cup \{-e_p - e_q - e_r + (1/3)(1,\ldots,1) \mid 1 \le p < q < r \le 9\}.$$

This is root system A_8 for $SU(9)$ (first 72 roots) and wts of rep $\wedge^3(\mathbb{C}^9)$ of $SU(9)$ (next 84 roots) and wts of dual rep $\wedge^6(\mathbb{C}^9)$ (last 84 roots).

What's special about 9 is wts of $\wedge^3$ have length 2.

$$R^\vee(\mathfrak{k}, T) = R(\mathfrak{k}, T).$$

$\#W(K,T) = 696729600$; $\#(\text{subgp } W(A_8)) = 362880$; index 1920.

Orbit of $W(E_8)$ on gp of $3^8 = 6561$ elts of order 3 in T.

Axioms for root data

Grothendieck: root data *completely* describe both cpt conn Lie gps and reductive alg gps.

Definition *Root datum* is $(X^*, R, X_*, R^\vee)$, subj to

1. X^* and X_* are dual lattices; write $\langle , \rangle$ for pairing.
2. $R \subset X^*$ finite, with given bij $\alpha \leftrightarrow \alpha^\vee$ with $R^\vee \subset X_*$.
3. $\langle \alpha, \alpha^\vee \rangle = 2$, all $\alpha \in R$.
   $\alpha \in R \rightsquigarrow s_\alpha \in \mathrm{Aut}(X^*)$; $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$.
4. $s_\alpha R = R$, $s_{\alpha^\vee} R^\vee = R^\vee$.
5. Root datum is *reduced* if $\alpha \in R \Rightarrow 2\alpha \notin R$.

Theorem. Every reduced root datum is root datum of unique compact group K.

Theorem. k any alg closed field; every reduced root datum is root datum of unique conn red alg gp $G(k)$.

Open problem: describe group morphisms with root data.

Suggests: probs about gps (diff geom, func analysis. . .)
$\rightsquigarrow$ probs about root data (combinatorics, lin alg$/\mathbb{Z}$. . .).

Complexifying K

Cplx alg group: subgp of $GL(E)$ def by poly eqns.

Reductive: $\nexists$ normal subgp of unip matrices. Sources...

Cpt Lie gp $K \rightsquigarrow$ faithful rep $E \rightsquigarrow$ embed $K \hookrightarrow GL(E)$.

$$K(\mathbb{C}) =_{\text{def}} \text{ Zariski closure of } K \text{ in } GL(E)$$

$$\begin{aligned} C(K)_K &= \text{alg of } K\text{-finite functions on } K \\ &= \text{matrix coeffs of fin-diml reps} \\ &\simeq \sum_{(\tau,E_\tau)\in\widehat{K}} \text{End}(E_\tau), \end{aligned}$$

finitely-generated commutative algebra $/\mathbb{C}$.

$$K(\mathbb{C}) = \text{Spec}(C(K)_K).$$

Second formulation says $K(\mathbb{C})$ is *domain of holomorphy* for matrix coeffs of fin-diml reps of K.

Theorem. Constructions give *all* cplx reductive alg gps.

Corollary. Root data $\leftrightarrow$ cplx conn reductive alg gps.

Realifying $K(\mathbb{C})$

Real alg gp: cplx alg gp, $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ action.

For $K(\mathbb{C})$ from $K \subset GL(E)$, can arrange Hilb space str on E, so $K \subset U(E)$. Defining eqns have Galois action "inverse conjugate transpose."

Modify this Galois action by alg auts of $K(\mathbb{C}) \rightsquigarrow$ other real forms of $K(\mathbb{C})$.

Example: $K = T = n$-diml torus

$$T(\mathbb{C}) = \{\mathrm{diag}(z_1, \ldots, z_n) \mid z_j \in \mathbb{C}^x\}.$$

Original Galois action: $\sigma_0(\mathrm{diag}(z)) = \mathrm{diag}(\overline{z}^{-1})$.

Automorphism of $T(\mathbb{C})$ given by $A \in GL(n, \mathbb{Z})$, $n \times n$ integer matrix of det ± 1:

$$\xi_A(z) =_{\mathrm{def}} (z_1^{a_{11}} z_2^{a_{12}} \cdots z_n^{a_{1n}}, \ldots, z_1^{a_{n1}} z_2^{a_{22}} \cdots z_n^{a_{nn}}).$$

General Galois action: $\sigma(\mathrm{diag}(z)) = \mathrm{diag}(\overline{z}^A)$, $A \in GL(n, \mathbb{Z})$, $A^2 = \mathrm{Id}$.

Examples of real tori

Continue with $T(\mathbb{C}) = (\mathbb{C}^\times)^n$, $\sigma(\mathrm{diag}(z)) = \mathrm{diag}(\overline{z}^A)$, $A \in GL(n, \mathbb{Z})$, $A^2 = \mathrm{Id}$.

Case $n = 1$, $A = +1$, $\sigma(z) = \overline{z}$, $T(\mathbb{R}) = \mathbb{R}^\times$.

Case $n = 1$, $A = -1$, $\sigma(z) = \overline{z}^{-1}$, $T(\mathbb{R}) = \text{circle}$.

Case $n = 2$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma(z, w) = (\overline{w}, \overline{z})$,

$$T(\mathbb{R}) = \{(z, \overline{z}) \mid z \in \mathbb{C}^\times\} \simeq \mathbb{C}^\times.$$

Taking diag blocks from these three cases, get

$$T(\mathbb{R}) = (\mathbb{R}^\times)^p \times (\text{circle})^q \times (\mathbb{C}^\times)^r.$$

That's all: canonical form for int matrices $A^2 = \mathrm{Id}$.

Root data: compact group representations

$T \subset K \rightsquigarrow (X^*, R, X_*, R^\vee)$ root datum, W Weyl group.

R^+ pos roots $\rightsquigarrow$ $(X^*, R^+, X_*, (R^\vee)^+)$ *based root datum*.

$X^{*,+} = \{\lambda \in X^* \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \;\; (\alpha \in R^+)$ *dominant wts*

Proposition. $X^{*,+}$ is fund domain for W on X^*.

Proposition. V fin-diml rep of $K \rightsquigarrow$ wts of V are W-invt:

$$[V] = \sum_{\mu \in X^*} m_V(\mu)\mu \in \mathbb{Z}[X^*(T)]^W.$$

Theorem. (Cartan-Weyl) If (E, τ) irr rep of K, then there is a unique $\lambda(\tau) \in \Delta(E, T)$ such that

$$\lambda + \alpha \notin \Delta(E, \tau) \qquad (\text{all } \alpha \in R^+).$$

Wt $\lambda(\tau)$ is dominant, has mult 1; defines bijection

$$\widehat{K} \leftrightarrow X^{*,+}, \qquad \tau \leftrightarrow \lambda(\tau).$$

Rep ring of K $=_{\text{def}}$ Groth gp of fin-diml reps of K

$\simeq \quad W$-invt gp ring of $X^*(T) = \mathbb{Z}[X^*(T)]^W$,

Finite-diml "translation family": problem

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$T \subset K \rightsquigarrow (X^*, R^+, X_*, (R^\vee)^+)$ based root datum, $X^{*,+}$ dom wts.

So far have family of irr reps $\{E(\lambda) \mid \lambda \in X^{*,+}\}$.

Trans princ $\leftrightsquigarrow$ nice props of family under $\otimes$.

Approximately true that for F any fin-diml rep of K

$$E(\lambda) \otimes F \overset{?}{\simeq} \sum_{\mu \in \Delta(F,T)} m_F(\mu) E(\lambda + \mu).$$

Precisely true for λ "large enough" wrt F; need

$$\langle \lambda, \alpha^\vee \rangle \geq \langle \mu, \alpha^\vee \rangle - 1 \qquad (\alpha \in R^+, \mu \in \Delta(F,T)).$$

False/meaningless for small λ.

Case $\lambda = 0$, $F = E(\gamma)$

- LHS is $E(\gamma)$
- RHS has term $E(\mu)$ for each wt μ of $E(\gamma)$
- undefined if μ not dominant.

Fin-diml "transl family": Steinberg solution

Think about construction of of finite-diml irr rep $E(\lambda)\ldots$

- $R^+ \leftrightsquigarrow K$-eqvt complex structure on K/T;
- weight $\lambda \leftrightsquigarrow K$-eqvt hol line bdle $\mathcal{L}_\lambda$ on K/T.
- $E(\lambda) = \text{hol secs} = H^0(K/T, \mathcal{L}_\lambda)$ $(\lambda$ dom$)$.

Suggests definition (Steinberg)

$$E(\lambda) = \sum_j (-1)^j H^j(K/T, \mathcal{L}_\lambda),$$

virtual rep of K defined for all $\lambda \in X^*$.

Explicitly... $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$

- EITHER $\exists \alpha^\vee \in R^\vee$ $\langle \lambda + \rho, \alpha^\vee\rangle = 0 \Rightarrow E(\lambda) = 0$.
- OR $\forall \alpha^\vee \in R^\vee$ $\langle \lambda + \rho, \alpha^\vee\rangle \neq 0$
- 2nd case: $\exists!\ w \in W$ $w(\lambda+\rho) - \rho$ dominant
- $E(\lambda) = \det(w) E(w(\lambda+\rho)-\rho)$.

For F fin-diml (virtual) rep of K, have as virtual reps

$$E(\lambda)\otimes F \ \simeq \sum_{\mu\in\Delta(F,T)} m_F(\mu) E(\lambda+\mu).$$

What's ρ got to do with it?

Formula $E(\lambda) = \det(w)E(w(\lambda+\rho)-\rho)$ is ugly.

Weight $\mu \in X^*$ (or $X^* \otimes_{\mathbb{Z}} \mathbb{C}$) called *singular* if $\langle \mu, \beta^\vee \rangle = 0$, some $\beta \in R$; else *regular*. μ singular $\leftrightarrow W^\mu \neq \{1\}$.

Use $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha \in X^* \otimes_{\mathbb{Z}} \mathbb{Q}$. Key fact:

Proposition. $\langle \rho, \alpha^\vee \rangle = 1$, all α simple.

Cor. dom wts in X^* $\leftrightarrow$ dom reg wts in $X^* + \rho$, $\quad \lambda \to \lambda + \rho$.

Definition $\pi(\lambda') =$ irr rep of highest wt $\lambda' - \rho$, any $\lambda' \in X^* + \rho$ dominant regular.

Better Definition $\pi(\lambda') =$ virtual rep with character

$$\left[\sum \det(w) e^{w \cdot \lambda'}\right] / [\text{Weyl denom}] \qquad (\lambda' \in X^* + \rho)$$

Proposition

$$\pi(\lambda') = \begin{cases} E(\lambda' - \rho), & \lambda' \text{ dom reg} \\ 0, & \lambda' \text{ singular} \\ \det(w)\pi(w\lambda'), & \text{always.} \end{cases}$$

$\{\pi(\lambda') \mid \lambda \in X^* + \rho\}$, is coherent family or translation family.

Summary of compact group coherent family

$T \subset K \rightsquigarrow (X^*, R^+, X_*, (R^\vee)^+)$ based root datum.

Coherent family of (virtual) reps of K defined by

(0) $\pi(\lambda')$ has character

$$\left[\sum \det(w) e^{w\cdot\lambda'}\right] / [\text{Weyl denom}] \qquad (\lambda' \in X^* + \rho)$$

(1) $\pi(\lambda') \otimes F \simeq \sum_{\mu\in\Delta(F,T)} m_F(\mu)\pi(\lambda'+\mu)$ (F fin-diml of K).

(2) Family $\{\pi(\lambda')\}$ has *one* (Weyl) char formula: plug in λ'.

(3) $\pi(\lambda')$ has “infinitesimal character” $W \cdot \lambda'$.

Properties (1)–(3) persist for coherent families of inf-diml reps of noncompact groups.

Infinitesimal characters

$G(\mathbb{R})$ = real points of complex connected reductive alg G

Seek to classify $\widehat{G(\mathbb{R})}$ = irr reps of $G(\mathbb{R})$.

Need good invariants for irr rep π...

Central character $\xi_Z(\pi)$ = restr of π to $Z = Z(G(\mathbb{R}))$.

Char of abelian gp Z; concrete, meaningful.

But Z too small. Harish-Chandra idea:
$\mathfrak{Z}$ = center of $U(\mathfrak{g})$.

Infinitesimal character $\lambda_{\mathfrak{Z}}(\pi)$ = restr of π to $\mathfrak{Z}$.

Homom of comm alg $\mathcal{Z}$ to $\mathbb{C}$; concrete, meaningful.

We'll partition $\widehat{G(\mathbb{R})}$ by infl char.

List of infinitesimal characters

To use partition of $\widehat{G}(\mathbb{R})$, need list of infl chars.

Fix maximal torus $H \subset G$, isom to $(\mathbb{C}^\times)^n$.

$X^*(H) =_{\text{def}} \text{Hom}_{\text{alg}}(H, \mathbb{C}^\times) =$ lattice of chars.

$X_*(H) =_{\text{def}} \text{Hom}_{\text{alg}}(\mathbb{C}^\times, H) =$ (dual) lattice of cochars.

$R = R(G, H) \subset X^* =$ roots of H in G.

$R^\vee = R^\vee(G, H) \subset X_* =$ coroots of H in G.

$\alpha \in R \rightsquigarrow s_\alpha \in \text{Aut}(X^*)$, $s_\alpha(\mu) = \mu - \langle \mu, \alpha^\vee \rangle \alpha$.

$W(G, H) =$ gp gen by $\{s_\alpha \mid \alpha \in R\} \subset \text{Aut}(X^*)$,
Weyl group of H in G.

$\mathfrak{Z} \simeq S(\mathfrak{h})^{W(G,H)}$, $\qquad$ $\text{Hom}(\mathfrak{Z}, \mathbb{C}) \simeq \mathfrak{h}^*/W(G, H)$

$\mathfrak{h}^* \simeq X^* \otimes_{\mathbb{Z}} \mathbb{C}$, cplx vector space def over $\mathbb{Z}$.

Infl char $=$ W orbit on $\mathfrak{h}^*$.

Translating modules

$\mathcal{M}_\lambda = \mathfrak{g}$-modules of generalized infl char $\lambda \in \mathfrak{h}^*$.

Theorem (Kostant)

Z $\mathfrak{g}$ module of infl char λ; F alg rep of G, weights $\Delta(F) \subset X^*$ (multiset). Then $Z \otimes F$ is annihilated by ideal

$$\prod_{\mu \in \Delta(F)} \ker(\textit{infl char } \lambda + \mu).$$

So $Z \otimes F$ = sum of submodules of gen infl char $\lambda + \mu$.

Definition (Transl functors: Jantzen, Zuckerman)

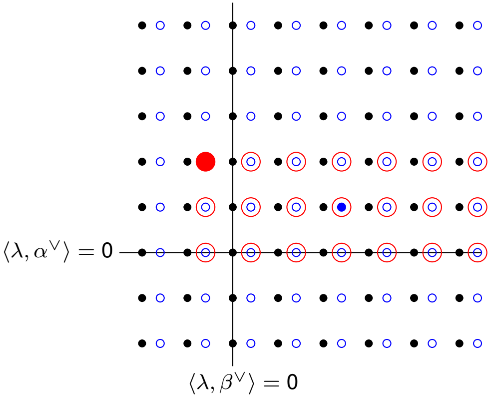
Suppose $\lambda \in \mathfrak{h}^*$, $\mu \in X^*$; define $F(\mu)$ = irr alg rep of G of extremal wt μ. Translation functor from λ to $\lambda + \mu$ is

$$\psi_\lambda^{\lambda+\mu}(Z) = \text{summand of } Z \otimes F(\mu) \text{ of gen infl char } \lambda + \mu.$$

$$\psi_\lambda^{\lambda+\mu} \colon \mathcal{M}_\lambda \to \mathcal{M}_{\lambda+\mu} \text{ is exact functor.}$$

Picture translating: good case

X^* and a coset for $SL(2) \times SL(2)$



Char lattice X^* **Base infl character λ_0** Coset $\lambda_0 + X^*$

Target infl character $\lambda_0 + \mu$ $\lambda_0 +$ (weights of $F(\mu)$)

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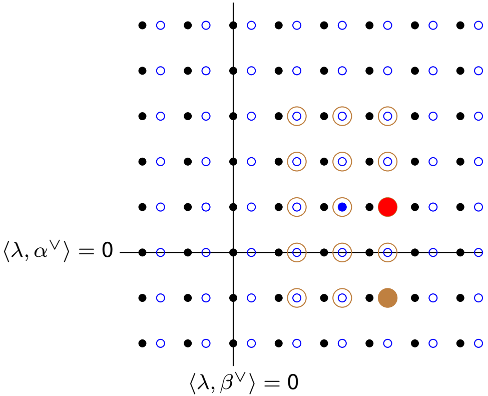

Char lattice X^* **Base infl character λ_0** Coset $\lambda_0 + X^*$

Target infl character $\lambda_0 + \mu'$ $\lambda_0 +$ (weights of $F(\mu')$)

Wall $\langle \lambda, \alpha^\vee \rangle$ difficult since $\langle \lambda_0, \alpha^\vee \rangle \in \mathbb{Z}$. Point $\lambda_0 + \mu''$ defines same infl char as $\lambda_0 + \mu'$. Translating to $\lambda_0 + \mu'$ inevitably gets extra copy of translation to $\lambda_0 + \mu''$

A little bookkeeping

$(X^*, R^+, X_*, (R^\vee)^+)$, $\lambda_0 \in \mathfrak{h}^* = X^* \otimes_{\mathbb{Z}} \mathbb{C}$.

Jantzen-Zuckerman $\rightsquigarrow$ study *transl fam* of infl chars $\lambda \in \lambda_0 + X^*$.

Definition Set of *integral roots for λ* is

$$R(\lambda) = \{\alpha \in R \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\}.$$

Set is CONSTANT on coset $\lambda_0 + X^*$.

Integral Weyl gp $W(\lambda_0) =_{\text{def}} W(R(\lambda_0))$ acts on $\lambda_0 + X^*$.

Easy fact: root datum $(X^*, R(\lambda), X_*, R(\lambda)^\vee)$ corrs to another cplx reductive group $G(\lambda)$. NOT ALWAYS A SUBGROUP OF G.

Deep fact: rep theory of real forms of G at infl char λ controlled by rep theory of real forms of $G(\lambda)$ at infl char λ.

$\lambda \in \mathfrak{h}^*$ *integrally dom* if dom for $R(\lambda)^+$; equiv, $\langle \lambda, \alpha^\vee \rangle$ never a negative integer.

Theorem Integrally dominant weights $(\lambda_0 + X^*)^+$ are a fundamental domain for action of $W(\lambda_0)$ on $\lambda_0 + X^*$.

Translating algebras

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$$U_\lambda = U(\mathfrak{g})/\left\langle \text{ker(infl char } \lambda)\right\rangle .$$

Definition (Translation functors for algebras)

$\lambda \in \mathfrak{h}^*$, $\mu \in X^*$, $F =$ irr alg rep of G of extremal wt μ.
Translated algebra from λ to $\lambda + \mu$ is

$$\Psi_\lambda^{\lambda+\mu}(U_\lambda) = \text{subalg of } U_\lambda \otimes \text{End}(F) \text{ of gen infl char } \lambda + \mu$$

Theorem (Bernstein-Gelfand)

1. $\psi_\lambda^{\lambda+\mu}$: $\{$*irr* U_λ *mods*$\} \to \{$*irr-or-zero* $\Psi_\lambda^{\lambda+\mu}(U_\lambda)$ *mods*$\}$.
2. **IF** *λ and $\lambda + \mu$ both integrally dominant, and λ not* **more** *singular than $\lambda + \mu$* **THEN** $\Psi_\lambda^{\lambda+\mu}(U_\lambda) = U_{\lambda+\mu}$.

Corollary (Jantzen-Zuckerman)

If λ and $\lambda + \mu$ both integrally dominant, vanish on same coroots, then $\psi_\lambda^{\lambda+\mu}$ is equivalence $\mathcal{M}_\lambda \to \mathcal{M}_{\lambda+\mu}$.

Can translate irr rep from int dom *regular* λ_0 to int dom $\lambda \in \lambda_0 + X^*$, getting irr rep or zero.

Translation families of representations

FIX $\lambda_0 \in \mathfrak{h}^*$ regular, integrally dominant $\rightsquigarrow$ infl char.

FIX Z_0 irr $\mathfrak{g}$ module of infl char λ_0.

DEFINE $Z(\lambda) = \psi_{\lambda_0}^{\lambda}(Z_0) \quad (\lambda \in \lambda_0 + X^* \text{ int dom})$,
translation family of irr-or-zero reps.

Example: $\lambda_0 = \rho$,

$$Z(\lambda) = \begin{cases} \text{fin diml, highest wt } \lambda - \rho & (\lambda \text{ reg}), \\ 0 & (\lambda \text{ singular}). \end{cases}$$

Example: $\lambda_0 = \rho$, $H(\mathbb{R})$ compact Cartan subgroup,
$Z(\lambda) =$ (limit of) discrete series repn, HC param λ.

PICTURE: nice family $Z(\lambda)$, $\lambda \in \lambda_0 + X^*$ INT DOM.

Other λ (other Weyl chambers)???

Crossing one wall

FIX $\lambda_0 \in \mathfrak{h}^*$ int dom regular representing infl char.

DEFINE $R^+(\lambda_0) = \{\alpha \in R(G,H) \mid \langle \lambda_0, \alpha^\vee \rangle = \text{pos int}\}$.

$\Pi(\lambda_0) =$ simple for $R(\lambda_0)$, $\quad W(\lambda_0) =$ Weyl gp of $R(\lambda_0)$.

FIX Z_0 irr $\mathfrak{g}$ module of infl char λ_0.

DEFINE $Z(\lambda) = \psi_{\lambda_0}^{\lambda}(Z_0)$ $\quad (\lambda \in \lambda_0 + X^*$ int dom). Other chambers?

FIX $\alpha \in \Pi(\lambda_0)$ simple root $\rightsquigarrow s = s_\alpha$ simple refl $\rightsquigarrow \lambda_s \in \lambda_0 + X^*$ fixed by $\{1, s\}$.

$\psi_s = \psi_{\lambda_0}^{\lambda_s}$ transl to $\alpha = 0$; $\psi_s(Z(\lambda_0)) = Z(\lambda_s)$ irr or 0.

$\phi_s = \psi_{\lambda_s}^{\lambda_0}$ transl from $\alpha = 0$; $\phi_s(Z(\lambda_s)) \stackrel{?}{=} Z(\lambda_0) + Z(s\lambda_0)$.

Try to define $Z(s\lambda_0) = \phi_s \psi_s(Z(\lambda_0)) - Z(\lambda_0)$.

Coherent families of virtual reps

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$\mathcal{M}^{\text{nice}}$ = "nice" $\mathfrak{g}$ mods: good char theory, virtual reps.

$K(\mathcal{M}^{\text{nice}})$ = Groth gp: free/$\mathbb{Z}$, basis = nice irreps.

Exs: BGG category $\mathcal{O}$, HC $(\mathfrak{g}, K)$-mods of fin length...

$\lambda_0 \in \mathfrak{h}^*$ regular, $R^+(\lambda_0) = \{\alpha \in R(G,H) \mid \langle \lambda_0, \alpha^\vee \rangle = \text{pos int}\}$.

$\Pi(\lambda_0) =$ simple for $R(\lambda_0)$, $\quad W(\lambda_0) =$ Weyl gp of $R(\lambda_0)$.

Z_0 nice irreducible of infinitesimal character λ_0.

Theorem (Schmid)

Attached to Z_0 is coherent family of virtual nice reps $\Theta \colon \lambda_0 + X^* \to K(\mathcal{M}^{nice})$ characterized by

1. $\Theta(\lambda_0) = [Z_0]$, $\qquad \Theta(\lambda)$ *has infl char* λ.
2. $\Theta(\lambda) \otimes F = \sum_{\mu \in \Delta(F)} \Theta(\lambda + \mu)$.

$$[\phi_s \psi_s(Z_0)] = \Theta(\lambda_0) + \Theta(s\lambda_0).$$

$W(\lambda_0)$ acts on $K(\mathcal{M}^{\text{nice}}_{\lambda_0})$, $w \cdot \Theta(\lambda_0) = \Theta(w^{-1} \cdot \lambda_0)$.

$$(1 + s) \cdot [Z_0] = [\phi_s \psi_s(Z_0)].$$

Action on reps of a simple reflection

Z_0 nice irr, reg infl char $\lambda_0 \rightsquigarrow W(\lambda_0)$ acts on $K(\mathcal{M}^{\text{nice}}_{\lambda_0})$,

$$(1+s)\cdot[Z_0] = [\phi_s\psi_s(Z_0)].$$

Have natural maps $Z_0 \to \phi_s\psi_s Z_0 \to Z_0$, induced by $\psi_s(Z_0) \to \psi_s(Z_0)$. Composition is zero.

Define $U^i_s(Z_0) =$ ith cohomology of complex

$$Z_0 \to \phi_s\psi_s Z_0 \to Z_0.$$

Dichotomy: **EITHER**

$s \in \tau(Z_0)$, $\psi_s(Z_0) = 0$, Z_0 dies on s wall, maps $=$ zero

$U^{-1}_s(Z_0) = U^1_s(Z_0) = Z_0$, $U^0_s(Z_0) = 0$, $s\cdot[Z_0] = -[Z_0]$,

OR

$s \notin \tau(Z_0)$, $\psi_s(Z_0) \neq 0$, Z_0 lives on s wall, maps $\neq$ zero

$U^{-1}_s(Z_0) = U^1_s(Z_0) = 0$, $U^0_s(Z_0) =$ sum of irrs with $s \in \tau$,

$$s\cdot[Z_0] = Z_0 + U^0_s(Z_0)$$

Either case: $s\cdot[Z_0] = Z_0 + \sum_i (-1)^i U^i_s(Z_0)$.

Why have you listened to this for three hours?

[Virt] reps in transl fam $\{\Theta(\lambda) \mid \lambda \in \lambda_0 + X^*\}$ all algebraically similar: same

- restr to K "at infinity"
- Gelfand-Kirillov dimension
- associated variety $\approx$ WF set of distribution char

$\Theta(\lambda)$ is irred for λ integrally dominant, regular.

char of $W(\lambda_0)$ rep gen by $\Theta(\lambda_0) \rightsquigarrow$ GK dimension.

Precise description of $W(\lambda_0)$ action $\rightsquigarrow$ Harish-Chandra char formula for $\Theta(\lambda_0)$.

Much less info about unitarity is preserved by translation, because fin diml reps not unitary.

But we know exactly how fin diml reps fail to be unitary, so there is hope for the future!

Thank you very much for dumplings, for skiing, for movies, for math, and most of all for your company!

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