Stronger arithmetic equivalence

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The Dedekind zeta function of a number field

Definition

Let $K = \mathbb{Q}(\alpha)$ be a number field. The Dedekind zeta function of K is defined by

$$\zeta_K(s) := \sum_{n\geq 1} a_n n^{-s} := \sum_I N(I)^{-s} = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1},$$

Each of the following is uniquely determined by the others:

- the Dedekind zeta function $\zeta_K(s)$;
- the integer coefficients $a_p, a_{p^2}, \ldots, a_{p^d}$ for all primes p, where $d = [K : \mathbb{Q}]$;
- the number of primes of K of degree r above p, for all p and $1 \leq r \leq d$.
- the cycle type of the permutation of Frob_p acting on $\{\sigma(\alpha) : \sigma \in G_K\}$ for all p.

One can replace "all" with "all but finitely many" throughout.

Arithmetic equivalence

Definition

Number fields K_1 and K_2 are arithmetically equivalent if $\zeta_{K_1}(s) = \zeta_{K_2}(s)$.
The fields $K_1 \sim K_2$ must have the same degree and Galois closure L.

Let $G := \mathrm{Gal}(L/\mathbb{Q})$, $H_1 := \mathrm{Gal}(L/K_1)$, and $H_2 := \mathrm{Gal}(L/K_2)$.

Definition

A Gassmann triple (G, H_1, H_2) is a triple of finite groups $H_1, H_2 \leq G$ for which we have $\#(H_1 \cap C) = \#(H_2 \cap C)$ for every G-conjugacy class C of elements of G. We then say that $H_1 \sim H_2$ are Gassmann equivalent (as subgroups of G).

Theorem (Gassmann 1926)

$K_1 \sim K_2$ if and only if $H_1 \sim H_2$.

Note that K_1 and K_2 are isomorphic if and only if H_1 and H_2 are conjugate.

Some examples of Gassmann triples

Example

Let $G = \mathrm{GL}_2(\mathbb{F}_3)$, let $H_1 = \{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \in G\}$, and let $H_2 = \{\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \in G\}$.
Then (G, H_1, H_2) is a non-trivial Gassmann triple (de Smit 2004).

Let $E/\mathbb{Q}$ be an elliptic curve with mod-3 Galois image G, and let $L = \mathbb{Q}(E[3])$.
Then $\mathrm{Gal}(L/\mathbb{Q}) \simeq G$, and $K_1 := L^{H_1}$ and $K_2 = L^{H_2}$ are non-conjugate arithmetically equivalent number fields of degree 8 (one can achieve 7 using $H_1, H_2 \leq \mathrm{SL}_3(\mathbb{F}_2)$).

Lemma

Finite groups H_1 and H_2 occur as elements of a Gassmann triple (G, H_1, H_2) if and only if they have the same order statistics.

It follows that Gassmann equivalence does not imply isomorphism: consider $(\mathbb{Z}/p\mathbb{Z})^3$ and $\mathrm{H}_3(\mathbb{F}_p) := \left\{ \left(\begin{smallmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{smallmatrix} \right) \right\}$ for any prime $p \geq 3$, or $\langle 16, 3 \rangle$ and $\langle 16, 10 \rangle$, for example.

Gassmann triples in other contexts

Gassmann triples (G, H_1, H_2) arise in many contexts that involve potentially non-isomorphic objects with the same "zeta function":

- If $\pi\colon M \to M_0$ is a normal finite Riemannian covering with deck group G, then M/H_1, and M/H_2 are isospectral (Sunada 1985).
- If Γ is a finite graph with $G = \mathrm{Aut}(\Gamma)$ then Γ/H_1 and Γ/H_2 are isospectral (Halbeisen–Hungerbühler 1995).
- If X/k is a smooth projective curve with $G = \mathrm{Aut}(X)$, then X/H_1 and X/H_2 have isogenous Jacobians (Prasad–Rajan 2003).
- If $G = \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, then the modular curves X_{H_1} and X_{H_2} parameterizing elliptic curves with "level H_i-structure" have the same L-function.
- If $\pi\colon X \to Y$ is a Galois étale cover of k-varieties then X/H_1 and X/H_2 have isomorphic Chow motives (Arapura–Katz–McReynolds–Solapurkar 2019).

Unlike the number field case, non-trivial Gassmann triples may yield isomorphic objects, and zeta function equality does not always force Gassmann equivalence.

How strong is arithmetic equivalence?

Theorem (Perlis 1977)

Arithmetically equivalent number fields K_1 and K_2 have the same degree, discriminant, signature, roots of unity, normal closure, and normal core.

The analytic class number formula

$$\lim_{s\to 1+}(s-1)\zeta_{K_i}(s) = \frac{2^{r_1}(2\pi)^{r_2}h_{K_i}R_{K_i}}{\#\mu_{K_i}|D_{K_i}|^{1/2}}$$

implies $h_{K_1}R_{K_1}=h_{K_2}R_{K_2}$, but the class numbers h_{K_i} and regulators R_{K_i} may differ.

There is a bijection of the places of K_1 and K_2 that preserves residue fields, but it may not be possible for this bijection to also preserve ramification indices.

In particular, the adele rings $\mathbb{A}_{K_1}$ and $\mathbb{A}_{K_2}$ need not be isomorphic.

Local isomorphism

Definition

Two number fields are locally isomorphic if there is a bijection of places in which corresponding completions are isomorphic (this forces arithmetic equivalence).

Proposition (Iwasawa 1953)

Number fields K_1, K_2 are locally isomorphic if and only if they have isomorphic rings of adèles $\mathbb{A}_{K_1} \simeq \mathbb{A}_{K_2}$ (as topological rings and as $\mathbb{A}_{\mathbb{Q}}$-algebras).

Proposition (Linowitz–McReynolds–Miller 2017)

Locally isomorphic number fields have isomorphic Brauer groups.

Locally isomorphic number fields may have distinct class numbers, as happens with $\mathbb{Q}(\sqrt[8]{-33})$ and $\mathbb{Q}(\sqrt[8]{-33 \cdot 16})$, with class numbers 256 and 128 (de Smit–Perlis, 1994).

Plan for the talk

- Define three stronger notions of Gassmann equivalence ($\mathbb{Q}$):
  - local integral equivalence ($\mathbb{Z}_p$)
  - integral equivalence ($\mathbb{Z}$)
  - solvable equivalence (🧪)
- Investigate their consequences beyond arithmetic equivalence ($\zeta_{K_1} = \zeta_{K_2}$):
  - class group isomorphism ($\mathrm{cl}_{K_1} \simeq \mathrm{cl}_{K_2}$)
  - local isomorphism ($\mathbb{A}_{K_1} \simeq \mathbb{A}_{K_2}$)
  - Galois group isomorphism ($\mathrm{Gal}(L/K_1) \simeq \mathrm{Gal}(L/K_2)$)
- Construct explicit examples and counterexamples

Gassmann equivalence ($\mathbb{Q}$)

Definition

Let $[H\backslash G]$ be the transitive (right) G-set consisting of (right) cosets of H.
Let $\chi_H\colon G \to \mathbb{Z}$ be the permutation character $g \mapsto \#[H\backslash G]^g$ (the character of 1_H^G).
Define $\chi_H(K) := \#[H\backslash G]^K$ for $K \leq G$ (note $\chi_H(K) \neq 0 \Leftrightarrow K \leq_G H$).

Proposition

For all $H_1, H_2 \leq G$ the following are equivalent:

- $\#(H_1 \cap C) = \#(H_2 \cap C)$ *for all* $C \in \mathrm{conj}(G)$;
- *there is a* G*-conjugacy preserving bijection* $H_1 \longleftrightarrow H_2$;
- $\chi_{H_1}(K) = \chi_{H_2}(K)$ *for all cyclic* $K \leq G$;
- *the* G*-sets* $[H_1\backslash G]$ *and* $[H_2\backslash G]$ *are isomorphic as* K*-sets for all cyclic* $K \leq G$;
- $\mathbb{Q}[H_1\backslash G] \simeq \mathbb{Q}[H_2\backslash G]$ *as* $\mathbb{Q}[G]$*-modules.*

One can replace "all $K \leq G$" with "all $K \leq H_1$ and all $K \leq H_2$".

Local integral equivalence ($\mathbb{Z}_p$)

Definition

$H_1, H_2 \leq G$ are locally integrally equivalent if $\mathbb{Z}_p[H_1\backslash G] \simeq \mathbb{Z}_p[H_2\backslash G]$ for all primes p.

Proposition

Call a group p-cyclic if its quotient by its p-core (largest normal p-subgroup) is cyclic. For all $H_1, H_2 \leq G$ the following are equivalent:

- *there is a G-conjugacy class preserving bijection of p-cyclic $K \leq H_1, H_2$;*
- *$\chi_{H_1}(K) = \chi_{H_2}(K)$ for all p-cyclic $K \leq G$ (or all $K \leq H_1, H_2$);*
- *$\mathbb{F}_p[H_1\backslash G] \simeq \mathbb{F}_p[H_2\backslash G]$ as $\mathbb{F}_p[G]$-modules;*
- *$\mathbb{Z}_p[H_1\backslash G] \simeq \mathbb{Z}_p[H_2\backslash G]$ as $\mathbb{Z}_p[G]$-modules.*

Theorem (Perlis 1978)

Locally integrally equivalent number fields have isomorphic class groups.

Integral equivalence ($\mathbb{Z}$)

Definition

$H_1, H_2 \leq G$ are integrally equivalent if $\mathbb{Z}[H_1\backslash G] \simeq \mathbb{Z}[H_2\backslash G]$.

Let $H_1, H_2 \leq G$ have index n, let $\rho_1, \rho_2 \colon G \to S_n$ be the representations corresponding to the permutation modules $\mathbb{Z}[H_1\backslash G]$, $\mathbb{Z}[H_2\backslash G]$.

Fix an ordering of $[H_1\backslash G]$ and $[H_2\backslash G]$. We may represent elements of $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[H_1\backslash G], \mathbb{Z}[H_2\backslash G])$ by matrices $M \in \mathbb{Z}^{n\times n}$ that satisfy

$$M_{ij} = M_{\rho_1(g)(i),\rho_2(g)(j)} \qquad \text{for all } g \in G.$$

($\mathbb{Q}$) rational equivalence: $\exists M \det(M) \neq 0$

($\mathbb{Z}_p$) local integral equivalence: $\exists M_i \gcd(\det(M_1), \ldots, \det(M_r)) = 1$

($\mathbb{Z}$) integral equivalence: $\exists M \det(M) = \pm 1$

What we know about integral equivalence

Theorem (Prasad 2017)

Let $\pi\colon X \to Y$ be a Galois cover of nice curves over k with Galois group G. If $H_1, H_2 \leq G$ are integrally equivalent then $\mathrm{Jac}(X/H_1) \simeq \mathrm{Jac}(X/H_2)$.

Remark: Infinite families of non-isomorphic curves of low genus with isomorphic Jacobians were previously known (Howe 2005).

Essentially only one non-trivial example of integral equivalence is known: $G = \mathrm{PSL}_2(\mathbb{F}_{29})$ with $H_1, H_2 \simeq A_5$ subgroups of index 203 (Scott 1992).

Scott proved this by writing down $M \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[H_1\backslash G], \mathbb{Z}[H_2\backslash G]) \subseteq \mathbb{Z}^{203\times 203}$ with $\det M = 1$ (most of the entries in M are zero, the nonzero entries are ± 1).

Similar triples exist for all primes $p \equiv \pm 29 \bmod 120$...
... but for $p = 149$ we need $M \in \mathbb{Z}^{27565\times 27565}$ (and none of the simplest M work).

What we don't know about integral equivalence

Two questions naturally arise from Prasad's result.

Question 1: Must integrally equivalent $H_1, H_2 \leq G$ be isomorphic?

We show that locally integrally equivalent $H_1, H_2 \leq G$ need not be, in general, but rationally equivalent subgroups of $\mathrm{PSL}_2(\mathbb{F}_p)$ are isomorphic (S 2016), so this necessarily holds for Scott's example.

Question 2: Must integrally equivalent number fields be locally isomorphic?

We show that locally integrally equivalent number fields need not be, in general, but locally integrally equivalent subgroups of $\mathrm{PSL}_2(\mathbb{F}_p)$ force local isomorphism, so this necessarily holds for Scott's example.

Remark: $\mathrm{PSL}_2(\mathbb{F}_p)$ can be realized as a Galois group over $\mathbb{Q}$ (Zywina 2015); this was previously known for infinitely many values of p including 29 (Shih 1974).

Solvable equivalence (🧪)

Definition

$H_1, H_2 \leq G$ are solvably equivalent if $\chi_{H_1}(K) = \chi_{H_2}(K)$ for all solvable $K \leq G$.

Solvable equivalence implies local integral equivalence (hence isomorphic class groups), and also guarantees that corresponding number fields are locally isomorphic.

Proposition

Number fields K_1, K_2 corresponding to solvably equivalent $H_1, H_2 \leq G$ are arithmetically equivalent, locally isomorphic, and have isomorphic class groups.

In particular, there is a bijection of the places of K_1 and K_2 that preserves residue fields and ramification indices, and yields isomorphic completions.

Remark: Solvable equivalence is stronger than necessary.

Results

Proposition

There are infinitely many non-isomorphic pairs of degree-32 number fields arising from locally (but not globally) integrally equivalent $H_1, H_2 \leq G$ (and none of degree < 32).

Proposition

There are infinitely many non-isomorphic pairs of degree-96 number fields arising from solvably (but not integrally) equivalent $H_1, H_2 \leq G$ (and none of degree < 48).

Proposition

For all primes $p \equiv \pm 29 \bmod 120$ the group $\mathrm{PSL}_2(\mathbb{F}_p)$ contains a pair of non-conjugate solvably equivalent subgroups $H_1, H_2 \simeq A_5$.

$H_1, H_2 \simeq A_5 \leq \mathrm{PSL}_2(\mathbb{F}_p)$ are integrally equivalent for $p = 29$; this is open for $p > 29$.

A minimal example of local integral equivalence

An exhaustive search of the 11,759,892 groups of order less than 1024 finds 74 that contain non-conjugate locally integrally equivalent subgroups with trivial normal core.

The smallest two have GAP ids $\langle 384, 18050\rangle$ and $\langle 384, 18046\rangle$, isomorphic to transitive permutation groups 32T9403 and 32T9408. Both are 2-extensions of $D_4 \times S_4$.

Example

The polynomials

$$x^{32} + 12x^{28} + 72x^{24} + 120x^{20} - 234x^{16} + 108x^{12} + 396x^{8} - 432x^{4} + 81,$$
$$x^{32} - 12x^{28} + 72x^{24} - 120x^{20} - 234x^{16} - 108x^{12} + 396x^{8} + 432x^{4} + 81$$

have the same splitting field, with Galois group $G =$32T9403.

They define non-isomorphic number fields K_1, K_2 that are the fixed fields of locally integrally equivalent subgroups $H_1, H_2 \leq G$ that are both isomorphic to D_6.

A minimal example of local integral equivalence

We can view each $M \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[H_1\backslash G], \mathbb{Z}[H_2\backslash G])$ as a 32×32 matrix with entries $a, b, c, \ldots, h \in \mathbb{Z}$, corresponding to the decomposition of G into double cosets $H_1 g H_2$. A (non-trivial) calculation finds that

$$\begin{aligned}\det M = &-\left(2(b-c)^2 + 3(e-f)^2\right)^8 \\ &\cdot \left(2(a-d) + (e+f-2g)\right)^6 \\ &\cdot \left(2(a+b+c+d) - (e+f+2g+4h)\right)^3 \\ &\cdot \left(2(a-b-c+d) - (e+f+2g-4h)\right)^3 \\ &\cdot \left(2(a-d) - 3(e+f-2g)\right)^2 \\ &\cdot \left(2(a+b+c+d) + 3(e+f+2g+4h)\right) \\ &\cdot \left(2(a-b-c+d) + 3(e+f+2g-4h)\right).\end{aligned}$$

One can choose $a, b, c, d, e, f, g, h \in \mathbb{Z}$ so that $\det M = 2^{32}$, and so that $\det M = 3^{12}$. H_1 and H_2 are not integrally equivalent because no $a, \ldots, h \in \mathbb{Z}$ make $\det M = \pm 1$.

This negatively answers Question 2.10 in (Guralnick–Weiss 1993).

$$M := \begin{bmatrix}
h & h & e & h & h & g & h & e & h & b & a & h & f & h & h & g & g & h & g & d & c & c & a & g & h & f & b & d & e & f & g & h \\
h & h & g & h & h & e & h & g & h & c & d & h & g & h & h & e & f & h & e & d & b & b & a & f & h & g & c & a & g & g & f & h \\
g & f & h & c & g & h & e & a & b & h & g & e & h & c & g & h & h & b & a & e & h & h & f & h & f & d & h & g & h & h & d & g \\
h & a & f & h & h & b & h & e & h & g & h & d & e & h & a & g & g & h & g & h & e & f & h & b & h & f & g & h & c & c & g & d \\
h & h & f & h & h & g & h & f & h & b & d & h & e & h & h & g & g & h & g & a & c & c & d & g & h & e & b & a & f & e & g & h \\
e & g & h & b & f & h & g & d & c & h & e & g & h & b & e & h & h & c & a & g & h & h & g & h & g & a & h & f & h & h & d & f \\
h & d & g & h & h & c & h & g & h & e & h & a & g & h & a & e & f & h & f & h & g & g & h & c & h & g & f & h & b & b & e & d \\
h & h & g & h & h & f & h & g & h & c & a & h & g & h & h & f & e & h & f & a & b & b & d & e & h & g & c & d & g & g & e & h \\
e & g & h & g & f & h & g & h & f & a & c & g & h & g & f & h & h & e & h & b & a & d & b & h & g & h & d & c & h & h & h & e \\
d & h & b & h & a & f & d & g & h & e & h & h & b & h & h & c & c & h & e & h & g & g & h & e & a & g & f & h & g & g & f & h \\
g & c & h & e & g & a & f & h & g & h & g & c & h & f & b & h & h & g & h & e & h & h & f & d & e & h & h & g & a & d & h & b \\
d & h & c & h & a & g & a & f & h & g & h & h & c & h & h & b & b & h & g & h & e & f & h & g & d & e & g & h & e & f & g & h \\
g & e & h & c & g & h & f & d & b & h & g & f & h & c & g & h & h & b & d & f & h & h & e & h & e & a & h & g & h & h & a & g \\
h & d & e & h & h & b & h & f & h & g & h & a & f & h & d & g & g & h & g & h & f & e & h & b & h & e & g & h & c & c & g & a \\
g & e & h & e & g & h & e & h & g & a & b & f & h & f & g & h & h & g & h & c & d & a & c & h & f & h & d & b & h & h & h & g \\
a & h & c & h & d & g & d & e & h & g & h & h & c & h & h & b & b & h & g & h & f & e & h & g & a & f & g & h & f & e & g & h \\
e & b & h & g & f & d & g & h & e & h & f & b & h & g & c & h & h & f & h & g & h & h & g & a & g & h & h & e & a & d & h & c \\
a & h & b & h & d & e & a & g & h & f & h & h & b & h & h & c & c & h & f & h & g & g & h & f & d & g & e & h & g & g & e & h \\
f & g & h & b & e & h & g & a & c & h & f & g & h & b & f & h & h & c & d & g & h & h & g & h & g & d & h & e & h & h & a & e \\
h & a & g & h & h & c & h & g & h & f & h & d & g & h & d & f & e & h & e & h & g & g & h & c & h & g & e & h & b & b & f & a \\
h & h & e & a & h & g & h & c & a & g & h & h & f & d & h & g & g & d & b & h & e & f & h & g & h & c & g & h & f & e & b & h \\
c & g & d & g & c & h & b & h & e & h & e & g & a & g & f & a & d & f & h & g & h & h & g & h & b & h & h & f & h & h & h & e \\
h & h & g & d & h & e & h & b & a & e & h & h & g & a & h & f & e & d & c & h & g & g & h & f & h & b & f & h & g & g & c & h \\
b & f & d & e & b & h & c & h & g & h & g & e & a & f & g & d & a & g & h & f & h & h & e & h & c & h & h & g & h & h & h & g \\
h & h & f & d & h & g & h & c & d & g & h & h & e & a & h & g & g & a & b & h & f & e & h & g & h & c & g & h & e & f & b & h \\
f & g & h & g & e & h & g & h & e & d & c & g & h & g & e & h & h & f & h & b & d & a & b & h & g & h & a & c & h & h & h & f \\
g & c & h & f & g & d & e & h & g & h & g & c & h & e & b & h & h & g & h & f & h & h & e & a & f & h & h & g & d & a & h & b \\
f & b & h & g & e & a & g & h & f & h & e & b & h & g & c & h & h & e & h & g & h & h & g & d & g & h & h & f & d & a & h & c \\
b & e & a & f & b & h & c & h & g & h & g & f & d & e & g & a & d & g & h & e & h & h & f & h & c & h & h & g & h & h & h & g \\
c & g & a & g & c & h & b & h & f & h & f & g & d & g & e & d & a & e & h & g & h & h & g & h & b & h & h & e & h & h & h & f \\
h & h & g & a & h & f & h & b & d & f & h & h & g & d & h & e & f & a & c & h & g & g & h & e & h & b & e & h & g & g & c & h \\
g & f & h & f & g & h & f & h & g & d & b & e & h & e & g & h & h & g & h & c & a & d & c & h & e & h & a & b & h & h & h & g
\end{bmatrix}$$

Locally integrally equivalent subgroups need not be isomorphic

Example

Let G be the symmetric group S_{21} and consider the following pair of subgroups:

$$\begin{aligned} H_1 := \langle\, &(4,5)(6,15,7,14)(8,17,9,16)(10,19,11,18)(12,21,13,20),\\ &(1,2)(3,5)(6,20,8,18)(7,21,9,19)(10,14,12,16)(11,15,13,17)\,\rangle,\\ H_2 := \langle\, &(4,5)(6,16,8,14)(7,17,9,15)(10,20,12,18)(11,21,13,19),\\ &(1,2)(3,5)(6,20,8,18)(7,21,9,19)(10,17,12,15)(11,16,13,14)\,\rangle. \end{aligned}$$

Then $\mathbb{Z}_p[H_1\backslash G] \simeq \mathbb{Z}_p[H_2\backslash G]$ for every prime p but $H_1 \not\simeq H_2$.
Indeed, the GAP identifiers of H_1 and H_2 are $\langle 48, 12\rangle$ and $\langle 48, 13\rangle$.

This example negatively answers Question 2.11 in (Guralnick–Weiss 1993).
It is the first of many examples that can be obtained by comparing $\mathcal{P}$-statistics, where $\mathcal{P}$ is the set of finite groups that are p-cyclic for some prime p.

Local integral equivalence does not imply local isomorphism

Example

The group $G := A_4 \times S_5$ contains locally integrally equivalent $H_1, H_2 \simeq D_6$.
Let L be the compositum of the splitting fields of the A_4 and S_5 polynomials $x^4 - 6x^2 - 8x + 60$ and $x^5 + 5x^3 + 10x - 2$, and let $K_1 := L^{H_1}$ and $K_2 := L^{H_2}$.
Above the ramified prime 2 we have

$$2\mathcal{O}_{K_1} = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4\mathfrak{p}_5^6\mathfrak{p}_6^6\mathfrak{p}_7^6\mathfrak{p}_8^6\mathfrak{p}_9^6\mathfrak{p}_{10}^6\mathfrak{p}_{11}^6\mathfrak{p}_{12}^6\mathfrak{p}_{13}^2\mathfrak{p}_{14}^2\mathfrak{p}_{15}^3\mathfrak{p}_{16}^3\mathfrak{p}_{17}^6\mathfrak{p}_{18}^6\mathfrak{p}_{19}^6\mathfrak{p}_{20}^6,$$
$$2\mathcal{O}_{K_2} = \mathfrak{q}_1^2\mathfrak{q}_2^2\mathfrak{q}_3^2\mathfrak{q}_4^2\mathfrak{q}_5^3\mathfrak{q}_6^3\mathfrak{q}_7^3\mathfrak{q}_8^3\mathfrak{q}_9^6\mathfrak{q}_{10}^6\mathfrak{q}_{11}^6\mathfrak{q}_{12}^6\mathfrak{q}_{13}\mathfrak{q}_{14}\mathfrak{q}_{15}^6\mathfrak{q}_{16}^6\mathfrak{q}_{17}^6\mathfrak{q}_{18}^6\mathfrak{q}_{19}^6\mathfrak{q}_{20}^6,$$

which shows that $K_1 \otimes_{\mathbb{Q}} \mathbb{Q}_2 \not\simeq K_2 \otimes_{\mathbb{Q}} \mathbb{Q}_2$.

This example also shows that the sums of the ramification indices can differ even when the products do not, complementing the example in (Mantilla-Soler 2019).

An example of solvable equivalence

The group $G =$ 16T1654 of order 5760 contains non-conjugate $H_1, H_2 \simeq A_5$ of index 96 such that every proper subgroup of H_1 is a proper subgroup of H_2.

It is the Galois group of an extension of $\mathbb{Q}[T]$, so Hilbert irreducibility gives infinitely many examples of corresponding number fields, including the splitting field of

$$x^{16} - 2x^{15} + 3x^{14} - 16x^{13} + 18x^{12} - 10x^{10} + 40x^9 - 39x^8 + 54x^7 + 23x^6 + 16x^5 - 140x^4 - 188x^3 - 28x^2 + 104x - 4.$$

Each $M \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[H_1\backslash G], \mathbb{Z}[H_2\backslash G])$ has entries $a, b, c, d, e \in \mathbb{Z}$ with

$$\det M = -(5a + 6b + 10c + 15d + 60e)(a - 6b - 10c + 3d + 12e)^5 (3a + 2b - 2c - 7d + 4e)^{15}(3a - 2b + 2c + d - 4e)^{30}(a + 2b - 2c + 3d - 4e)^{45}$$

No $a, b, c, d, e \in \mathbb{Z}$ yield $\det M = \pm 1$, so H_1 and H_2 are not integrally equivalent.

This example partially addresses Remark 4.3a in (Scott 1992) by providing a rank-5 example of locally isomorphic permutation modules that are not globally isomorphic (Scott proves a lower bound of 4 and an upper bound of 8 on the minimal rank).

Summary

subgroups $H_1, H_2 \le G$	number fields $K_1, K_2 \le L$
$(\mathbb{Q})$ rational equivalence	(ζ_K) arithmetic equivalence
$(\mathbb{Z}_p)$ local integral equivalence	(cl_K) class group isomorphism
$(\mathbb{Z})$ integral equivalence	$(\mathbb{A}_K)$ local isomorphism
(🧪) solvable equivalence	$(\simeq)$ $\mathrm{Gal}(L/K)$-isomorphism

(🧪) $\Rightarrow$ (ζ_K)	$(\mathbb{Z}) \Rightarrow (\zeta_K)$	$(\mathbb{Z}_p) \Rightarrow (\zeta_K)$	$(\mathbb{Q}) \Rightarrow (\zeta_K)$
(🧪) $\Rightarrow$ (cl_K)	$(\mathbb{Z}) \Rightarrow (\mathrm{cl}_K)$	$(\mathbb{Z}_p) \Rightarrow (\mathrm{cl}_K)$	$(\mathbb{Q}) \not\Rightarrow (\mathrm{cl}_K)$
(🧪) $\Rightarrow$ $(\mathbb{A}_K)$	$(\mathbb{Z})$? $(\mathbb{A}_K)$	$(\mathbb{Z}_p) \not\Rightarrow (\mathbb{A}_K)$	$(\mathbb{Q}) \not\Rightarrow (\mathbb{A}_K)$
(🧪) ? $(\simeq)$	$(\mathbb{Z})$? $(\simeq)$	$(\mathbb{Z}_p) \not\Rightarrow (\simeq)$	$(\mathbb{Q}) \not\Rightarrow (\simeq)$