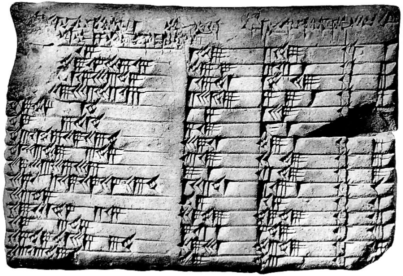
Diophantine computations

Andrew Sutherland

Massachusetts Institute of Technology
Simons Collaboration in Arithmetic Geometry, Number Theory, and Computation



2022 Arf Lecture

A Diophantine problem

Many of the oldest problems in number theory involve equations of the form

$$P(x_1, \ldots, x_n) = k,$$

where P is a polynomial with integer coefficients and k is a fixed integer.
We seek integer solutions in $x_1, \ldots, x_n$. Some notable examples:

- $x^2 + y^2 = z^2$ [Babylonians?]
  $(119, 120, 169)$, $(4601, 4800, 6649)$, $\ldots$ [Babylonians ∼1800 BCE]
- $x^2 - 4729494y^2 = 1$ [Archimedes 251 BCE]
  $776\ldots800$ cattle [Amthor 1880, German-Williams-Zarnke 1965]
- $x^3 + y^3 = z^3$ [Fermat 1637]
  No solutions with $xyz \neq 0$. [Euler 1753]
- $w^4 + x^4 + y^4 = z^4$ [Euler 1769]
  $(2682440, 15365639, 18796760, 20615673)$ [Elkies 1986]
- $v^5 + w^5 + x^5 + y^5 = z^5$ [Euler 1769]
  $(27, 84, 110, 133, 144)$ [Lander-Parkin 1966]

Algorithm to find (or determine existence of) solutions?

Q: Is there an algorithm that can answer all such questions? [Hilbert 1900]
A: No! [Davis, Robinson, Davis-Putnam, Robinson, Matiyasevich 1970]

What if we restrict the degree of the polynomial P?

Q: How about degree one? [Euclid $\sim$250 BCE, Diophantus $\sim$250]
A: Yes! [Euclid $\sim$250 BCE, Brahmagupta 628]

Q: How about degree two? [Babylonians, Diophantus, Hilbert 1900]
A: Yes! [Babylonians, Diophantus, Fermat, Euler, Lagrange, Legendre, Gauss, Siegel 1972]

Q: How about degree three, say, sums of cubes? [Waring 1770]
A: For sums of positive cubes, yes (we can bound the possible solutions).
But for the "easier" Waring problem with no sign constraints, this is an open problem.

It is the simplest example of a potentially undecidable Diophantine equation.

Sums of two cubes

Let us now consider any positive integer k. If we have

$$k = x^3 + y^3 = (x+y)(x^2 - xy + y^2),$$

then we can write $k = rs$ with $r = x + y$ and $s = x^2 - xy + y^2$.
If we now put $y = r - x$, we obtain the quadratic equation

$$s = 3x^2 - 3rx + r^2,$$

whose integer solutions we can find using the quadratic formula.

This yields an algorithm to determine all integer solutions to $x^3 + y^3 = k$:

- Factor the integer k.
- Use this factorization to enumerate all positive integers r, s for which $k = rs$.
- If $t := \sqrt{12s - 3r^2} \in \mathbb{Z}$ then output $x = (3r + t)/6$ and $y = (3r - t)/6$.

For $k = 1729 = 19 \cdot 91$ we find $t = 3$, yielding $x = 10$ and $y = 9$.
For $k = 1729 = 13 \cdot 133$ we find $t = 33$, yielding $x = 12$ and $y = 1$.

Sums of four or more cubes

Every integer has infinitely many representations as the sum of five cubes.

This follows from the identity

$$6m = (m+1)^3 + (m-1)^3 - m^3 - m^3.$$

If we write $k = 6n + r$, then $r^3 \equiv r \bmod 6$ and, we can apply this identity to $m = f(n) := (k - (6n+r)^3)/6$ for any integer n, yielding the parameterization

$$k = (6n+r)^3 + (f(n)+1)^3 + (f(n)-1)^3 - f(n)^3 - f(n)^3.$$

A more complicated collection of similar identities shows that all $k \not\equiv \pm 4 \bmod 9$ can be represented as a sum of four cubes in infinitely many ways [Demjanenko 1966].

It is conjectured that in fact every integer k has infinitely many representations as a sum of four cubes [Sierpinski 1960], but the case $k \equiv \pm 4 \bmod 9$ remains open.

Sums of three cubes

Not every integer is the sum of three cubes. Indeed, if $x^3 + y^3 + z^3 = k$ then

$$x^3 + y^3 + z^3 \equiv k \bmod 9$$

The cubes modulo 9 are $0, \pm 1$; we cannot write ± 4 as a sum of three elements of $\{0, \pm 1\}$. This rules out all $k \equiv \pm 4 \bmod 9$, including $4, 5, 13, 14, 22, 23, 31, 32, \ldots$

There are infinitely many ways to write $k = 0, 1, 2$ as sums of three cubes. For all $n \in \mathbb{Z}$,

$$\begin{aligned} n^3 + (-n)^3 + 0^3 &= 0, \\ (9n^4)^3 + (3n - 9n^4)^3 + (1 - 9n^3)^3 &= 1, \\ (1 + 6n^3)^3 + (1 - 6n^3)^3 + (-6n^2)^3 &= 2. \end{aligned}$$

Multiplying by m^3 yields similar parameterizations for k of the form m^3 or $2m^3$.
For $k \not\equiv \pm 4 \bmod 9$ not of the form m^3 or $2m^3$ the question is completely open.

Remark 1: The parameterizations above are not exhaustive [Payne and Vaserstein 1992].
Remark 2: Every $k \in \mathbb{Z}$ is the sum of three rational cubes in infinitely many ways [Ryley 1825].

Mordell's challenge

There are two easy ways to write 3 as a sum of three cubes:

$$1^3 + 1^3 + 1^3 = 3 \qquad \text{and} \qquad (-5)^3 + 4^3 + 4^3 = 3.$$

In his paper *On the integer solutions of the equation $x^2 + y^2 + z^2 + 2xyz = n$* Mordell wrote:

> *I do not know anything about the integer solutions of $x^3 + y^3 + z^3 = 3$ beyond the existence of... it must be very difficult indeed to find out anything about any other solutions. One may wonder if the problem of finding other solutions is comparable in difficulty with that of finding when an assigned sequence, e.g. 123456789, occurs in the decimal expansion of π*

This remark sparked a 65 year search for additional solutions.

None were found, but researchers did have success with many other values of $k \not\equiv \pm 4 \bmod 9$. But some proved to be particularly difficult.

20th century timeline for $x^3 + y^3 + z^3 = k$ with $k > 0$ and $|x|, |y|, |z| \leq N$

- 1908 Werebrusov finds a parametric solution for $k = 2$.
- 1936 Mahler finds a parametric solution for $k = 1$.
- 1942 Mordell proves any other parameterization has degree at least five (likely none exist).
- 1953 Mordell asks about $k = 3$.
- 1955 Miller, Woollett check $k \leq 100$, $N = 3200$, solve all but nine $k \leq 100$.
- 1963 Gardiner, Lazarus, Stein: $k \leq 1000$, $N = 2^{16}$, crack $k = 87$, all but seventy $k \leq 1000$.
- 1992 Heath-Brown, Lioen, te Riele crack $k = 39$.
- 1992 Heath-Brown conjectures infinity of solutions for all $k \not\equiv \pm 4 \bmod 9$.
- 1994 Koyama checks $k \leq 1000$, $N = 2^{21} - 1$, finds 16 new solutions.
- 1994 Koyama checks $k \leq 1000$, $N = 3414387$, finds 2 new solutions.
- 1994 Conn, Vaserstein crack $k = 84$.
- 1995 Jagy cracks $k = 478$.
- 1995 Bremner cracks $k = 75$ and $k = 768$.
- 1995 Lukes cracks $k = 110$, $k = 435$, and $k = 478$.
- 1996 Elkies checks $k \leq 1000$, $N = 10^7$ finding several new solutions (follow up by Bernstein).
- 1997 Koyama, Tsuruoka, Sekigawa check $k \leq 1000$, $N = 2 \cdot 10^7$ finding five new solutions.
- 1999-2000 Bernstein checks $k \leq 1000$, $N \geq 2 \cdot 10^9$, cracks $k = 30$ and ten other $k \leq 1000$.
- 1999-2000 Beck, Pine, Tarrant, Yarbrough Jensen also crack $k = 30$, and $k = 52$.

Poonen's challenge

In 2008 Bjorn Poonen opened his AMS Notices article *Undecidability in number theory* (winner of the Chauvenet prize) with the following challenge:

> *Does the equation* $x^3 + y^3 + z^3 = 29$ *have a solution in integers?*
> *Yes:* $(3, 1, 1)$*, for instance. How about* $x^3 + y^3 + z^3 = 30$*?*
> *Again yes, although this was not known until 1999: the smallest solution is* $(283059965, -2218888517, 2220422932)$*.*
> *And how about 33? This is an unsolved problem.*

This spurred another 10 years of searches for solutions to 33 (as well as 3).
Elsenhans and Jahnel searched to $N = 10^{14}$ cracking nine more $k \leq 1000$.
Huisman pushed on to $N = 10^{15}$ and cracked $k = 74$ in 2016.

In the spring of 2019 Andrew Booker finally answered Poonen's challenge with

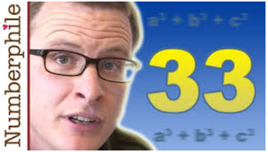
$$8866128975287528^3 - 8778405442862239^3 - 2736111468807040^3 = 33,$$

leaving 42 as the only unresolved case below 100 (and ten other $k \leq 1000$).
But still no progress on Mordell's challenge, even with $N = 10^{16}$ [Booker19].

Popularization

Numberphile host Brady Haran has made several YouTube videos popularizing this problem.

74 is Cracked!
(Sander Huisman)



The uncracked problem with 33
(Tim Browning)

42 is the new 33
(Andrew Booker)

Booker's breakthrough with 33 received international press coverage.

Mathematician solves 64-year-old 'Diophantine puzzle' (Newsweek):

> *"...the mathematician is now working with ... in an attempt to find the solution for the final unsolved number below a hundred: 42."*

The significance of 42 (according to Douglas Adams)

"O Deep Thought computer...We want you to tell us....The Answer."
"The Answer to what?" asked Deep Thought.
"Life!" urged Fook. "The Universe!" said Lunkwill. "Everything!" they said in chorus.

Deep Thought paused for a moment's reflection...
"There is an answer. But, I'll have to think about it."

seven and a half million years pass...

"Good Morning," said Deep Thought at last. "Er...good morning, O Deep Thought" said Loonquawl nervously, "do you have..."

"An Answer for you?" interrupted Deep Thought. "I have."

"Forty-two," said Deep Thought, with infinite majesty and calm.

Deep Thought then designs Earth to compute the Ultimate Question whose answer is 42.

Search algorithms

We seek solutions to $x^3 + y^3 + z^3 = k$ for some fixed k (such as $k = 3$ or $k = 42$).
How long does it take to check all $x, y, z \in \mathbb{Z}$ with $\max(|x|, |y|, |z|) \leq N$?

1. Naive brute force: $O(N^3)$ arithmetic operations.
2. Less naive brute force (is $x^3 + y^3 - k$ a cube?): $O(N^{2+o(1)})$.
3. Apply the sum of two cubes algorithm to $k - z^3$: $O(N^{1+o(1)})$ expected time.

None of these is fast enough to go past $N = 10^{16}$ in a reasonable amount of time.

We instead use the approach suggested in [Heath-Brown 1989], which seeks solutions for a fixed k (by contrast, Elkies' approach seeks solutions to $x^3 + y^3 + z^3 \leq b$ with b small).

With suitable optimizations this gives a heuristic complexity of $O(N(\log\log N)^{1+o(1)})$ arithmetic operations (each takes less than a nanosecond in the practical range of interest).

The asymptotic bit complexity is $O(N(\log N)(\log\log N)^{2+o(1)})$.

Assume $x^3 + y^3 + z^3 = k > 0, \quad |x| > |y| > |z| \geq \sqrt{k}, \quad k \equiv \pm 3 \bmod 9$ cube free, and put

$$k - z^3 = x^3 + y^3 = (x+y)(x^2 - xy + y^2).$$

Define $d := |x+y|$ so that z is a cube root of k modulo d. Then

$$\{x, y\} = \left\{ \frac{\operatorname{sgn}(k - z^3)}{2} \left(d \pm \sqrt{\frac{4|k - z^3| - d^3}{3d}} \right) \right\},$$

Thus d, z determine x, y, and one finds that $d < \alpha|z|$, where $\alpha := \sqrt[3]{2} - 1 \approx 0.26$.
One also finds that $3 \nmid d$ and $\operatorname{sgn}(z)$ is determined by $d \bmod 3$ and $k \bmod 9$.

Given N, our strategy is to enumerate all $d \in \mathbb{Z} \cap (0, \alpha N)$ coprime to 3, and for each d enumerate all $z \in \mathbb{Z}$ satisfying $z^3 \equiv k \bmod d$ with $|z| \leq N$ such that

$$3d(4 \operatorname{sgn}(z)(z^3 - k) - d^3) = \square. \tag{1}$$

Every such (d, z) yields a solution (x, y, z), and we will find all solutions satisfying our assumptions with $|z| \leq N$, even when $|x|, |y| > N$.

Complexity obstacles

problem: To compute cube roots of k mod d we need the factorization of d.

solution: Enumerate d combinatorially, as a product of prime powers along with the cube roots of k mod d (this also lets us efficiently skip d for which there are none).

problem: There are $\Omega(N \log N)$ pairs (d, z) we potentially need to consider.

solution: For $d \leq N^{3/4}$ (say) we sieve arithmetic progressions of z mod d using auxiliary $p \nmid d$. Each reduces the number of pairs (d, z) by a factor of ≈ 2 and $O(\log \log N)$ suffice.

We don't literally sieve, we use the CRT to lift progressions mod d to progressions mod pd, but only use the lifts that yield solutions modulo p (about half, on average).

With this approach the total number of pairs (d, z) with $d \leq N^{3/4}$ we need to consider becomes $o(N)$, and for $d > N^{3/4}$ we heuristically expect $O(N)$.

CRT sieving

For $k = 33$ and $d = 5$ we have $z \equiv 2 \bmod d$ and $\operatorname{sgn}(z) = +1$ and $z \equiv k + d \equiv 0 \bmod 2$, and only $z \equiv 0 \bmod 7$ satisfies $3d(4\operatorname{sgn}(z)(z^3 - k) - d^3) = \square \bmod 7$.

p	modulus	residue classes	$\lvert z\rvert \le 10^{16}$ to check
	5	1	2.0×10^{15}
2	10	1	1.0×10^{15}
7	70	1	1.4×10^{14}
13	910	3	3.3×10^{13}
17	15470	27	1.7×10^{13}
23	355810	324	9.1×10^{12}
29	10318490	4860	4.7×10^{12}
43	443695070	92340	2.1×10^{12}
67	29727569690	2493180	8.4×10^{11}
103	3061939678070	107206740	3.5×10^{11}

Cubic reciprocity constraints allow only 14 residue classes modulo $27k = 891$.
This further reduces the number of z to check by another factor of 63.6.
This leaves only 5.5×10^9 values of z to check, which takes about a minute.

The conjecture of Heath-Brown

[Heath-Brown 1992] uses products of local densities to heuristically estimate

$$R_k(N_1, N_2) := \#\Big\{(x,y,z) \in \mathbb{Z}^3 : x^3 + y^3 + z^3 = k,\ N_1 \le \max(|x|,|y|,|z|) \le N_2\Big\}.$$

Assume k is cube free, and for each prime power $q = p^n$ define

$$N(q) := \#\Big\{(x,y,z) \bmod q : x^3 + y^3 + z^3 \equiv k \bmod q\Big\},$$

$$\sigma_p := \frac{N(p)}{p^2}\ (p \ne 3), \qquad \sigma_3 = \frac{N(9)}{81}, \qquad \sigma_\infty := 4\int_1^\infty \int_{N_1/t}^{N_2/t} \frac{dz}{z} \frac{dt}{\left(t^3+1\right)^{2/3}} = c \log \frac{N_2}{N_1},$$

where $c = \frac{2\Gamma(1/3)^2}{3\Gamma(2/3)} \approx 3.5332$. For $N_2 \gg N_1 \gg 0$ we should then expect

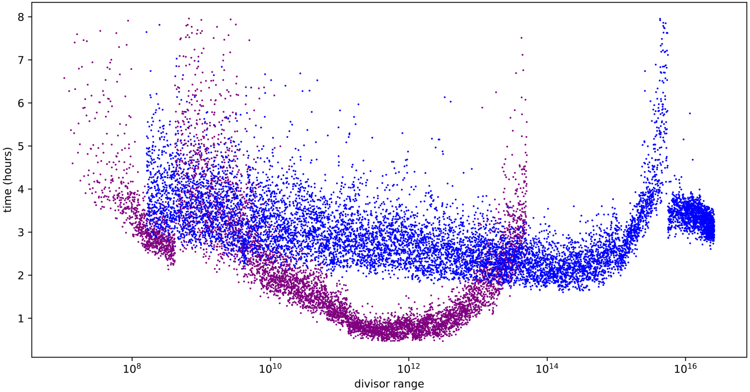
$$R_k(N_1, N_2) \sim \prod_{p \le \infty} \sigma_p = \delta_k \log \frac{N_2}{N_1},$$

where δ_k is an explicit constant that depends only on k. If we put $\omega_k := \exp(6/\delta_k)$ then we should expect one solution with $|x| > |y| > |z|$ in $[N, \omega_k N]$ on average (as $N \to \infty$).

Heath-Brown's predictions for $3 \leq k < 100$ compared to Huisman's data

			$N = 10^5$		$N = 10^{10}$		$N = 10^{15}$	
k	$\delta_k/6$	ω_k	expect	actual	expect	actual	expect	actual
3	0.061	12969857	0.7	2	1.4	2	2.1	2
93	0.072	1185438	0.8	2	1.6	3	2.5	3
74	0.086	106692	1.0	0	2.0	0	3.0	1
33	0.089	77368	1.0	0	2.0	0	3.1	0
30	0.090	68020	1.0	0	2.1	1	3.1	3
39	0.090	68358	1.0	0	2.1	1	3.1	1
12	0.100	22518	1.1	1	2.3	2	3.4	2
87	0.104	14593	1.2	1	2.4	2	3.6	3
75	0.112	7287	1.3	0	2.6	1	3.9	4
42	0.113	6728	1.3	0	2.6	0	3.9	0
60	0.119	4531	1.4	3	2.7	5	4.1	8
...								
9	0.427	11	4.9	3	9.8	8	14.8	15
44	0.434	11	5.0	1	10.0	7	15.0	16
7	0.437	10	5.0	3	10.1	11	15.1	18
...								
83	1.210	3	13.9	16	27.9	32	41.8	49
90	1.854	2	21.3	20	42.7	36	64.0	48
99	1.989	2	22.9	21	45.8	35	68.7	56

The search for 42



Each dot represents 50 cores, approximately 90 core-years.
Purple dots correspond to smooth values of d, blue dots do not.

The result for 42

$$-80538738812075974^3 + 80435758145817515^3 + 12602123297335631^3 = 42$$

$$d = |x + y| = 11 \cdot 43 \cdot 215921 \cdot 1008323 = 102980666258459 \approx 1.030 \times 10^{14}$$

$$x \approx -8.053873 \times 10^{16}, \quad y \approx 8.043575 \times 10^{16}, \quad z \approx 1.260212 \times 10^{16}$$

$$\begin{array}{rr} - & 522413599036979150280966144853653247149764362110424 \\ + & 520412211582497361738652718463552780369306583065875 \\ + & \underline{2001387454481788542313426390100466780457779044591} \\ & 42 \end{array}$$

The result for 3

$$569936821221962380720^3 - 569936821113563493509^3 - 472715493453327032^3 = 3$$

$$d = |x + y| = 167 \cdot 649095133 = 108398887211 \approx 1.084 \times 10^{11}$$

$$x \approx 5.699368 \times 10^{20}, \quad y \approx -5.699368 \times 10^{20}, \quad z \approx -4.727155 \times 10^{17}$$

$$\begin{array}{r} 185131426470358721030003064550489120286063150089838997749248000 \\ -185131426364725746289073278168542399539619802127338908944671229 \\ - \quad \underline{105632974740929786381946720746443347962500088804576768} \\ 3 \end{array}$$

Heath-Brown's predictions for $100 < k \leq 1000$

			$N = 10^5$		$N = 10^{10}$		$N = 10^{15}$	
k	$\delta_k/6$	ω_k	expect	actual	expect	actual	expect	actual
858	0.029	1720798182665417	0.3	1	0.7	2	1.0	2
276	0.032	42958715811596	0.4	1	0.7	1	1.1	2
390	0.033	15332443619105	0.4	0	0.8	0	1.1	0
516	0.033	13632255817671	0.4	0	0.8	1	1.1	1
663	0.033	12076668982001	0.4	0	0.8	1	1.1	1
975	0.039	163996624946	0.5	0	0.9	0	1.3	0
165	0.040	90472906051	0.5	0	0.9	0	1.4	0
555	0.043	14746456526	0.5	1	1.0	2	1.5	2
921	0.044	6885076231	0.5	0	1.0	0	1.5	0
348	0.045	5369191063	0.5	2	1.0	2	1.5	3
906	0.050	536676769	0.6	0	1.1	0	1.7	0
366	0.051	324767552	0.6	0	1.2	0	1.8	1
579	0.051	348505529	0.6	0	1.2	0	1.8	0
654	0.057	46795226	0.7	2	1.3	2	2.0	3
114	0.058	26824751	0.7	0	1.3	0	2.0	0
705	0.062	8959243	0.7	1	1.4	2	2.2	2
732	0.063	7553865	0.7	0	1.5	0	2.2	0
402	0.079	321328	0.9	1	1.8	2	2.7	3
633	0.080	282820	0.9	0	1.8	0	2.8	0
537	0.089	80345	1.0	2	2.0	3	3.1	3
795	0.089	71223	1.0	0	2.1	0	3.1	0
641	0.128	2519	1.5	1	2.9	1	4.4	2
627	0.130	2248	1.5	0	3.0	0	4.5	0
956	0.217	102	2.5	3	5.0	6	7.5	8
782	0.453	10	5.2	3	10.4	5	15.7	11
855	2.641	2	30.4	27	60.8	51	91.2	77

A better search strategy

To check $|z| \leq N$ we need to check $d \leq B := (\sqrt[3]{2} - 1)N \approx N/4$.

The value of B determines the number of arithmetic progressions (about $B/2$).
The value of N/B determines the length of these arithmetic progressions.

It is much cheaper to increase N than it is to increase B.

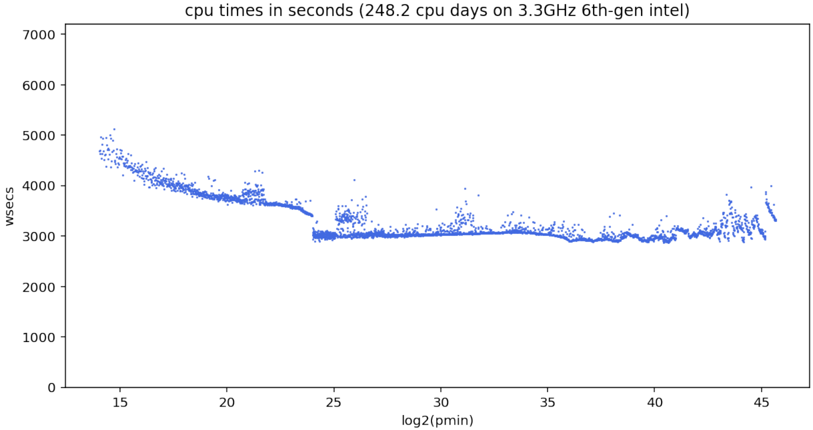
On the other hand, one heuristically expects the density of solutions to decay exponentially with N/B. This leads to an optimization problem. We want to choose $R := N/B$ to minimize $T(B, N) = T(B, RB)$. The optimal R should satisfy

$$T_B(B, RB)\frac{\partial B}{\partial R} + T_N(B, RB)(B + R\frac{\partial B}{\partial R}) = 0,$$

where T_B and T_N denote partial derivatives of $T(B, N)$. We typically want $R \in [50, 250]$.

We should also skip prime values of d close to B, which produce few progressions.

The search for 42 redux



Each dot represents 2 cores, approximately 0.7 core years.