Modular polynomials and isogeny volcanoes

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Isogenies

An *isogeny* $\phi : E_1 \to E_2$ is a morphism of elliptic curves,
a nonzero rational map that preserves the identity.

Over a finite field, E_1 and E_2 are isogenous if and only if

$$\#E_1(\mathbb{F}_q) = \#E_2(\mathbb{F}_q).$$

Some applications of isogenies

Isogenies make hard problems easier:

- Counting the points on E.
  Polynomial time (Schoof-Elkies-Atkin).
- Constructing E with the CM method.
  $|D| > 10^{15}$ (BBEL, S, Enge-S).
- Computing the endomorphism ring of E.
  Heuristically subexponential time (Bisson-S).

These algorithms all rely on modular polynomials $\Phi_\ell(X, Y)$.

Properties of isogenies

Degree

The kernel of $\phi : E_1 \to E_2$ is a finite subgroup of $E_1(\overline{F})$.
When ϕ is separable, we have $|\ker \phi| = \deg \phi$.

An ℓ-isogeny is a (separable) isogeny of degree ℓ.
For prime ℓ, the kernel is necessarily cyclic.

Orientation

We say that $\phi : E_1 \to E_2$ is *horizontal* if $\mathrm{End}(E_1) \cong \mathrm{End}(E_2)$.
Otherwise ϕ is *vertical*.

Isogenies from kernels

Any finite subgroup G of $E(\overline{F})$ determines a separable isogeny with G as its kernel

Given G, we can compute ϕ explicitly using Vélu's formula.

The complexity depends both on the size of $\ker \phi$,
and the field in which the points of $\ker \phi$ are defined.

If E is defined over F, so is ϕ, but the points in $\ker \phi$ may have coordinates in an extension of degree up to $\ell^2 - 1$.

The classical modular polynomial Φ_ℓ

The modular function $j : \mathbb{H} \to \mathbb{C}$ is a complex analytic function

$$j(z) = 1/q + 744 + 196884q + 21493760q^2 + \ldots,$$

where $q = e^{2\pi i z}$. The function $j(\ell z)$ is algebraic over $\mathbb{C}(j)$, and its minimal polynomial $\Phi_\ell(X)$ has coefficients in $\mathbb{Z}[j]$.

$$\Phi_\ell(X, Y) = \Phi_\ell(Y, X); \qquad \deg_X \Phi_\ell = \ell + 1.$$

The modular equation $\Phi_\ell(X, Y) = 0$ parameterizes pairs of elliptic curves related by a cyclic ℓ-isogeny.

Parametrizing isogenies

Assuming char $F \neq \ell$, for all elliptic curves E_1/F and E_2/F:

$$\Phi_\ell\big(j(E_1), j(E_2)\big) = 0 \quad \Longleftrightarrow \quad E_1 \text{ and } E_2 \text{ are } \ell\text{-isogenous.}$$

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The ℓ-isogeny graph G_ℓ has vertex set $\{j(E) : E/F\}$,
and edges (j_1, j_2) whenever $\Phi_\ell(j_1, j_2) = 0$.

The neighbors of j_0 are the roots of $\Phi_\ell(X, j_0)$ that lie in F.

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Φ_ℓ is big: $O(\ell^3 \log \ell)$ bits.

ℓ	coefficients	largest	average	total
127	8258	7.5kb	5.3kb	5.5MB
251	31880	16kb	12kb	48MB
503	127262	36kb	27kb	431MB
1009	510557	78kb	60kb	3.9GB
2003	2009012	166kb	132kb	33GB
3001	4507505	259kb	208kb	117GB
4001	8010005	356kb	287kb	287GB
5003	12522512	454kb	369kb	577GB
10007	50085038	968kb	774kb	4.8TB
20011	200250080	2.0Mb	1.6Mb	40TB*

Size of $\Phi_\ell(X, Y)$

*Estimated

Algorithms to compute Φ_ℓ

***q*-expansions**:

(Atkin ?, Elkies ’92, ’98, LMMS ’94, Morain ’95, Müller ’95, BCRS ’99)

Φ_ℓ:	$O(\ell^4 \log^{3+\epsilon} \ell)$	(via the CRT)
Φ_ℓ mod p:	$O(\ell^3 \log \ell \log^{1+\epsilon} p)$	$(p > \ell + 1)$

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isogenies: (Charles-Lauter 2005)

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evaluation-interpolation: (Enge 2009)

Φ_ℓ:	$O(\ell^3 \log^{4+\epsilon} \ell)$	(floating-point)
Φ_ℓ mod m:	$O(\ell^3 \log^{4+\epsilon} \ell)$	(reduces Φ_ℓ)

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For certain p we can compute Φ_ℓ mod p in expected time

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Computing Φ_ℓ mod m uses $O(\ell^2 \log(\ell m))$ space.

In practice the algorithm is much faster than other methods.
It is probabilistic, but its output is unconditionally correct.

Explicit Chinese Remainder Theorem

Suppose $c \equiv c_i \bmod p_i$ for distinct primes $p_i, \ldots, p_n$. Then

$$c \equiv \sum c_i a_i M_i \bmod M,$$

where $M = \prod p_i$, $M_i = M/p_i$ and $a_i = 1/M_i \bmod p_i$.

With $M > 4c$, the explicit CRT computes $c \bmod m$ directly via

$$c \equiv \left(\sum c_i a_i M_i - rM\right) \bmod m,$$

where the integer $r \approx \sum a_i c_i / p_i$ (use $O(\log n)$ bits of precision).

Using an online algorithm, this can be applied to N coefficients c in parallel, using $O\big(\log M + n \log m + N(\log m + \log n)\big)$ space.

Montgomery-Silverman, Bernstein, S.

Some performance highlights

Level records

1. **5003**: Φ_ℓ
2. **20011**: Φ_ℓ mod m
3. **60013**: $\Phi_\ell^{\mathfrak{f}}$

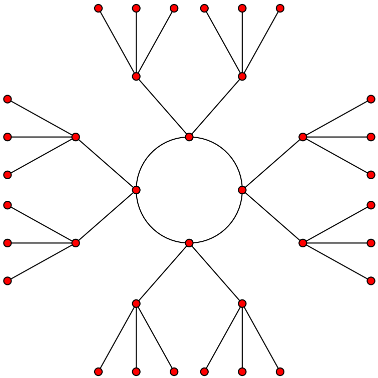
Speed records

1. **251**: Φ_ℓ in 28s Φ_ℓ mod m in 4.8s (vs 688s)
2. **1009**: Φ_ℓ in 2830s Φ_ℓ mod m in 265s (vs 107000s)

Single core CPU times (AMD 3.0 GHz), using $m \approx 2^{256}$.

Effective throughput when computing Φ_{1009} mod m is over 100 Mb/s.

A 3-volcano of depth 2



ℓ-volcanoes

An ℓ-*volcano* is a connected undirected graph whose vertices are partitioned into levels $V_0, \ldots, V_d$, such that:

1. The subgraph on V_0 (the *surface*) is a regular connected graph of degree at most 2.
2. For $i > 0$, each $v \in V_i$ has exactly one neighbor in V_{i-1}. All edges not on the surface arise in this manner.
3. For $i < d$, each $v \in V_i$ has degree ℓ+1.

The integers ℓ, d, and $|V_0|$ uniquely determine the shape.

The ℓ-isogeny graph G_ℓ

Some facts about G_ℓ (Kohel, Fouquet-Morain):

- The ordinary components of G_ℓ are ℓ-volcanoes
  (provided they don't contain $j = 0, 1728$).
- The curves in level V_i of a given ℓ-volcano all have the same endomorphism ring, isomorphic to an imaginary quadratic order $\mathcal{O}_i$.
- The order $\mathcal{O}_0$ is maximal at ℓ, and $[\mathcal{O}_0 : \mathcal{O}_i] = \ell^i$.

Curves in the same ℓ-volcano are necessarily isogenous, but isogenous curves need not lie in the same ℓ-volcano.

The CM action

Let $E/\mathbb{F}_q$ be an ordinary elliptic curve with $\mathrm{End}(E) \cong \mathcal{O}$.

The class group $\mathrm{cl}(\mathcal{O})$ acts on the set

$$\{j(E/\mathbb{F}_q) : \mathrm{End}(E) \cong \mathcal{O}\}.$$

Horizontal ℓ-isogenies are the action of an ideal with norm ℓ.

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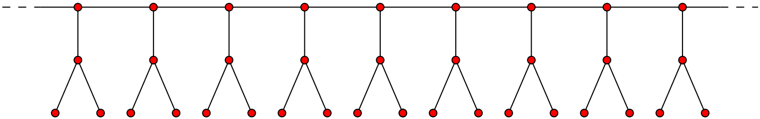
A horizontal isogeny of large degree may be equivalent to a sequence of isogenies of small degree, via relations in $\mathrm{cl}(\mathcal{O})$.

Under the ERH this is always true, and "small" $= O(\log^2 |D|)$.

Running the rim



$$\Phi_2(X,Y) = X^3 + Y^3 - X^2Y^2 + 1488X^2Y - 162000X^2 + 1488XY^2 + 40773375XY$$
$$+ 8748000000X - 162000Y^2 + 8748000000Y - 157464000000000$$

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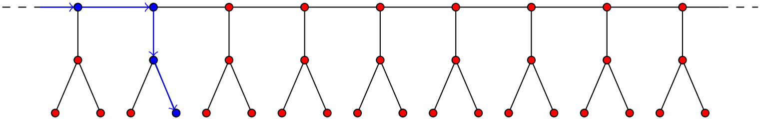

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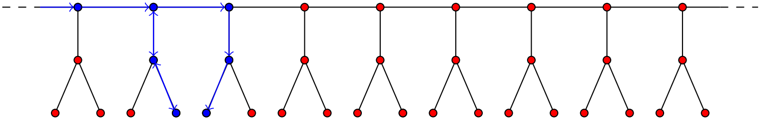

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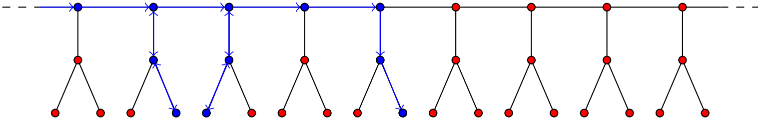

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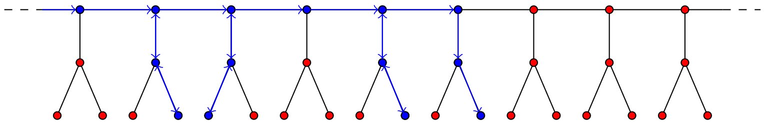

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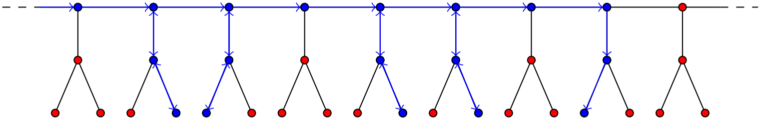

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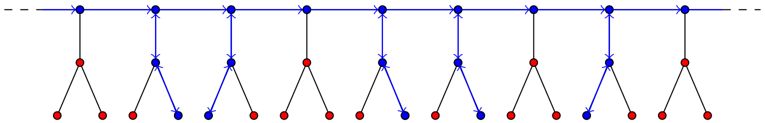

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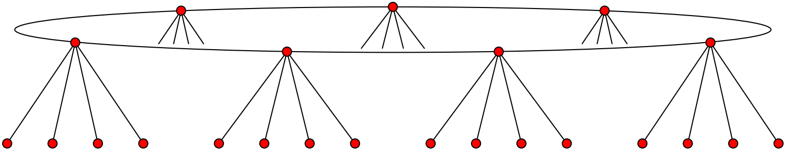
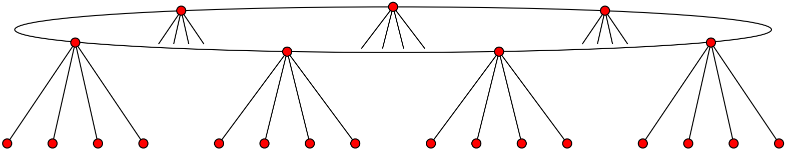
Mapping a volcano



Mapping a volcano

Example
$\ell = 5, \quad p = 4451, \quad D = -151$

General requirements
$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$



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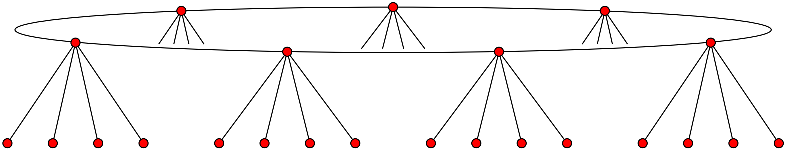

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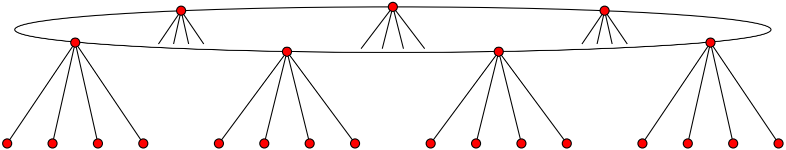

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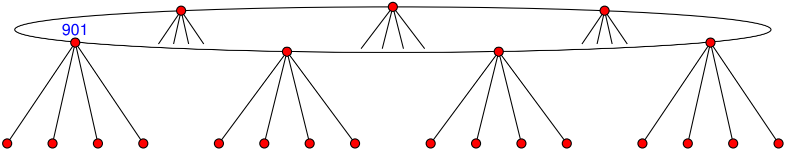

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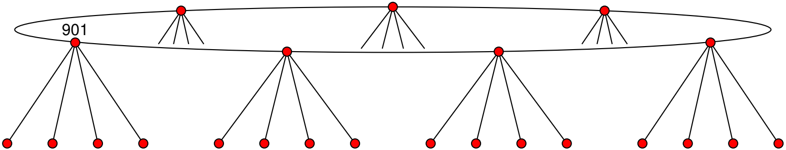

2. Enumerate surface using the action of α_{ℓ_0}

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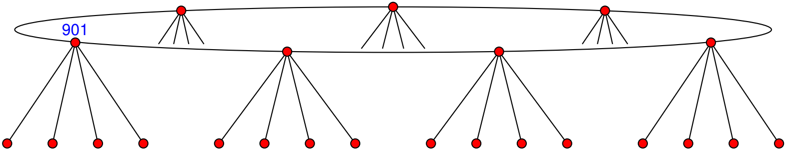

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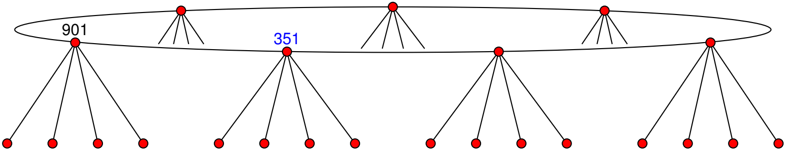

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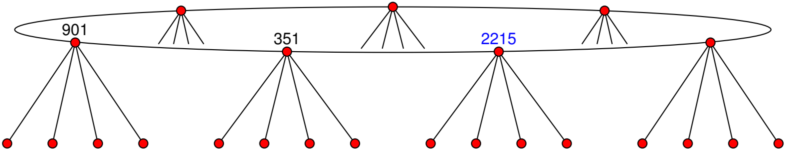

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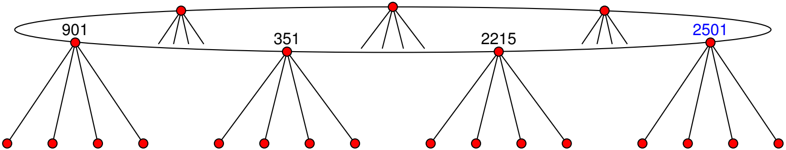

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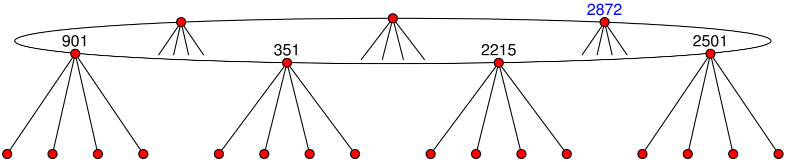
$\ell_0 \neq \ell, \left(\frac{D}{\ell_0}\right) = 1, \alpha_\ell = \alpha_{\ell_0}^k$



2. Enumerate surface using the action of α_{ℓ_0}

$901 \xrightarrow{2} 1582 \xrightarrow{2} 2501 \xrightarrow{2} 351 \xrightarrow{2} 701 \xrightarrow{2} 2872 \xrightarrow{2} 2215 \xrightarrow{2}$

Mapping a volcano

Example

$\ell = 5, \quad p = 4451, \quad D = -151$

$t = 52, \quad v = 2, \quad h(D) = 7$

$\ell_0 = 2, \quad \alpha_5 = \alpha_2^3$

General requirements

$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$

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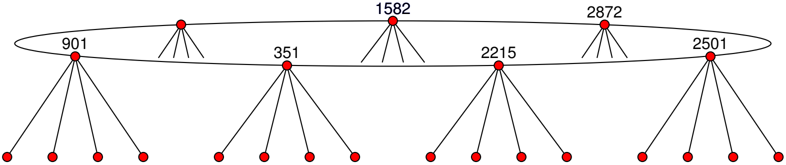

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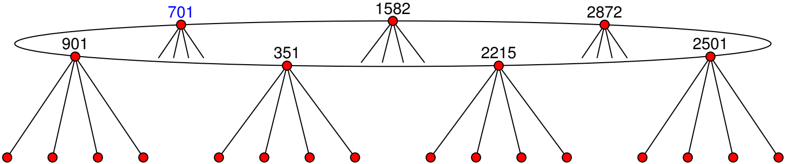

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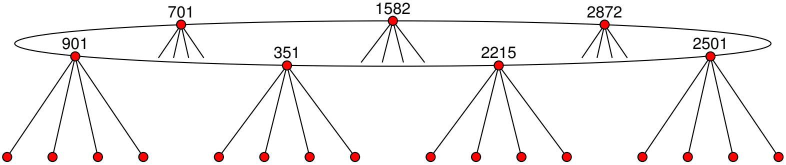

3. Descend to the floor using Vélu's formula

Mapping a volcano

Example

$\ell \;= 5, \quad p = 4451, \quad D = -151$

$t \;= 52, \quad v = 2, \quad h(D) = 7$

$\ell_0 = 2, \quad \alpha_5 = \alpha_2^3$

General requirements

$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$

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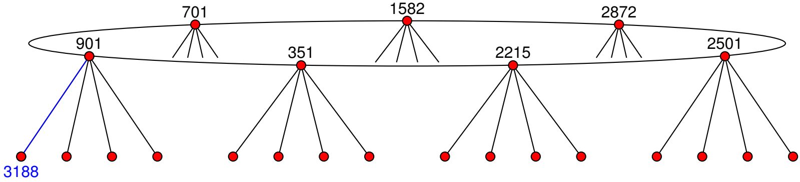

3. Descend to the floor using Vélu's formula: $901 \xrightarrow{5} 3188$

Mapping a volcano

Example

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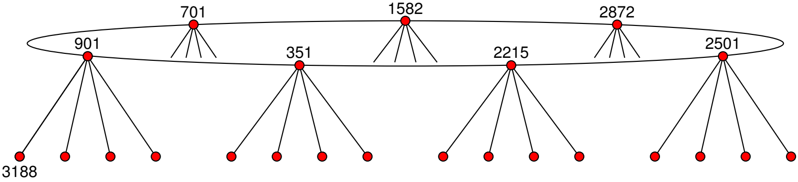

4. Enumerate floor using the action of β_{ℓ_0}

Mapping a volcano

Example

$\ell = 5, \quad p = 4451, \quad D = -151$
$t = 52, \quad v = 2, \quad h(D) = 7$
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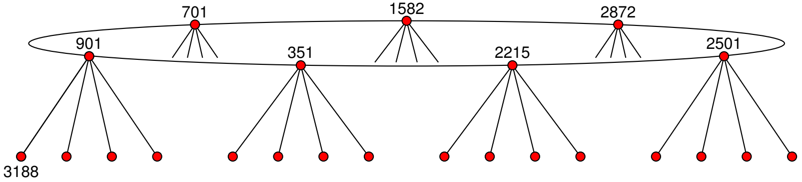

4. Enumerate floor using the action of β_{ℓ_0}

$$
\begin{array}{llllllll}
3188 \xrightarrow{2} & 945 \xrightarrow{2} & 3144 \xrightarrow{2} & 3508 \xrightarrow{2} & 2843 \xrightarrow{2} & 1502 \xrightarrow{2} & 676 \xrightarrow{2} \\
2970 \xrightarrow{2} & 3497 \xrightarrow{2} & 1180 \xrightarrow{2} & 2464 \xrightarrow{2} & 4221 \xrightarrow{2} & 4228 \xrightarrow{2} & 2434 \xrightarrow{2} \\
1478 \xrightarrow{2} & 3244 \xrightarrow{2} & 2255 \xrightarrow{2} & 2976 \xrightarrow{2} & 3345 \xrightarrow{2} & 1064 \xrightarrow{2} & 1868 \xrightarrow{2} \\
3328 \xrightarrow{2} & 291 \xrightarrow{2} & 3147 \xrightarrow{2} & 2566 \xrightarrow{2} & 4397 \xrightarrow{2} & 2087 \xrightarrow{2} & 3341 \xrightarrow{2}
\end{array}
$$

Mapping a volcano

Example

$\ell = 5, \quad p = 4451, \quad D = -151$

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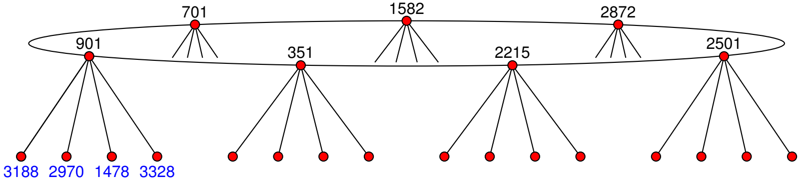

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Mapping a volcano

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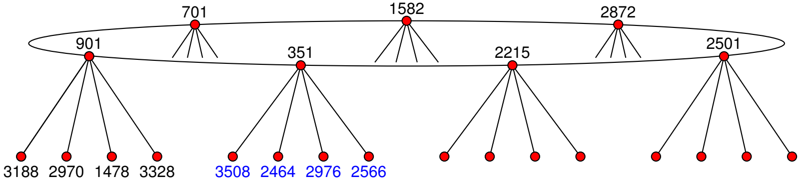

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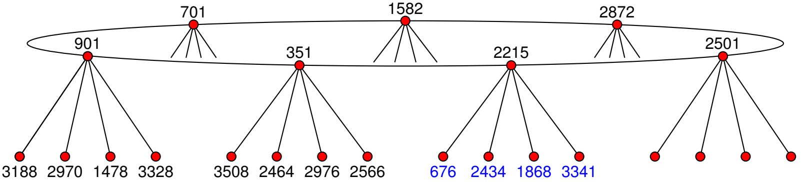

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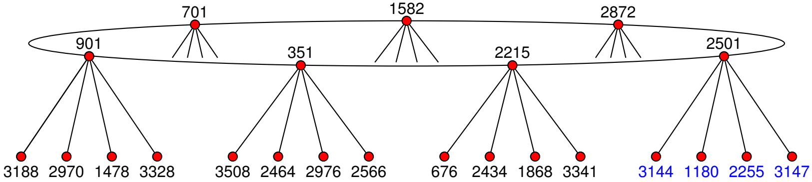

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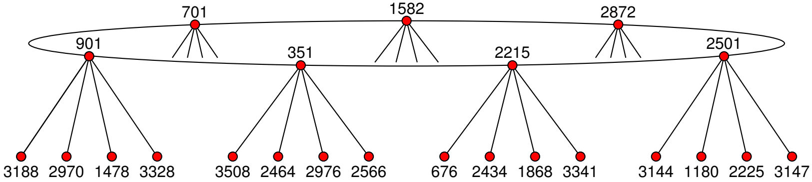

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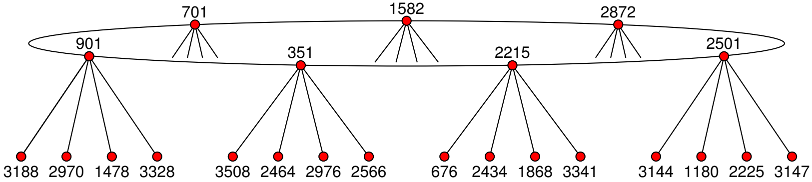

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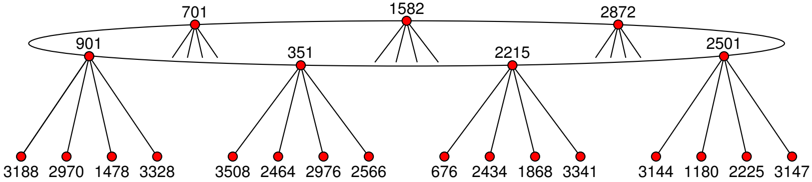

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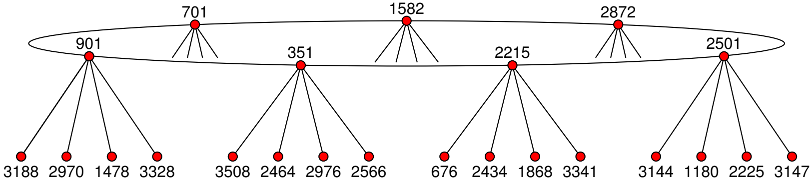
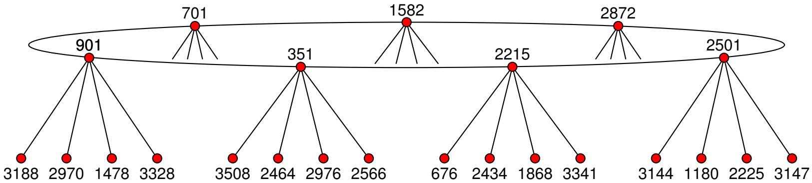
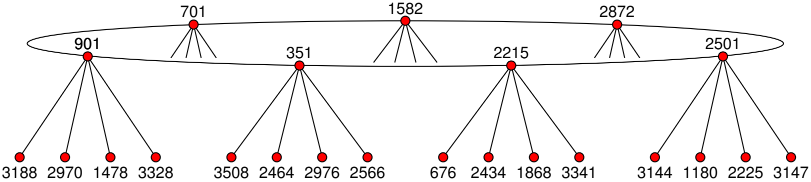


Interpolation



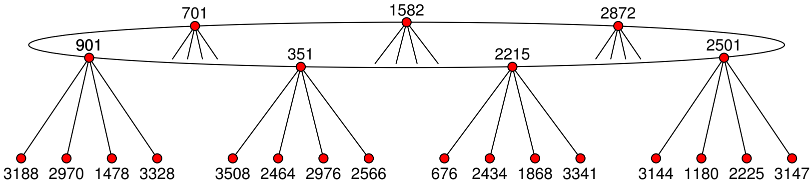
$$
\begin{aligned}
\Phi_5(X,\ 901) &= (X-\ 701)(X-\ 351)(X-3188)(X-2970)(X-1478)(X-3328)\\
\Phi_5(X,\ 351) &= (X-\ 901)(X-2215)(X-3508)(X-2464)(X-2976)(X-2566)\\
\Phi_5(X,2215) &= (X-\ 351)(X-2501)(X-3341)(X-1868)(X-2434)(X-\ 676)\\
\Phi_5(X,2501) &= (X-2215)(X-2872)(X-3147)(X-2255)(X-1180)(X-3144)\\
\Phi_5(X,2872) &= (X-2501)(X-1582)(X-1502)(X-4228)(X-1064)(X-2087)\\
\Phi_5(X,1582) &= (X-2872)(X-\ 701)(X-\ 945)(X-3497)(X-3244)(X-\ 291)\\
\Phi_5(X,\ 701) &= (X-1582)(X-\ 901)(X-2843)(X-4221)(X-3345)(X-4397)
\end{aligned}
$$

Interpolation



$$
\begin{aligned}
\Phi_5(X,\ \ 901) &= X^6 + 1337X^5 + \ \ 543X^4 + \ \ 497X^3 + 4391X^2 + 3144X + 3262\\
\Phi_5(X,\ \ 351) &= X^6 + 3174X^5 + 1789X^4 + 3373X^3 + 3972X^2 + 2932X + 4019\\
\Phi_5(X, 2215) &= X^6 + 2182X^5 + \ \ 512X^4 + \ \ 435X^3 + 2844X^2 + 2084X + 2709\\
\Phi_5(X, 2501) &= X^6 + 2991X^5 + 3075X^5 + 3918X^3 + 2241X^2 + 3755X + 1157\\
\Phi_5(X, 2872) &= X^6 + \ \ 389X^5 + 3292X^4 + 3909X^3 + \ \ 161X^2 + 1003X + 2091\\
\Phi_5(X, 1582) &= X^6 + 1803X^5 + \ \ 794X^4 + 3584X^3 + \ \ 225X^2 + 1530X + 1975\\
\Phi_5(X,\ \ 701) &= X^6 + \ \ 515X^5 + 1419X^4 + \ \ 941X^3 + 4145X^2 + 2722X + 2754
\end{aligned}
$$

Interpolation



$$
\begin{aligned}
\Phi_5(X,Y) = X^6 &+ (4450Y^5 + 3720Y^4 + 2433Y^3 + 3499Y^2 + 70Y + 3927)X^5 \\
&(3720Y^5 + 3683Y^4 + 2348Y^3 + 2808Y^2 + 3745Y + 233)X^4 \\
&(2433Y^5 + 2348Y^4 + 2028Y^3 + 2025Y^2 + 4006Y + 2211)X^3 \\
&(3499Y^5 + 2808Y^4 + 2025Y^3 + 4378Y^2 + 3886Y + 2050)X^2 \\
&(70Y^5 + 3745Y^4 + 4006Y^3 + 3886Y^2 + 905Y + 2091)X \\
(Y^6 + 3927Y^5 &+ 233Y^4 + 2211Y^3 + 2050Y^2 + 2091Y + 2108)
\end{aligned}
$$

Computing $\Phi_\ell(X, Y)$ mod p

Assume D and p are suitably chosen with $D = O(\ell^2)$ and $\log p = O(\log \ell)$, and that $H_D(X)$ has been precomputed.

1.	Find a root of $H_D(X)$ over $\mathbb{F}_p$.	$O(\ell \log^{3+\epsilon} \ell)$
2.	Enumerate the surface(s) using cl(D)-action.	$O(\ell \log^{2+\epsilon} \ell)$
3.	Descend to the floor using Vélu.	$O(\ell \log^{1+\epsilon} \ell)$
4.	Enumerate the floor using cl($\ell^2 D$)-action.	$O(\ell^2 \log^{2+\epsilon} \ell)$
5.	Build each $\Phi_\ell(X, j_i)$ from its roots.	$O(\ell^2 \log^{3+\epsilon} \ell)$
6.	Interpolate $\Phi_\ell(X, Y)$ mod p.	$O(\ell^2 \log^{3+\epsilon} \ell)$

Time complexity is $O(\ell^2 \log^{3+\epsilon} \ell)$.
Space complexity is $O(\ell^2 \log \ell)$.

After computing $\Phi_5(X, Y) \bmod p$ for the primes:

$$4451, 6911, 9551, 28111, 54851, 110051, 123491, 160591, 211711, 280451, 434111, 530851, 686051, 736511,$$

we apply the CRT to obtain

$$
\begin{aligned}
\Phi_5(X, Y) = {} & X^6 + Y^6 - X^5Y^5 + 3720(X^5Y^4 + X^4Y^5) - 4550940(X^5Y^3 + X^3Y^5) \\
& + 2028551200(X^5Y^2 + X^2Y^5) - 246683410950(X^5Y + XY^5) + 1963211489280(X^5 + Y^5) \\
& + 1665999364600X^4Y^4 + 107878928185336800(X^4Y^3 + X^3Y^4) \\
& + 383083609779811215375(X^4Y^2 + X^2Y^4) + 128541798906828816384000(X^4Y + XY^4) \\
& + 1284733132841424456253440(X^4 + Y^4) - 4550940(X^3Y^5 + X^5Y^3) \\
& - 441206965512914835246100X^3Y^3 + 26898488858380731577417728000(X^3Y^2 + X^2Y^3) \\
& - 192457934618928299655108231168000(X^3Y + XY^3) \\
& + 280244777828439527804321565297868800(X^3 + Y^3) \\
& + 5110941777552418083110765199360000X^2Y^2 \\
& + 36554736583949629295706472332656640000(X^2Y + XY^2) \\
& + 6692500042627997708487149415015068467200(X^2 + Y^2) \\
& - 264073457076620596259715790247978782949376XY \\
& + 53274330803424425450420160273356509151232000(X + Y) \\
& + 141359947154721358697753474691071362751004672000.
\end{aligned}
$$

After computing $\Phi_5(X, Y)$ mod p for the primes:

4451, 6911, 9551, 28111, 54851, 110051, 123491, 160591, 211711, 280451, 434111, 530851, 686051, 736511,

we apply the CRT to obtain

$$\begin{aligned}
\Phi_5(X, Y) = {} & X^6 + Y^6 - X^5Y^5 + 3720(X^5Y^4 + X^4Y^5) - 4550940(X^5Y^3 + X^3Y^5) \\
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& + 280244777828439527804321565297868800(X^3 + Y^3) \\
& + 5110941777552418083110765199360000X^2Y^2 \\
& + 36554736583949629295706472332656640000(X^2Y + XY^2) \\
& + 6692500042627997708487149415015068467200(X^2 + Y^2) \\
& - 264073457076620596259715790247978782949376XY \\
& + 53274330803424425450420160273356509151232000(X + Y) \\
& + 141359947154721358697753474691071362751004672000.
\end{aligned}$$

(but note that $\Phi_5^{\mathfrak{f}}(X, Y) = X^6 + Y^6 - X^5Y^5 + 4XY$).

Computing Φ_ℓ mod m

Given a prime $\ell > 2$ and an integer $m > 0$:

1. Pick a discriminant D suitable for ℓ.
2. Select a set of primes S suitable for ℓ and D.
3. Precompute H_D, cl(D), cl($\ell^2 D$), and CRT data.
4. For each $p \in S$, compute Φ_ℓ mod p and update CRT data.
5. Perform CRT postcomputation and output Φ_ℓ mod m.

To compute Φ_ℓ over $\mathbb{Z}$, just use $m = \prod p$.

For "small" m, use explicit CRT mod m.
For "large" m, standard CRT for large m.
For m in between, use a hybrid approach.

Complexity

Theorem (GRH)

For every prime $\ell > 2$ there is a suitable discriminant D with $|D| = O(\ell^2)$ for which there are $\Omega(\ell^3 \log^3 \ell)$ primes $p = O(\ell^6 (\log \ell)^4)$ that are suitable for ℓ and D.

Heuristically, $p = O(\ell^4)$. In practice, $\lg p < 64$.

Theorem (GRH)

The expected running time is $O(\ell^3 \log^3 \ell \log\log \ell)$.
The space required is $O(\ell^2 \log(\ell m))$.

An explicit height bound for Φ_ℓ

Let ℓ be a prime.
Let $h(\Phi_\ell)$ be the (natural) logarithmic height of Φ_ℓ.

Asymptotic bound: $h(\Phi_\ell) = 6\ell \log \ell + O(\ell)$ (Paula Cohen 1984).

An explicit height bound for Φ_ℓ

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Let $h(\Phi_\ell)$ be the (natural) logarithmic height of Φ_ℓ.

Asymptotic bound: $h(\Phi_\ell) = 6\ell\log\ell + O(\ell)$ (Paula Cohen 1984).

Explicit bound: $h(\Phi_\ell) \le 6\ell\log\ell + 17\ell$ (Bröker-S 2009).

Conjectural bound: $h(\Phi_\ell) \le 6\ell\log\ell + 12\ell$ (for $\ell > 30$).

The explicit bound holds for all ℓ.
The conjectural bound is known to hold for $30 < \ell < 3600$.

Other modular functions

We can compute polynomials relating $f(z)$ and $f(\ell z)$ for other modular functions, including the Weber $\mathfrak{f}$-function.

The coefficients of $\Phi_\ell^{\mathfrak{f}}$ are roughly 72 times smaller.
This means we need 72 fewer primes.

The polynomial $\Phi_\ell^{\mathfrak{f}}$ is roughly 24 times sparser.
This means we need 24 times fewer interpolation points.

We get a better than 1728-fold speedup using $\Phi_\ell^{\mathfrak{f}}$.

Modular polynomials for $\ell = 11$

Classical:

$X^{12} + Y^{12} - X^{11}Y^{11} + -1X^{11}Y^{11} + 8184X^{11}Y^{10} - 28278756X^{11}Y^9 + 53686822816X^{11}Y^8$
$- 61058988656490X^{11}Y^7 + 42570393135641712X^{11}Y^6 - 17899526272883039048X^{11}Y^5$
$+ 4297837238774928467520X^{11}Y^4 - 529134841844639613861795X^{11}Y^3 + 27209811658056645815522600X^{11}Y^2$
$- 374642006356701393515817612X^{11}Y + 296470902355240575283200000X^{11}$

. . . 8 pages omitted . . .

$+$ 392423345094527654908696 . . . 100 digits omitted . . . 000

Atkin:

$X^{12} - X^{11}Y + 744X^{11} + 196680X^{10} + 187X^9Y + 21354080X^9 + 506X^8Y + 830467440X^8$
$- 11440X^7Y + 16875327744X^7 - 57442X^6Y + 208564958976X^6 + 184184X^5Y + 1678582287360X^5$
$+ 1675784X^4Y + 9031525113600X^4 + 1867712X^3Y + 32349979904000X^3 - 8252640X^2Y + 74246810880000X^2$
$- 19849600XY + 98997734400000X + Y^2 - 8720000Y + 58411072000000$

Weber:

$X^{12} + Y^{12} - X^{11}Y^{11} + 11X^9Y^9 - 44X^7Y^7 + 88X^5Y^5 - 88X^3Y^3 + 32XY$

Weber modular polynomials

For $\ell = 1009$, the size of $\Phi_\ell^{\mathfrak{f}}$ is 2.3MB, versus 3.9GB for Φ_ℓ, and computing $\Phi_\ell^{\mathfrak{f}}$ takes 1.5s, versus 2840s for Φ_ℓ.

The current record is $\ell = 60013$.
Working mod *m*, level $\ell > 100000$ is feasible.

The polynomials $\Phi_\ell^{\mathfrak{f}}$ for all $\ell < 5000$ are available for download:

`http://math.mit.edu/~drew`