The translation principle and Hermitian forms

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Outline

Infinitesimal characters

$G(\mathbb{R})$ = real points of complex connected reductive alg G

Seek to classify $\widehat{G(\mathbb{R})}_u =$ irr unitary reps of $G(\mathbb{R})$.

Need good invariants for irr rep π...

Central character $\xi_Z(\pi)$ = restr of π to $Z = Z(G(\mathbb{R}))$.

Concrete, meaningful, has info about unitarity

Z too small. Harish-Chandra idea:
$\mathfrak{Z} =$ center of $U(\mathfrak{g})$.

Infinitesimal character $\lambda_{\mathfrak{Z}}(\pi) =$ restr of π to $\mathfrak{Z}$.

Concrete, meaningful, has info about unitarity.

We'll partition $\widehat{G(\mathbb{R})}_u$ by infl char.

List of infinitesimal characters

To use partition of $\widehat{G(\mathbb{R})}_u$, need list of infl chars.

Fix maximal torus $H \subset G$, isom to $(\mathbb{C}^\times)^n$.

$X^*(H) =_{\text{def}} \text{Hom}_{\text{alg}}(H, \mathbb{C}^\times) =$ lattice of chars.

$X_*(H) =_{\text{def}} \text{Hom}_{\text{alg}}(\mathbb{C}^\times, H) =$ (dual) lattice of cochars.

$R = R(G, H) \subset X^* =$ roots of H in G.

$R^\vee = R^\vee(G, H) \subset X_* =$ coroots of H in G.

$\alpha \in R \rightsquigarrow s_\alpha \in \text{Aut}(X^*)$, $s_\alpha(\mu) = \mu - \langle \mu, \alpha^\vee \rangle \alpha$.

$W(G, H) =$ gp gen by $\{s_\alpha \mid \alpha \in R\} \subset \text{Aut}(X^*)$,
Weyl group of H in G.

$\mathfrak{Z} \simeq S(\mathfrak{h})^{W(G,H)}$, $\qquad \text{Hom}(\mathfrak{Z}, \mathbb{C}) \simeq \mathfrak{h}^*/W(G, H)$

$\mathfrak{h}^* \simeq X^* \otimes_{\mathbb{Z}} \mathbb{C}$, cplx vector space def over $\mathbb{Z}$.

Infl char $=$ W orbit on $\mathfrak{h}^*$.

Translating modules

$\mathcal{M}_\lambda = \mathfrak{g}$-modules of generalized infl char $\lambda \in \mathfrak{h}^*$.

Theorem (Kostant)

Z $\mathfrak{g}$ module of infl char λ; F alg rep of G, weights $\Delta(F) \subset X^*$ (multiset). Then $Z \otimes F$ is annihilated by ideal

$$\prod_{\mu \in \Delta(F)} \ker(\textit{infl char } \lambda + \mu).$$

So $Z \otimes F$ = sum of submodules of gen infl char $\lambda + \mu$.

Definition (Transl functors: Jantzen, Zuckerman)

Suppose $\lambda \in \mathfrak{h}^*$, $\mu \in X^*$; define F = irr alg rep of G of extremal wt μ. Translation functor from λ to $\lambda + \mu$ is

$$\psi_\lambda^{\lambda+\mu}(Z) = \text{summand of } Z \otimes F \text{ of gen infl char } \lambda + \mu.$$

$$\psi_\lambda^{\lambda+\mu} \colon \mathcal{M}_\lambda \to \mathcal{M}_{\lambda+\mu} \text{ is exact functor.}$$

Translating algebras

$$U_\lambda = U(\mathfrak{g}) / \langle \ker(\text{infl char } \lambda) \rangle .$$

Definition (Translation functors for algebras)

$\lambda \in \mathfrak{h}^*$, $\mu \in X^*$, F = irr alg rep of G of extremal wt μ.
Translated algebra from λ to $\lambda + \mu$ is

$$\Psi_\lambda^{\lambda+\mu}(U_\lambda) = \text{subalg of } U_\lambda \otimes \text{End}(F) \text{ of gen infl char } \lambda + \mu$$

Theorem (Bernstein-Gelfand)

1. $\psi_\lambda^{\lambda+\mu} \colon \{\textit{irr } U_\lambda \textit{ mods}\} \to \{\textit{irr-or-zero } \Psi_\lambda^{\lambda+\mu}(U_\lambda) \textit{ mods}\}$.
2. **IF** $\nexists$ *root* α *s.t.* $\langle \lambda, \alpha^\vee \rangle \leq 0$ *int,* $\langle \lambda + \mu, \alpha^\vee \rangle < 0$,
   THEN $\Psi_\lambda^{\lambda+\mu}(U_\lambda) = U_{\lambda+\mu}$.

Corollary (Jantzen-Zuckerman)

If λ and $\lambda + \mu$ define same positive integral roots, then $\psi_\lambda^{\lambda+\mu}$ is equiv of categories $\mathcal{M}_\lambda \to \mathcal{M}_{\lambda+\mu}$.

Translation families of representations

FIX $\lambda_0 \in \mathfrak{h}^*$ regular representing infl char.

DEFINE $R^+(\lambda_0) = \{\alpha \in R(G, H) \mid \langle \lambda_0, \alpha^\vee \rangle = \text{pos int}\}$.

$\Pi(\lambda_0) =$ simple roots for $R(\lambda_0)$.

$W(\lambda_0) =$ Weyl group of $R(\lambda_0)$.

FIX Z_0 irr $\mathfrak{g}$ module of infl char λ_0.

DEFINE $Z(\lambda) = \psi_{\lambda_0}^{\lambda}(Z_0) \quad (\lambda \in \lambda_0 + X^* \text{ dom})$,

translation family of irreps.

Example: $\lambda_0 = \rho$,

$$Z(\lambda) = \begin{cases} \text{fin diml, highest wt } \lambda - \rho & (\lambda \text{ reg}), \\ 0 & (\lambda \text{ singular}). \end{cases}$$

Example: $\lambda_0 = \rho$, $H(\mathbb{R})$ compact Cartan subgroup, $Z(\lambda) =$ (limit of) discrete series repn, HC param λ.

BIG PICTURE: nice family $Z(\lambda)$, $\lambda \in X^* + \lambda_0$ DOM.

Other λ (other Weyl chambers)???

Crossing one wall

FIX $\lambda_0 \in \mathfrak{h}^*$ regular representing infl char.

DEFINE $R^+(\lambda_0) = \{\alpha \in R(G,H) \mid \langle \lambda_0, \alpha^\vee \rangle = \text{pos int}\}$.

$\Pi(\lambda_0) =$ simple for $R(\lambda_0)$, $\quad W(\lambda_0) =$ Weyl gp of $R(\lambda_0)$.

FIX Z_0 irr $\mathfrak{g}$ module of infl char λ_0.

DEFINE $Z(\lambda) = \psi_{\lambda_0}^{\lambda}(Z_0)$ $\quad$ (λ $R(\lambda_0)$-dom). Other chambers?

FIX $\alpha \in \Pi(\lambda_0)$ simple root $\rightsquigarrow s = s_\alpha$ simple refl $\rightsquigarrow \lambda_s \in \lambda_0 + X^*$ fixed by $\{1, s\}$.

$\psi_s = \psi_{\lambda_0}^{\lambda_s}$ transl to $\alpha = 0$; $\psi_s(Z(\lambda_0)) = Z(\lambda_s)$ irr or 0.

$\phi_s = \psi_{\lambda_s}^{\lambda_0}$ transl from $\alpha = 0$; $\phi_s(Z(\lambda_s)) \stackrel{?}{=} Z(\lambda_0) + Z(s\lambda_0)$.

Try to define $Z(s\lambda_0) = \phi_s\psi_s(Z(\lambda_0)) - Z(\lambda_0)$.

Coherent families of virtual reps

$\mathcal{M}^{\text{nice}}$ =“nice” $\mathfrak{g}$ mods: good char theory, virtual reps.

$K(\mathcal{M}^{\text{nice}})$ = Groth gp: free/$\mathbb{Z}$, basis = nice irreps.

Exs: BGG category $\mathcal{O}$, HC $(\mathfrak{g}, K)$-mods of fin length...

$\lambda_0 \in \mathfrak{h}^*$ regular, $R^+(\lambda_0) = \{\alpha \in R(G, H) \mid \langle \lambda_0, \alpha^\vee \rangle = \text{pos int}\}$.

$\Pi(\lambda_0)$ = simple for $R(\lambda_0)$, $\quad W(\lambda_0)$ = Weyl gp of $R(\lambda_0)$.

Z_0 nice irreducible of infinitesimal character λ_0.

Theorem (Schmid)

Attached to Z_0 is coherent family of virtual nice reps $\Theta\colon \lambda_0 + X^* \to K(\mathcal{M}^{nice})$ characterized by

1. $\Theta(\lambda_0) = [Z_0]$, $\qquad \Theta(\lambda)$ *has infl char* λ.
2. $\Theta(\lambda) \otimes F = \sum_{\mu \in \Delta(F)} \Theta(\lambda + \mu)$.

$$[\phi_s \psi_s(Z_0)] = \Theta(\lambda_0) + \Theta(s\lambda_0).$$

$W(\lambda_0)$ acts on $K(\mathcal{M}^{\text{nice}}_{\lambda_0})$, $w \cdot \Theta(\lambda_0) = \Theta(w^{-1} \cdot \lambda_0)$.

$$(1 + s) \cdot [Z_0] = [\phi_s \psi_s(Z_0)].$$

Action on reps of a simple reflection

Z_0 nice irr, reg infl char $\lambda_0 \rightsquigarrow W(\lambda_0)$ acts on $K(\mathcal{M}^{\text{nice}}_{\lambda_0})$,

$$(1+s)\cdot[Z_0] = [\phi_s\psi_s(Z_0)].$$

Have natural maps $Z_0 \to \phi_s\psi_s Z_0 \to Z_0$, induced by $\psi_s(Z_0) \to \psi_s(Z_0)$. Composition is zero.

Define $U^i_s(Z_0) =$ ith cohomology of complex

$$Z_0 \to \phi_s\psi_s Z_0 \to Z_0.$$

Dichotomy: **EITHER**

$s \in \tau(Z_0)$, $\psi_s(Z_0) = 0$, Z_0 dies on s wall, maps $=$ zero
$U^{-1}_s(Z_0) = U^1_s(Z_0) = Z_0$, $U^0_s(Z_0) = 0$, $s\cdot[Z_0] = -[Z_0]$,

OR

$s \notin \tau(Z_0)$, $\psi_s(Z_0) \neq 0$, Z_0 lives on s wall, maps $\neq$ zero
$U^{-1}_s(Z_0) = U^1_s(Z_0) = 0$, $U^0_s(Z_0) =$ sum of irrs with $s \in \tau$,

$$s\cdot[Z_0] = Z_0 + U^0_s(Z_0)$$

Either case: $s\cdot[Z_0] = Z_0 + \sum_i (-1)^i U^i_s(Z_0)$.

As easy as ABC…

(A) By reading Harish-Chandra, or better by having Gregg read him to you, can calculate action of $\phi_s\psi_s$ on "standard representations."

(B) By reading Kazhdan and Lusztig and Beilinson and Bernstein and Brylinski and Kashiwara and Deligne and Gabber and Grothendieck and his friends… can easily calculate action of $\phi_s\psi_s$ on irreducible representations.
For instance, if $s \notin \tau(Z_0)$,

$$[\phi_s\psi_s(Z_0)] = 2[Z_0] + \sum_{X_0 \text{ s.t. } s\in\tau(X_0)} (\text{top coeff of } P_{X_0,Z_0}(q))[X_0].$$

(C) By combining the answers to (A) and (B), can write character formulas for irreducible representations…

Almos DEF...

(D) By reading Knapp-Stein, or better by having Gregg read them to you, can calculate action of $\phi_s\psi_s$ on *Hermitian forms* on standard representations.

(E) By reading Jantzen and Beilinson and Bernstein and Kazhdan and Lusztig and Deligne and Gabber and Grothendieck and his friends... can calculate action of $\phi_s\psi_s$ on Hermitian forms on irreducible representations.

For instance, if $s \notin \tau(Z_0)$,

$$[\phi_s\psi_s(Z_0)] = [Z_0 \otimes (\text{hyperbolic plane})] \\ +[\text{Zuckerman transl form on } U_s^0(Z_0)].$$

(F) By combining answers to (D) and (E), can write signature formulas for irreducible representations...

Character formulas

Can decompose Verma module into irreducibles

$$V(\lambda) = \sum_{\mu \leq \lambda} m_{\mu,\lambda} L(\mu) \qquad (m_{\mu,\lambda} \in \mathbb{N})$$

or write a formal character for an irreducible

$$L(\lambda) = \sum_{\mu \leq \lambda} M_{\mu,\lambda} V(\mu) \qquad (M_{\mu,\lambda} \in \mathbb{Z})$$

Can decompose standard HC module into irreducibles

$$I(x) = \sum_{y \leq x} m_{y,x} J(y) \qquad (m_{y,x} \in \mathbb{N})$$

or write a formal character for an irreducible

$$J(x) = \sum_{y \leq x} M_{y,x} I(y) \qquad (M_{y,x} \in \mathbb{Z})$$

Matrices m and M upper triang, ones on diag, mutual inverses. Entries are KL polynomials eval at 1.

Defining Hermitian dual representation

V $(\mathfrak{g}, K)$-mod); write π for repn map.

Hermitian dual of V

$$V^h = \{\xi : V \to \mathbb{C} \text{ additive}, \xi(zv) = \overline{z}\xi(v), K\text{-finite}\}$$

Want to construct functor

$$\text{cplx linear rep } (\pi, V) \rightsquigarrow \text{cplx linear rep } (\pi^h, V^h)$$

using Hermitian transpose map of operators. REQUIRES twisting by conjugate linear automorphism of $\mathfrak{g}$.

Assume $\sigma \colon G \to G$ antiholom aut, $\sigma(K) = K$.

Define $(\mathfrak{g}, K)$-module structure $\pi^{h,\sigma}$ on V^h,

$$\pi^{h,\sigma}(X) \cdot \xi = [\pi(-\sigma(X))]^h \cdot \xi \qquad (X \in \mathfrak{g}, \xi \in V^h).$$
$$\pi^{h,\sigma}(k) \cdot \xi = [\pi(\sigma(k)^{-1})]^h \cdot \xi \qquad (k \in K, \xi \in V^h).$$

Unitary representation theory $\leftrightsquigarrow$

$$\sigma_0 = \text{ real form with complexified maximal compact } K.$$

We need critically

$$\sigma_c = \text{ compact real form of } G \text{ preserving } K.$$

Invariant Hermitian forms

$V = (\mathfrak{g}, K)$-module, σ antihol aut of G preserving K.

A σ-invt sesq form on V is sesq pairing $\langle , \rangle$ such that

$$\langle X \cdot v, w \rangle = \langle v, -\sigma(X) \cdot w \rangle, \quad \langle k \cdot v, w \rangle = \langle v, \sigma(k^{-1}) \cdot w \rangle$$

$$(X \in \mathfrak{g}; k \in K; v, w \in V).$$

Proposition

σ-invt sesq form on V $\leftrightsquigarrow$ $(\mathfrak{g}, K)$-map $T \colon V \to V^{h,\sigma}$:

$$\langle v, w \rangle_T = (Tv)(w).$$

Form is Hermitian iff $T^h = T$.
Assume V is irreducible.
$V \simeq V^{h,\sigma} \Leftrightarrow \exists$ invt sesq form $\Leftrightarrow \exists$ invt Herm form
A σ-invt Herm form on V is unique up to real scalar.

Map $T \to T^h$ defines a real form of the cplx line $\mathrm{Hom}_{\mathfrak{g},K}(V, V^{h,\sigma})$.

Invariant forms on standard reps

Recall multiplicity formula

$$I(x) = \sum_{y \leq x} m_{y,x} J(y) \qquad (m_{y,x} \in \mathbb{N})$$

for standard $(\mathfrak{g}, K)$-mod $I(x)$.

Want parallel formulas for σ-invt Hermitian forms.

Need forms on standard modules.

Form on irr $J(x) \xrightarrow{\text{deformation}}$ Jantzen filt $I_n(x)$ on std, nondeg forms $\langle , \rangle_n$ on I_n/I_{n+1}.

Details (proved by Beilinson-Bernstein):

$$I(x) = I_0 \supset I_1 \supset I_2 \supset \cdots, \qquad I_0/I_1 = J(x)$$

$$I_n/I_{n+1} \text{ completely reducible}$$

$$[J(y) \colon I_n/I_{n+1}] = \text{coeff of } q^{(\ell(x)-\ell(y)-n)/2} \text{ in KL poly } Q_{y,x}$$

Hence $\langle , \rangle_{I(x)} \overset{\text{def}}{=} \sum_n \langle , \rangle_n$, nondeg form on gr $I(x)$.

Restricts to original form on irr $J(x)$.

Virtual Hermitian forms

$$\mathbb{Z} = \text{Groth group of vec spaces.}$$

Think of integers as mults of irr reps in virtual reps $\rightsquigarrow$

$$\mathbb{Z}[\text{set of irreps}] = \text{Groth grp of finite length reps.}$$

For invariant forms...

$$\mathbb{W} = \mathbb{Z} \oplus \eta\mathbb{Z} = \text{Groth grp of fin diml forms.}$$

Ring structure

$$(p + q\eta)(p' + q'\eta) = (pp' + qq') + (pq' + q'p)\eta.$$

Regard mult of irr-with-form in virt-with-form as $\in \mathbb{W}$:

$$\mathbb{W}[\text{set of irreps}] \approx \text{Groth grp of reps with invt forms.}$$

Two problems: invt form $\langle , \rangle_J$ may not exist for irr J; and $\langle , \rangle_J$ may not be preferable to $-\langle , \rangle_J$.

Hermitian KL polynomials: multiplicities

Fix σ-invt Hermitian form $\langle,\rangle_{J(x)}$ on each irr admitting one; recall Jantzen form $\langle,\rangle_n$ on $I(x)_n/I(x)_{n+1}$.

MODULO problem of irrs with no invt form, write

$$(I_n/I_{n-1}, \langle,\rangle_n) = \sum_{y \leq x} w_{y,x}(n)(J(y), \langle,\rangle_{J(y)}),$$

coeffs $w(n) = (p(n), q(n)) \in \mathbb{W}$; summand means

$$p(n)(J(y), \langle,\rangle_{J(y)}) \oplus q(n)(J(y), -\langle,\rangle_{J(y)})$$

Define Hermitian KL polynomials

$$Q^{\sigma}_{y,x} = \sum_n w_{y,x}(n) q^{(l(x)-l(y)-n)/2} \in \mathbb{W}[q]$$

Eval in $\mathbb{W}$ at $q = 1 \leftrightarrow$ form $\langle,\rangle_{I(x)}$ on std.

Reduction to $\mathbb{Z}[q]$ by $\mathbb{W} \to \mathbb{Z} \leftrightarrow$ KL poly $Q_{y,x}$.

Hermitian KL polynomials: characters

Matrix $Q^\sigma_{y,x}$ is upper tri, 1s on diag: INVERTIBLE.

$$P^\sigma_{x,y} \stackrel{\text{def}}{=} (-1)^{l(x)-l(y)}((x,y) \text{ entry of inverse}) \in \mathbb{W}[q].$$

Definition of $Q^\sigma_{x,y}$ says

$$(\operatorname{gr} I(x), \langle,\rangle_{I(x)}) = \sum_{y\le x} Q^\sigma_{x,y}(1)(J(y), \langle,\rangle_{J(y)});$$

inverting this gives

$$(J(x), \langle,\rangle_{J(x)}) = \sum_{y\le x} (-1)^{l(x)-l(y)} P^\sigma_{x,y}(1)(\operatorname{gr} I(y), \langle,\rangle_{I(y)})$$

Next question: how do you compute $P^\sigma_{x,y}$?

Herm KL polys for σ_c

$\sigma_c =$ cplx conj for cpt form of G, $\sigma_c(K) = K$.

Plan: study σ_c-invt forms, relate to σ_0-invt forms.

Proposition

Suppose $J(x)$ irr $(\mathfrak{g}, K)$-module, real infl char. Then $J(x)$ has σ_c-invt Herm form $\langle , \rangle^c_{J(x)}$, characterized by

$\langle , \rangle^c_{J(x)}$ is pos def on the lowest K-types of $J(x)$.

Proposition $\implies$ Herm KL polys $Q^{\sigma_c}_{x,y}$, $P^{\sigma_c}_{x,y}$ well-def.

Coeffs in $\mathbb{W} = \mathbb{Z} \oplus \eta\mathbb{Z}$; $\eta = (0, 1) \leftrightsquigarrow$ one-diml neg def form.

Conj: $Q^{\sigma_c}_{x,y}(q) = Q_{x,y}(q\eta), \quad P^{\sigma_c}_{x,y}(q) = P_{x,y}(q\eta).$

Equiv: if $J(y)$ appears at level n of Jantzen filt of $I(x)$, then Jantzen form is $(-1)^{(l(x)-l(y)-n)/2}$ times $\langle , \rangle_{J(y)}$.

Conjecture is false. . . but not seriously so. Need an extra power of η on the right side.

Deforming to $\nu = 0$

Have computable conjectural formula (omitting ℓ_o)

$$(J(x), \langle , \rangle^c_{J(x)}) = \sum_{y \le x} (-1)^{l(x)-l(y)} P_{x,y}(s)(\mathrm{gr}\, I(y), \langle , \rangle^c_{I(y)})$$

for σ^c-invt forms in terms of forms on stds, same inf char.

Polys $P_{x,y}$ are KL polys, computed by `atlas` software.

Std rep $I = I(\nu)$ deps on cont param ν. Put $I(t) = I(t\nu),\ t \ge 0$.

If std rep $I = I(\nu)$ has σ-invt form so does $I(t)$ ($t \ge 0$).

(signature for $I(t)$) = (signature on $I(t+\epsilon)$), $\epsilon \ge 0$ suff small.

Sig on $I(t)$ differs from $I(t-\epsilon)$ on odd levels of Jantzen filt:

$$\langle , \rangle_{\mathrm{gr}\, I(t-\epsilon)} = \langle , \rangle_{\mathrm{gr}\, I(t)} + (s-1) \sum_m \langle , \rangle_{I(t)_{2m+1}/I(t)_{2m+2}}.$$

Each summand after first on right is known comb of stds, all with cont param strictly smaller than $t\nu$. ITERATE...

$$\langle , \rangle^c_J = \sum_{I'(0) \text{ std at } \nu' = 0} v_{J,I'} \langle , \rangle^c_{I'(0)} \qquad (v_{J,I'} \in \mathbb{W}).$$

From σ_c to σ_0

Cplx conjs σ_c (compact form) and σ_0 (our real form) differ by Cartan involution θ: $\sigma_0 = \theta \circ \sigma_c$.

Irr $(\mathfrak{g}, K)$-mod $J \rightsquigarrow J^\theta$ (same space, rep twisted by θ).

Proposition

J admits σ_0-invt Herm form if and only if $J^\theta \simeq J$. If $T_0 \colon J \xrightarrow{\sim} J^\theta$, and $T_0^2 = \mathrm{Id}$, then

$$\langle v, w \rangle_J^0 = \langle v, T_0 w \rangle_J^c.$$

$T \colon J \xrightarrow{\sim} J^\theta \Rightarrow T^2 = z \in \mathbb{C} \Rightarrow T_0 = z^{-1/2} T \rightsquigarrow \sigma$-invt Herm form.

To convert formulas for σ_c invt forms $\rightsquigarrow$ formulas for σ_0-invt forms need intertwining ops $T_J \colon J \xrightarrow{\sim} J^\theta$, consistent with decomp of std reps.

Equal rank case

rk K = rk $G \Rightarrow$ Cartan inv inner: $\exists \tau \in K$, $\mathrm{Ad}(\tau) = \theta$.

$\theta^2 = 1 \Rightarrow \tau^2 = \zeta \in Z(G) \cap K$.

Study reps π with $\pi(\zeta) = z$. Fix square root $z^{1/2}$.

If ζ acts by z on V, and $\langle , \rangle_V^c$ is σ_c-invt form, then
$\langle v, w \rangle_V^0 \stackrel{\text{def}}{=} \langle v, z^{-1/2}\tau \cdot w \rangle_V^c$ is σ_0-invt form.

$$\langle , \rangle_J^c = \sum_{I'(0) \text{ std at } \nu' = 0} v_{J,I'} \langle , \rangle_{I'(0)}^c \qquad (v_{J,I'} \in \mathbb{W}).$$

translates to

$$\langle , \rangle_J^0 = \sum_{I'(0) \text{ std at } \nu' = 0} v_{J,I'} \langle , \rangle_{I'(0)}^0 \qquad (v_{J,I'} \in \mathbb{W}).$$

I' has LKT $\mu' \Rightarrow \langle , \rangle_{I'(0)}^0$ definite, sign $z^{-1/2}\mu'(\tau)$.

J unitary $\Leftrightarrow$ each summand on right pos def.

Computability of $v_{J,I'}$ needs conjecture about $P_{x,y}^{\sigma_c}$.

General case

Fix "distinguished involution" δ_0 of G inner to θ

Define extended group $G^\Gamma = G \rtimes \{1, \delta_0\}$.

Can arrange $\theta = \mathrm{Ad}(\tau\delta_0)$, some $\tau \in K$.

Define $K^\Gamma = \mathrm{Cent}_{G^\Gamma}(\tau\delta_0) = K \rtimes \{1, \delta_0\}$.

Study $(\mathfrak{g}, K^\Gamma)$-mods $\leftrightsquigarrow$ $(\mathfrak{g}, K)$-mods V with $D_0 \colon V \xrightarrow{\sim} V^{\delta_0}$, $D_0^2 = \mathrm{Id}$.

Beilinson-Bernstein localization: $(\mathfrak{g}, K^\Gamma)$-mods $\leftrightsquigarrow$ action of δ_0 on K-eqvt perverse sheaves on G/B.

Should be computable by mild extension of Kazhdan-Lusztig ideas. Not done yet!

Now translate σ_c-invt forms to σ_0 invt forms

$$\langle v, w\rangle_V^0 \overset{\text{def}}{=} \langle v, z^{-1/2}\tau\delta_0 \cdot w\rangle_V^c$$

on $(\mathfrak{g}, K^\Gamma)$-mods as in equal rank case.

Possible unitarity algorithm

Hope to get from these ideas a computer program; enter

- real reductive Lie group $G(\mathbb{R})$
- general representation π

and ask whether π is unitary.

Program would say either

- π has no invariant Hermitian form, or
- π has invt Herm form, indef on reps μ_1, μ_2 of K, or
- π is unitary, or
- I'm sorry Dave, I'm afraid I can't do that.

Answers to finitely many such questions $\rightsquigarrow$ complete description of unitary dual of $G(\mathbb{R})$.

This would be a good thing.

Historical notes

some quotations from something

. . . And Langlands showed unto the world the Food Chain. And God agreed to these conditions, and called the Food Chain the Langlands Classification. And Langlands said, “Lo, so it is not only on this small world, but at other places as well.” And we heeded him not, choosing rather to start reading in section 3.

And God created Schmid, and said unto him, “Lo, the Fruit of the land is sweet, yet it is hard to gather; make for my people an easier life.” And Schmid planted an orchard; nine years he labored there, gathering into it all the fruits of the earth. And God saw that it was very good, and that students wouldn’t need to heed the hard parts of the prophet Harish-Chandra anymore.

And (late in a long day) God created Zuckerman. And Zuckerman domesticated Cattle (possibly by slipping something into their water supply) and called them Cohomologically Induced Representations (possibly after slipping something into his own water supply).

And there were lectures, but not always preprints; and there were preprints, but not always papers; and it was the 1970s.

Historical notes continued

And the Beasts are not tamed nor are they counted, nor is any brand upon them. They lurk in the orchards of Schmid, in the cattleyards of Zuckerman, and the aviaries of Harish-Chandra; and they roam the earth as they will.

And God created Barbasch, to tame the Beasts and to number them. And Barbasch wrote upon the wall MENE MENE TEKEL PARSIN. Then came the king's wise men, but they could not read the writing or make known its interpretation. And all the lords were perplexed.

And God saw everything he had made, and behold, it was sundown Friday. So God rested from the work which he had done.

And Harish-Chandra walks with God, and Langlands returns not his calls, and Schmid works no longer in this department, and Zuckerman works no longer in this dimension.

By the waters of San Francisco, there we sat down and wept,
we wept when we remembered Princeton.
On the blackboards there we laid down our chalk.
For our organizers required of us seminars, and our students, lectures, saying,
"Prove for us one of the theorems of Zuckerman!"
How shall we follow Zuckerman, when we cannot even smoke tobacco;
when flowers sprout not from our hair, nor acid rock from our radios?

And there was evening and there was morning, the 1980s.